Copenhagen, Fall school on GGT
November 2023
Masterclass: BUILDINGS

References:

- Abramenko–Brown: Buildings – Thry & App
- Brown: Buildings
- Serre: Trees
- Davis: Geometry of Coxeter groups
- Ronan: lectures on buildings

other books: Garrett, Thomas

- Weiss: Structure of spherical bldgs } class
- — " — affine — " —
- MST: Dimensions of ADLVs (AMS Memoirs)
- Schwer: Shadows in the wild
- Tits: Lecture notes on spherical bldgs and BN-pairs

Buildings

grps via actions on spaces

- introduced as analogs of symm. spaces for reductive grps over non-Archimedian fields (with valuation)
- later axiomatized and classified
- encode a lot of information about these grps and associated Grassmannians, flag- and other varieties
- prime examples for CAT(0) spaces
  more generally κ

Part 1: First examples and a definition
Part 2: Coxeter groups & shadows
Part 3: BN-pairs and retractions
Part 4: Connection with CAT(0) geometry

Part 1 First examples and a definition

① The Fano plane / projective planes

Given a finite field $\mathbb{F}_q$ of order q

We construct a geometric object on which $GL_3(\mathbb{F}_q)$ acts and which encodes properties of (certain subgrps of) this grp

$G := GL_3(\mathbb{F}_q)$ or $SL_3(\mathbb{F}_q)$

$V := \mathbb{F}_q^3$ 3-dim VS over $\mathbb{F}_q$

$\mathcal{P} := \{1\text{-dim subspaces of } V\}$ points

$\mathcal{L} := \{2\text{-dim} \quad \text{—"—} \quad \}$ lines

We say a point p is incident to a line l if (as linear subspaces of V) $p \subset l$.

The incidence graph Δ_q of the projective plane is given as follows:

- vertices correspond to elements of $\mathcal{P} \sqcup \mathcal{L}$
- a pair $\{p, l\}$ forms an edge if $p \in \mathcal{P}$, $l \in \mathcal{L}$ and p incident with l.

Exercise: prove the following:

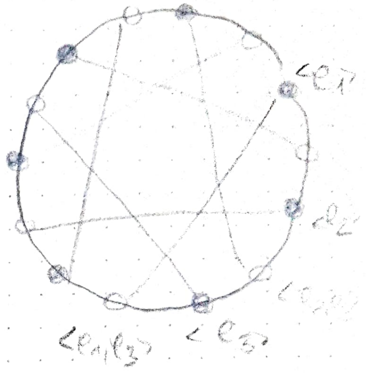
a) $|\mathcal{L}| = |\mathcal{P}| = q^2 + q + 1$

b) each point is contained in $q+1$ distinct lines

c) " line contains " ——— points

<u>Example:</u> $q=2$: # points & lines is $2^2+2+1=7$

$\Delta_2 =$ <u>Heawood graph</u>

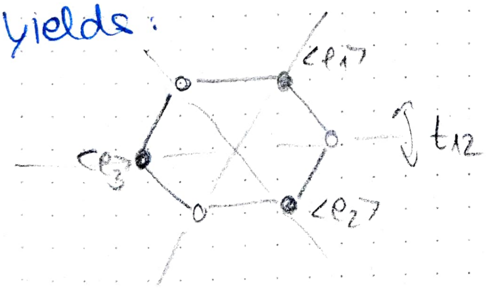


- $\mathcal{P} = \{\langle e_i\rangle, \langle e_i+e_j\rangle_{i\neq j}, \langle e_1+e_2+e_3\rangle\}$
- $\mathcal{L} = \{$spans of pairs of distinct points$\}$

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

The graph Δ_q is a union of hexagons (subgraphs in the form of), called <u>apartments</u>. Each of them corresponds to a basis in V.

Basis $\{e_1, e_2, e_3\}$ yields:



Sym(3) acts on every apartment by permuting the basis vectors.

<u>Connection between G and Δ:</u>

As a matrix group $G = GL_3(\mathbb{F}_q)$ acts on $V = \mathbb{F}_q^3$ by multiplication.

$\Rightarrow$ G acts on Δ and permutes points, resp. lines and apartments. (colors are preserved)

This action is transitive on pairs edge $\subset$ apartment.

What is stab(e)?

$$e = \underset{\langle e_1\rangle}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{\langle e_1, e_2\rangle}{\bullet}$$

Vertex stabilizers:

- $\mathrm{stab}_G(\langle e_1\rangle) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\} =: P_2$
- $\mathrm{stab}_G(\langle e_1, e_2\rangle) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} =: P_1$

P_1, P_2 are the <u>standard parabolic subgps</u>

- $\mathrm{stab}_G(e) = P_1 \cap P_2 =: B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}$

  the <u>standard Borel subgroup</u>.

Properties A: (Proof: exercise)

a) $\mathcal{L} \underset{1:1}{\hat{=}} G/P_1 \qquad \mathcal{P} \underset{1:1}{\hat{=}} G/P_2 \qquad \underset{\text{in } \Delta_q}{\text{edges}} \hat{=} G/B$

i.e. simplices in Δ_p are labeled by a unique left-coset of P_1, P_2 or B.

b) Edges gB and hB share a P_i-vertex iff $gP_i = hP_i$ (or, equiv. iff $g^{-1}h \in P_i$)

Stabilizers of Apartments:

Consider the apartment A wrt the standard basis $\{e_1, e_2, e_3\}$.

- pointwise stab. of A is the torus of G
  $$T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \in G \right\}$$
- setwise stabilizer is the subgrp
  $$N = \left\{ \begin{matrix} \text{monomial} \\ \text{mxes} \end{matrix} \right\}$$

Properties B (Proof: exercise)

a) N is the normalizer of T in G

b) $N/T \cong Sym(3)$ and in particular a Coxeter group

Put $W := N/T$ then W is generated by the matrices $s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ satisfying $(s_1 s_2)^3 = \mathbb{1}$.

c) $P_i = B \sqcup B s_i B$ for $i = 1, 2$

d) $G = \bigsqcup_{w \in W} BwB$ Bruhat decomposition of G

Note: the pair B, N above is an instance of a spherical BN-pair. more later in the course

$\rightarrow$ char. 0, compl. … $\mathbb{Q}$ $v_q(\frac{a}{b}) = n$ where $\frac{a}{b} = \frac{a'}{b'} p^n$ wrt and $p \nmid a', b'$

② A tree for $SL_2(\mathbb{Q}_p)$

(write elements as $\sum_{n \geq N} a_n p^n$ with $a_n \in \mathbb{F}_p$, $a_N \neq 0$

Let K be a field with **discrete valuation** $\nu : K^* \to \mathbb{Z} \cup \{\infty\}$ s.th.

$$\nu(x+y) \geq \inf\{\nu(x), \nu(y)\} \quad \forall x, y \in K$$

"=" if $\nu(x) \neq \nu(y)$

$$\nu(x \cdot y) = \nu(x) + \nu(y)$$

$$\nu(x) = \infty \text{ iff } x = 0$$

Ex: p-adic valuation on $\mathbb{Q}$:

$\nu_p(\frac{a}{b}) := n$ where $\frac{a}{b} = p^n \cdot \frac{a'}{b'}$ and $p \nmid a', b'$

Define further: (N=0 above)

- valuation ring $\mathcal{O} := \{x \in K \mid \nu(x) \geq 0\}$ $\mathbb{Z}_p$
- uniformizer $\pi \in K$: $\nu(\pi) = 1$ $\pi = p$
- residue field $k := \mathcal{O}/\pi\mathcal{O}$ $\cong \mathbb{F}_p$

Consider $V := K^2$ 2-dim VS

A lattice L in V is a f.g. $\mathcal{O}$-submodule of V

Write $[L]$ for its homothety class

$\nwarrow$ equiv. up to rescaling with elts in K^*

Let $X :=$ set of classes $[L]$ of lattices in the space V.

Define a <u>graph Δ</u> as follows:

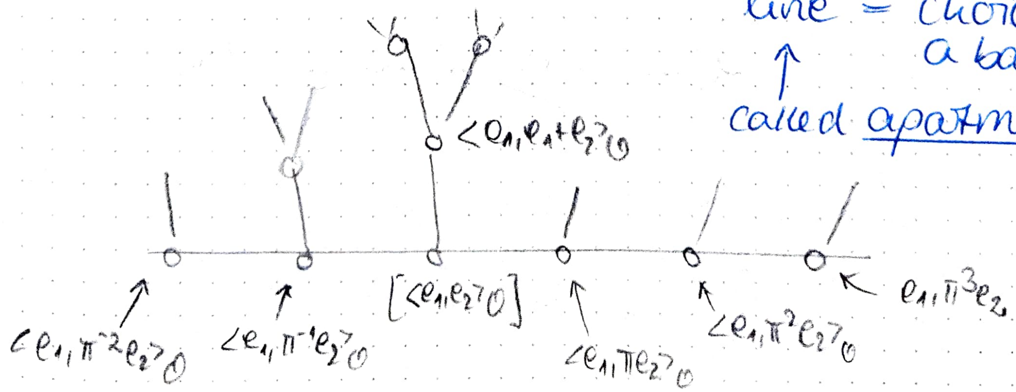
- vertices $\hat{=}$ elements in X
- edges $\underset{[L]}{\circ}\!\!\longrightarrow\!\!\underset{[L']}{\circ}$ exist if there exist representatives L, L' of the classes s.th. $\pi L \subset L' \subset L$

or, equiv: $L' \subset L$ and $L/L' \cong k$

<u>Thm</u> The graph Δ is a tree without leafs. Every pair of vertices (or edges) is contained in a common bi-infinite line.

Every pair of lattices L, L' has a common $\mathcal{O}$-basis s.th. $\{b_1, b_2\}$ is a basis for L and there exist $\underbrace{a_1, a_2}_{\in \mathbb{Z}}$ s.th. $\{b_1 \pi^{a_1}, b_2 \pi^{a_2}\}$ is a basis for L'.

line $\hat{=}$ choice of a basis
$\uparrow$
called <u>apartments</u>



Properties C

a) $Sl_2(K)$ acts on Δ. This action is transitive on pairs of edges contained in apartments.

b) A line is setwise stabilized by the semidirect product of $\langle \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=s} \rangle$ and $T(\mathcal{O}) = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \in Sl_2(\mathcal{O}) \right\}$

c) s reflects the line corresponding to e_1, e_2 at the origin $\langle e_1, e_2 \rangle_{\mathcal{O}}$

d) Stabilizer of the edge $\underset{\langle e_1, e_2\rangle_{\mathcal{O}}}{\circ}$ —— $\underset{\langle e_1, e_2 \pi\rangle_{\mathcal{O}}}{\circ}$ is given by

$$I := \left\{ \begin{pmatrix} \mathbb{Z}_p & \overset{=\mathcal{O}}{\mathbb{Z}_p} \\ \underset{\pi\mathcal{O}}{p\mathbb{Z}_p} & \mathbb{Z}_p \end{pmatrix} \in Sl_2(\mathbb{Z}_p) \right\}$$

= inverse image of the Borel $B(k)$ under the proj. $\mathcal{O} \twoheadrightarrow k$

For a proof see Serre chapter II, Brown VI 8B

Put again $N(K) :=$ normalizer of $T(K)$, then $(I, N(K))$ is an affine BN-pair for $Sl_2(K)$

3) A first formal definition of a BUILDING:

Def: A building is a simplicial complex $\Delta \neq \emptyset$ together with a collection of sub-complexes $A \in \mathcal{A}$, called apartments, s.th. the following hold: $\Delta = \bigsqcup_{A \in \mathcal{A}} A$ and

(B0) Each apartment is a Coxeter complex

(B1) For every pair of simplices a, b in Δ there exists an apartment $A \in \mathcal{A}$ containing a and b.

(B2) if A, A' are two apartments containing a and b then $\exists$ isomorph. $A \to A'$ fixing $A \cap A'$ pointwise.

Maximal simplices in Δ are called chambers or alcoves.

Play with Bram Bekker's buildings gallery