Problem Sheet

Problems on Coxeter groups

Problem 1
Let $W = \mathrm{Sym}_4$ be the symmetric group on four letters (i.e., the Coxeter group of type A_3.) Find all reduced decompositions of the unique longest element $w_0 = s_1s_3s_2s_1s_3s_2 \in W$.

Problem 2
Show that the direct product $W_1 \times W_2$ of two Coxeter groups W_1 and W_2 is a Coxeter group.

Problem 3
Use the deletion condition to prove the following statements (and give geometric interpretations).

(a) Let (W, S) be a Coxeter system and $S' \subset S$, $W' := \langle S' \rangle$. Then the word length of an element of W' with respect to S' is the same as its length with respect to S.

(b) Let W' be as above. Show that every coset wW' has a unique representative w_m of minimal length in W and that $l_S(w_m w') = l_S(w_m) + l_S(w')$ for all $w' \in W'$.

(c) Suppose W is finite. Show that W contains a unique element w_0 of maximal length and that $l(w_0) = l_S(w) + l_S(w^{-1}w_0)$.

Problem 4
Draw the Cayley graphs of the triangle groups $(2, 3, 5)$ and $(2, 3, 7)$ with respect to the standard generators.

Here a triangle group (a, b, c) is a group generated by three elements s_1, s_2, s_3 such that the order of s_1s_2 is a, the order of s_2s_3 is b and the order of s_3s_1 is c.

Problem 5
Show that the following two Coxeter presentations define isomorphic Coxeter groups

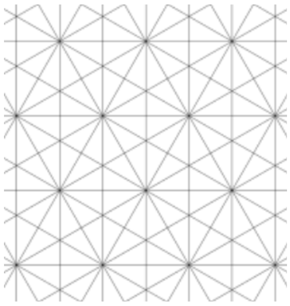
$$\langle s_1, s_2 \mid s_i^2 = (s_1, s_2)^6 = 1\rangle, \quad \langle t_1, t_2, t_3 \mid t_i^2 = (t_1, t_2)^3 = (t_2, t_3)^2 = (t_1t_3)^2 = 1\rangle.$$

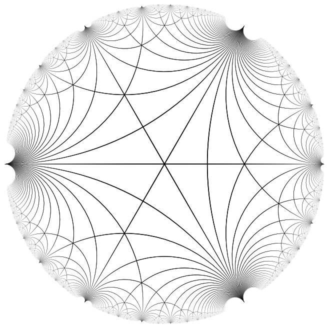
Problem 6

Prove that there exist only finitely many Euclidean triangle groups and list all of them.

Problem 7

Write out the Coxeter presentations which correspond to the following tessellations:





Problem 8

Prove that for every Coxeter system (W, S) there exists an epimorphism $\epsilon : W \to \mathbb{Z}/2\mathbb{Z}$ with $\epsilon(s) = [1]$ for all s in S.

Problem 9

Let (W, S) be a Coxeter system. Define the set of reflections R as

$$R := \{wsw^{-1} \mid s \in S, w \in W\}.$$

Further, let $l_R : W \to \mathbb{N}_0$ be the corresponding length function.

(a) Show that $l_R(w) \leq l_S(w)$ for all $w \in W$.

(b) Show that l_R is constant on every conjugacy class in W.

(c) Show that for all $u, v \in W$ we have $l_R(uv) = l_R(u) + l_R(v) \mod 2$.

Problem 10

Let (W, S) be a Coxeter system and $\Gamma = (V, E)$ be the corresponding Coxeter graph. Define $\Gamma' = (V', E')$ to be the subgraph of Γ with $V' = V$ and E' the set of edges in E with odd label. Show that the set of connected components of Γ' and the set of conjugacy classes of the generators in S are bijective.

Problem 11
Let (W, S) be a Coxeter system. Prove that the shadow of an element $w \in W$ with respect to the trivial positive orientation is the same as the interval $[1, w]$ in Bruhat order.

Problem 12
Play with the shadow-app on the following website:
`https://www.mathelabor.ovgu.de/shadows`

Problems on buildings

Problem 13
Prove that every complete bipartite graph is a building. Name the underlying Coxeter group.

Problem 14
Prove that the Heawood graph and the SL_2-tree constructed in the lecture are buildings. Moreover, prove that every tree without leafs is a building of type D_∞.

Problem 15
Suppose Γ is a metric realization of a graph with the following properties:

- every vertex is contained in at least two edges, and
- Γ is δ-hyperbolic with respect to every $\delta > 0$.

Prove that then Γ is a tree without leafs, i.e. a building.

Problem 16
Fix $n \in \mathbb{N}$ and let V be a vector space of dimension $(n+1)$ over a field K. Let $\Delta(V)$ be the flag-complex of V and let $\phi : GL(V) \to Aut(\Delta(V))$ be the natural action of $GL(V)$ on $\Delta(V)$. (See below for a definition). Prove that the following hold true:

(a) Every chamber, i.e. maximal flag, in $\Delta(V)$ has length n and every flag $\{U_1, \ldots, U_k\}$ is contained in a chamber.

(b) The complex $\Delta(V)$ is the union of all apartments, i.e.

$$\Delta(V) = \bigcup_{B \text{ basis of } V} \{\Sigma(B)\},$$

where $\Sigma(B)$ is the set of all flags spanned by the basis B.

(c) The kernel of ϕ consists of all non-trivial scalar multiples of the identity matrix, i.e.

$$\ker(\phi) = \{\lambda E_n \mid \lambda \in K^*\}.$$

(d) $GL(V)$ acts transitively on the set of all apartments, the set of chambers and the set of vertices in $\Delta(V)$.

Fix $n \in \mathbb{N}$ and let V be a vector space of dimension $(n+1)$ over a field K. A sequence of k ascending proper, non-trivial sub-vector spaces V_i in V, i.e. $V_1 \subset V_2 \subset \cdots \subset V_k \subset V$, is called a *flag* of length k. A *maximal flag* is a flag of length n. We typically refer to those as *chambers*. The *flag-complex* of V, denoted by $\Delta(V)$, is the set of all flags in V. The group $GL(V)$ naturally acts on the flag complex by left-multiplication on the elements V_i of the flags.

Problem 17

Let Δ_q denote the projective plane over F_q. Prove that

(a) $|\mathcal{L}| = |\mathcal{P}| = q^2 + q + 1$

(b) Each point is contained in $q + 1$ distinct lines.

(b) Each line contains $q + 1$ distinct points.

Draw a completely labeled picture of D_2. (If you are brave also of Δ_3.)

Problem 18

Let $V = F_q^3$, $G = GL_3(F_q)$, e_i be the i-th standard basis vector in V and put $P_1 = \text{stab}_G(\langle e_1 \rangle)$, $P_2 = \text{stab}_G(\langle e_1, e_2 \rangle)$ and $B = P_1 \cap P_2$. Prove that

(a) $\mathcal{L}$ is in bijection with G/P_1.

(b) $\mathcal{P}$ is in bijection with G/P_2.

(c) Edges in the projective plane are in bijection with G/B.

(d) Two edges gB and hB share a vertex of type P_i (i.e. a coset of P_i representing either a line if $i = 1$ or a point if $i = 2$) if and only if $gP_i = hP_i$. The condition $gP_i = hP_i$ is equivalent to saying that $g^{-1}h \in P_i$.

Problem 19

Let $G = GL_3(F_q)$, denote by N the set of monomial matrices and let T denote the diagonal matrices in G. The groups P_i and B are as in Problem 16. Prove that

(a) N is the normalizer of T in G.

(b) $W := N/T$ is isomorphic to $Sym(3)$ the symmetric group on three letter.

(c) $P_i = B \cup Bs_iB$, i=1,2, where $s_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$.

Problem 20

Let $G = GL_3(F_q)$ and let B and W be as in Problem 17. Prove that G admits the Bruhat decomposition, that is

$$G = \bigsqcup_{w \in W} BwB.$$

Problem 21

Prove that apartments are convex in the following sense: Suppose A is an apartment in a building Δ. Let x be a chamber in A and σ some other simplex in A. Then every minimal gallery connecting x and σ is contained in A. (Why is it important to have a chamber x and a simplex and not just two arbitrary simplices in A?)

Problem 22

Let A be an apartment of an affine building Δ of type (W, S). Let ϕ_c be an orientation on A determined by an alcove c and let ϕ_∞ be an orientation determined by a chamber in ∂A. Fix a base-alcove c_0 in A and let γ be a folded gallery of type w, where $\vec{w}$ is minimal. Prove the following:

(a) if γ is ϕ_c-positively folded, there exists a minimal gallery τ in δ starting in c_0 such that $\rho_{d,A}(\tau) = \gamma$.

(b) if γ is ϕ_∞-positively folded, there exists a minimal gallery τ in δ starting in c_0 such that $\rho_{\infty,A}(\tau) = \gamma$.

(c) for $\phi \in \{\phi_d, \phi_\infty\}$, the ϕ-shadow of w is the image of all end-alcoves of minimal galleries in Δ of type $\vec{w}$ starting in c_0.