EIGENVALUE PROBLEMS AND FREE BOUNDARY MINIMAL SURFACES IN SPHERICAL CAPS

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1. Introduction

In these notes we are going to discuss the work of the author joint with Ana Menezes [46], where a link is stablished between free boundary minimal surfaces in geodesic balls of round spheres and Steklov-type eingenvalue problems, and this is applied to uniqueness questions in minimal surface theory.

2. Free-boundary minimal immersions

Consider a compact manifold Σ and a Riemannian manifold with boundary $(M, \langle \cdot, \cdot \rangle)$. Let $\Phi_t : \Sigma \to (M, \langle \cdot, \cdot \rangle)$ be a smooth one parameter family of immersions for $t \in (-\epsilon, \epsilon)$ and denote $X = \frac{\partial \Phi}{\partial t}$. We assume $\Phi_t(\partial\Sigma) \subset \partial M$, thus $X|_{\partial\Sigma}$ is tangent to ∂M.

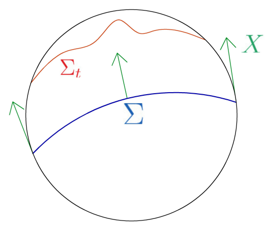

Figure 1. A smooth variation of a submanifold.

The first variation formula gives:

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}\big(\Phi_t(\Sigma)\big) = -\int_\Sigma \langle \vec{H}, X\rangle dA + \int_{\partial\Sigma} \langle \nu, X\rangle dL, \tag{2.1}$$

where $\vec{H}$ is the mean curvature of the immersion Φ_0 and ν is the outward pointing unit conormal of $\Phi(\partial\Sigma)$. Therefore Φ_0 is a critical point of the volume functional if and only if $\vec{H} = 0$ and $\nu \perp \partial\mathcal{N}$.

Exercise 1. Prove (2.1) and the caracterization of critical points of vol.

Definition 1. Let $\Phi : \Sigma \to zM$ be an immersion such that $\Phi(\partial\Sigma) \subset \partial\mathcal{M}$. We say Φ is a *free boundary minimal immersion* if the following conditions hold:

(i) $\vec{H} = 0$;
(ii) $\Phi(\Sigma)$ meets ∂M orthogonally along $\Phi(\partial\Sigma)$, i.e., $\nu \perp \partial M$.

It is also commom to identify Σ with its image $\Phi(\Sigma)$ and say Σ is a free-boundary minimal submanifold of $(M, \langle \cdot, \cdot \rangle)$.

Before we continue, we need to fix some notation. A point $x \in \mathbb{R}^{n+1}$ will be described in terms of the canonical cartesian coordinates as $x = (x_0, x_1, \ldots, x_n)$. The canonical basis of $\mathbb{R}^{n+1}$ will be denoted by $\{\partial_0, \partial_1, \ldots, \partial_n\}$. Let $\varepsilon \in \{-1, 0, 1\}$ and consider the following metric on $\mathbb{R}^{n+1}$,

$$g_\varepsilon = \varepsilon dx_0^2 + \sum_{i=1}^{n} (dx_i)^2. \tag{2.2}$$

Define

$$\mathbb{E}^n_\varepsilon = \left\{ x \in \mathbb{R}^{n+1} : x_0^2 + \varepsilon \sum_{i=1}^{n} (x_i)^2 = 1 \text{ and } x_0 > 0 \right\}.$$

Then $\mathbb{E}^n_\varepsilon$ endowed with the Riemannian metric induced by g_ε correponds to the Euclidean space $\mathbb{R}^n$ when $\varepsilon = 0$, to the round hemisphere $\mathbb{S}^n_+$ when $\varepsilon = 1$ and to the hyperbolic space $\mathbb{H}^n$ when $\varepsilon = -1$.

From now on, along all the text, when $\varepsilon = 1$ we will always assume $r < \pi/2$. Define

$$c_\varepsilon(r) = \begin{cases} \cos r, & \text{if } \varepsilon = 1, \\ 1, & \text{if } \varepsilon = 0, \\ \cosh r, & \text{if } \varepsilon = -1, \end{cases} \qquad s_\varepsilon(r) = \begin{cases} \sin r, & \text{if } \varepsilon = 1, \\ r, & \text{if } \varepsilon = 0, \\ \sinh r, & \text{if } \varepsilon = -1, \end{cases}$$

and also,

$$\tan_\varepsilon r = s_\varepsilon(r)/c_\varepsilon(r) \quad \text{and} \quad \cot_\varepsilon r = c_\varepsilon(r)/s_\varepsilon(r).$$

Observe

$$c_\varepsilon(r)^2 + \varepsilon s_\varepsilon(r)^2 = 1 \tag{2.3}$$

We will denote by $\mathbb{B}^n_\varepsilon(r)$ the closed geodesic ball of radius r and center $(1, 0, \ldots, 0)$ in $\mathbb{E}^n_\varepsilon$. We have

$$\mathbb{B}^n_\varepsilon(r) = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x_i)^2 \leq s^2_\varepsilon(r) \right\}. \tag{2.4}$$

Equivalently, for $\varepsilon \in \{-1, 1\}$ it holds

$$\mathbb{B}^n_\varepsilon(r) = \left\{ x \in \mathbb{R}^{n+1} : \varepsilon x_0 \geq \varepsilon c_\varepsilon(r) \right\}. \tag{2.5}$$

On the case of Euclidean balls, using an homothety we can send $\mathbb{B}^3_0(r)$ to $\mathbb{B}^3$ and vice-versa, so we can assume $r = 1$. Because of this we use the simplified notation $\mathbb{B}^n = \mathbb{B}^n_0(1)$.

2.1. **Free-boundary minimal disks in $\mathbb{B}^n_\varepsilon(r)$.** Let Π a vector subspace of $\mathbb{R}^{n+1}$ of dimension 3, containing ∂_0. Then $\Pi \cap \mathbb{B}^n_\varepsilon(r)$ is a totally geodesic disk inside $\mathbb{B}^n_\varepsilon(r)$ which satisfies the free-boundary condition. When $\varepsilon = 0$ we call these surfaces *equatorial disks.*

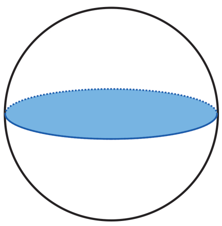

FIGURE 2. An equatorial disk in $\mathbb{B}^3$.

In the following, $\mathbb{D}$ denotes the disk in $\mathbb{R}^2$, of radius 1 and center at origin. A fundamental result due to Nitsche [52] stablishes that a free-boundary minimal immersion of $\mathbb{D}$ in $\mathbb{B}^3$ is totally geodesic. This was later generalized by Souam [57] to the case of immersions in $\mathbb{B}^3_\varepsilon(r)$, $\varepsilon \in \{-1, 1\}$. This was strenghtened in the following result.

Theorem 1 (Fraser-Schoen [24])**.** *Let $\Phi : \mathbb{D} \to \mathbb{B}^n_\varepsilon(r)$ be a branched free boundary minimal immersion which is proper, i.e. $\Phi(\mathbb{D}) \cap \partial\mathbb{B}^n_\varepsilon(r) = \Phi(\partial\mathbb{D})$. Then $\Phi(\mathbb{D})$ is totally geodesic.*

Proof. We will use the following notation: ∇ is the pull back connection in the pullback bundle $\Phi^*(T\mathbb{E}^n_\varepsilon)$; $\nabla^\perp$ is the connection in the pullback $\Phi^*(N\Sigma)$ of the normal bundle to $\Sigma = \Phi(\mathbb{D})$ in $\mathbb{B}^n_\varepsilon(r)$; $\mathrm{I\!I}$ is the second fundamental form of Σ in $\mathbb{E}^n_\varepsilon$. Consider any complex coordinate $z = x_1 + ix_2$ on Σ, and

$$\mathrm{I\!I}(\partial_z, \partial_z) = \frac{1}{4}\left[(\mathrm{I\!I}_{11} - \mathrm{I\!I}_{22}) - 2i\mathrm{I\!I}_{12}\right] \tag{2.6}$$

where $\mathrm{I\!I}_{ij} = \mathrm{I\!I}(\partial_{x_i}, \partial_{x_j})$. Denote $\mathrm{I\!I}_{ij;k} = (\nabla^\perp_{\partial_{x_k}}\mathrm{I\!I})(\partial_{x_i}, \partial_{x_j})$. Since Φ is conformal and minimal we have $\mathrm{I\!I}_{11} + \mathrm{I\!I}_{22} = 0$. This together with the Codazzi equations implies

$$\nabla^\perp_{\partial_{\bar z}}\mathrm{I\!I}(\partial_z, \partial_z) = \frac{1}{8}\left[\mathrm{I\!I}_{11;1} - \mathrm{I\!I}_{22;1} + 2\mathrm{I\!I}_{12;2} + i(\mathrm{I\!I}_{11;2} - \mathrm{I\!I}_{22;2} - 2\mathrm{I\!I}_{12;1})\right] = 0.$$

It follows from metric compatibility of the connection that the following function is holomorphic on $\mathbb{D}$,

$$\phi(z) := \big\langle \mathrm{I\!I}(\partial_z, \partial_z), \mathrm{I\!I}(\partial_z, \partial_z)\big\rangle.$$

Consider polar coordinates (r, θ) on the disk, so $z = re^{i\theta}$. Set $w = \log z = \log r + i\theta$. Then

$$\partial_z = \frac{\partial w}{\partial z}\partial_w = \frac{1}{z}\partial_w \quad \text{and} \quad \mathrm{I\!I}(\partial_z, \partial_z) = \frac{1}{z^2}\mathrm{I\!I}(\partial_w, \partial_w).$$

On $\partial\mathbb{D}$ we have $\partial_w = \partial_r + i\partial_\theta$, and

$$z^2\mathrm{I\!I}(\partial_z, \partial_z) = \mathrm{I\!I}(\partial_w, \partial_w) = \mathrm{I\!I}(\partial_r, \partial_r) - \mathrm{I\!I}(\partial_\theta, \partial_\theta) - 2i\mathrm{I\!I}(\partial_r, \partial_\theta). \tag{2.7}$$

By the free boundary condition, the outward unit normal ν along $\partial\Sigma$ is normal to the sphere $\partial\mathbb{B}^n_\varepsilon(r)$. Thus, since $d\Phi(\partial_r) = \lambda\nu$, where $\lambda = |d\Phi(\partial_r)|$, we have

$$\mathrm{I\!I}(\partial_r, \partial_\theta) = \nabla^\perp_{\partial_\theta} d\Phi(\partial_r) = [-\lambda \cot_\varepsilon r d\Phi(\partial_\theta)]^\perp,$$

Therefore,

$$\mathrm{I\!I}(\partial_r, \partial_\theta) = 0 \text{ on } \partial\mathbb{D}, \tag{2.8}$$

and so $z^2\mathrm{I\!I}(\partial_z, \partial_z)$ is real on $\partial\mathbb{D}$.

Thus, $z^4\phi(z)$ is a holomorphic function on $\mathbb{D}$ that is real on $\partial\mathbb{D}$. This implies that $z^4\phi(z)$ is constant, and since it vanishes at the origin, one has $z^4\phi(z) \equiv 0$. In particular, by (2.7) and (2.8) we see that on $\partial\mathbb{D}$,

$$0 = z^4\phi(z) = \left[\mathrm{I\!I}(\partial_r, \partial_r) - \mathrm{I\!I}(\partial_\theta, \partial_\theta)\right]^2.$$

Therefore, $z^2\mathrm{I\!I}(\partial_z, \partial_z) = 0$ on $\partial\mathbb{D}$. We claim $z^2\mathrm{I\!I}(\partial_z, \partial_z) \equiv 0$ on $\mathbb{D}$. To see this, choose a global frame $s_1, \ldots, s_{n-2}$ for the normal bundle $\Phi^*(N\Sigma)$. Then,

$$z^2\mathrm{I\!I}(\partial_z, \partial_z) = \sum_{j=1}^{n-2} f_j s_j$$

for some complex-valued functions f_j on $\mathbb{D}$, $j = 1, \ldots, n-2$, and

$$0 = \nabla^\perp_{\partial_{\bar z}}\left(z^2\mathrm{I\!I}(\partial_z, \partial_z)\right) = \nabla^\perp_{\partial_{\bar z}}\left(\sum_{j=1}^{n-2} f_j s_j\right) = \sum_{j=1}^{n-2}\left(\frac{\partial f_j}{\partial \bar z} + \sum_{k=1}^{n-2} a_{kj} f_k\right)s_j,$$

where a_{jk}, j, $k = 1, \ldots n-2$, are given by $\nabla^\perp_{\partial_{\bar z}} s_j = \sum_{k=1}^{n-2} a_{jk} s_k$. Hence, for $j = 1, \ldots, n-2$ we have

$$\begin{cases} \frac{\partial f_j}{\partial \bar z} + \sum_{k=1}^{n-2} a_{kj} f_k = 0, & \text{on } \mathbb{D} \\ f_j = 0, & \text{on } \partial\mathbb{D}. \end{cases}$$

This implies that $f_j \equiv 0$ on $\mathbb{D}$ for $j = 1, \ldots, n-2$, and so $z^2 \mathrm{I\!I}(\partial_z, \partial_z) \equiv 0$ on $\mathbb{D}$.

Hence, $\mathrm{I\!I}(\partial_z, \partial_z) \equiv 0$ on $\mathbb{D}$. By (2.6), this implies that $\mathrm{I\!I}_{11} = \mathrm{I\!I}_{22}$ and $\mathrm{I\!I}_{12} = 0$ on $\mathbb{D}$. Therefore Σ is totally geodesic. $\square$

There is an strinking analogy between minimal submanifolds of spheres and free boundary minimal submanifolds of the unit Euclidean ball. In particular, we can see Nitsche's Theorem as analogous to a result of Almgren [1] which says that a minimal $\mathbb{S}^2$ in $\mathbb{S}^3$ is totally geodesic. Calabi [3] proved that there are many minimal immersions of $\mathbb{S}^2$ in $\mathbb{S}^n$ for $n \geq 4$, which are not totally geodesic. So, from the point of view of the mentioned analogy, the previous result is surprising. However, one has the following version of uniqueness of minimal 2-spheres of higher dimensional spheres.

Corollary 1. *Let Σ be an immersed minimal $\mathbb{S}^2$ in $\mathbb{S}^n$ that is invariant under reflection through a totally geodesic $\mathbb{S}^{n-1}$. Then Σ is totally geodesic.*

Proof. Since the surface Σ intersects the equatorial $\mathbb{S}^{n-1}$ orthogonally, it is transversal to it. Thus the intersection consists of smooth circles. If we consider an inner-most circle (one which bounds a disk disjoint from the other circles), then we get a free boundary minimal disk in a hemisphere, and by Theorem 2.2 this disk is totally geodesic. It follows that Σ is the corresponding totally geodesic minimal $\mathbb{S}^2$ since it is assumed to be reflection symmetric. $\square$

2.2. **Free-boundary minimal annuli in $\mathbb{B}^n_\varepsilon(r)$.** In $\mathbb{B}^3$ there exist a family of free-boundary minimal annuli called *critical catenoids*, which are characterized as the only non-flat free boundary minimal surfaces in $\mathbb{B}^3$ that are invariant under rotations around a given axis. They can be defined as follows. Fix an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and let t_0 be the unique positive solution to the equation $t\sinh(t) = \cosh(t)$. The critical catenoids are the images of the maps $\Phi : [-t_0, t_0] \times \mathbb{S}^1 \to \mathbb{R}^3$,

$$\Phi(t, \theta) = a_0 \cosh(t)\cos(\theta)e_1 + a_0 \cosh(t)\sin(\theta)e_2 + a_0 t e_3,$$

where $a_0 = (t_0 \cosh(t_0))^{-1}$. The constants t_0 and a_0 are chosen in such way that the map Φ is conformal and its image is a piece of a catenoid contained in $\mathbb{B}^3$, meeting $\partial\mathbb{B}^3$ orthogonally, and whose axis of symmetry is the line generated by the vector e_3.

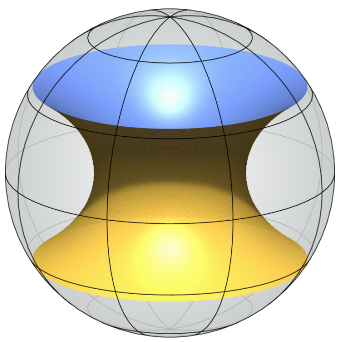

FIGURE 3. A critical catenoid in $\mathbb{B}^3$. Picture by Mario Schulz.

Inspired by a result due to Chern-do Carmo-Kobayashi [12] and Lawson [38] concerning closed minimal surfaces, Lucas Ambrozio and Ivaldo Nunes proved a uniqueness result for free-boundary minimal disks and annuli.

Theorem 2 (Ambrozio-Nunes [4]). *Let Σ be a compact free boundary minimal surface in the Euclidean ball $\mathbb{B}^3$. Assume that for all points $x \in \Sigma$, we have*

$$|\mathrm{II}|^2(x)\langle x, N(x)\rangle^2 \le 2, \tag{2.9}$$

where $N(x)$ denotes a unit normal vector at the point $x \in \Sigma$ and A denotes the second fundamental form of Σ. Then

(1) either $|\mathrm{II}|^2\langle x, N\rangle^2 \equiv 0$ and Σ is a flat equatorial disk;
(2) or $|\mathrm{II}|^2(p)\langle p, N(p)\rangle^2 = 2$ at some point $p \in \Sigma$ and Σ is a critical catenoid.

We say Σ is a rotation hypersurface of $\mathbb{E}^n_\varepsilon$ if the following holds: there exist subspaces Π^2, Π^3 of $\mathbb{R}^{n+1}$ of dimensions 2 and 3 respectivelly, and a regular curve $\Gamma \subset \mathbb{R}^{n+1}$, such that $\Pi^2 \subset \Pi^3$, $\Gamma \subset \Pi^3 \cap \mathbb{E}^n_\varepsilon$ and Σ is the orbit of Γ under the action of the subgroup of isometries of $\mathbb{E}^n_\varepsilon$ which preserves Π^2. In [13] do Carmo and Dajczer described the family of rotational minimal hypersurfaces in the round sphere $\mathbb{S}^n$ and in the hyperbolic space $\mathbb{H}^n$. For the case $n = 3$, the parametrization Φ_a is given by,

$$\begin{aligned}\Phi_a(s,\theta) = \Bigg(&\sqrt{\frac{1}{2} - \varepsilon a c_\varepsilon(2s)}\, c_\varepsilon\varphi(s), \sqrt{\frac{1}{2} - \varepsilon a c_\varepsilon(2s)}\, s_\varepsilon\varphi(s),\\ &\sqrt{\frac{\varepsilon}{2} + a c_\varepsilon(2s)}\cos\theta, \sqrt{\frac{\varepsilon}{2} + a c_\varepsilon(2s)}\sin\theta\Bigg), \quad (s,\theta) \in \mathbb{R}\times\mathbb{S}^1,\end{aligned} \tag{2.10}$$

where a is a constant such that

$$\begin{cases} -\frac{1}{2} < a \le 0, & \text{if } \varepsilon = 1,\\ a > \frac{1}{2}, & \text{if } \varepsilon = -1,\end{cases}$$

and $\varphi(s)$ is given by

$$\varphi(s) = \sqrt{\varepsilon\left(\frac{1}{4} - a^2\right)} \int_0^s \frac{1}{\left[\frac{1}{2} - \varepsilon a c_\varepsilon(2t)\right]\sqrt{\frac{\varepsilon}{2} + a c_\varepsilon(2t)}}\,dt.$$

Remark 1. *For other parametrizations of these surfaces see [50, 53].*

As observed by Haizhong Li and Changwei Xiong [39], for fixed r there exists $a = a(r,\varepsilon)$ and a positive number $s_0 = s_0(r,\varepsilon)$ such that

$$\Phi_{a,\varepsilon} : [-s_0, s_0] \times \mathbb{S}^1 \to \mathbb{B}^3_\varepsilon(r) \tag{2.11}$$

is a free boundary minimal immersion in $\mathbb{B}^3_\varepsilon(r)$, where $r \in (0, \frac{\pi}{2})$ if $\varepsilon = 1$. We call these surfaces *critical rotational annuli.*

In the following we will need another model for $\mathbb{E}^3_\varepsilon$. In $[0, R_\infty) \times \mathbb{S}^2$ consider the warped product metric

$$g = dr^2 + \lambda(\rho)^2 g_{\mathbb{S}^2}, \tag{2.12}$$

where $\lambda(\rho)$ and R_∞ are given by

$$\lambda(\rho) = \begin{cases} \rho, & R_\infty = +\infty,\\ \sinh\rho, & R_\infty = +\infty,\\ \sin\rho, & R_\infty = \pi/2.\end{cases}$$

For any surface $\Sigma \subset \mathbb{E}^3_\varepsilon$, there are two important functions,: the potential function λ' and the support function $\langle X, N\rangle$. Here $X = \lambda(\rho)\partial_r$ is the natural conformal vector field on $\mathbb{E}^3_\varepsilon$ and N is a unit normal of the surface. Li and Xiong proved the following generalization of Theorem 2.

Theorem 3 (Li-Xiong [39]). *Let Σ be a compact free boundary minimal surface in $\mathbb{B}^3_\varepsilon(r)$. Assume that for all points x in Σ, we have*

$$(\lambda')^{-2}|\mathrm{I\!I}|^2\langle N(x), X\rangle^2 \leq 2, \tag{2.13}$$

where $N(x)$ denotes a unit normal vector at the point $x \in \Sigma$ and A denotes the second fundamental form of Σ. Then

(1) either $|\mathrm{I\!I}|^2\langle N(x), X\rangle^2 \equiv 0$ and Σ is a totally geodesic disk;
(2) or $(\lambda')^{-2}|\mathrm{I\!I}|^2\langle N, X\rangle^2(p) = 2$ at some point $p \in \Sigma$ and Σ is a critical rotational annulus.

Proof. Consider the real function f_ε defined by

$$f_\varepsilon(s) = \begin{cases} s, & \text{if } \varepsilon \in \{0, -1\}, \\ 1 - \sqrt{1-s}, & \text{if } \varepsilon = 1. \end{cases}$$

Then $\psi_\varepsilon : \Sigma \to \mathbb{R}$ given by $\psi_\varepsilon(x) = f_\varepsilon\big(\lambda(\rho(x))^2\big)$ satisfies

$$\nabla_\Sigma \psi_\varepsilon = 2f_\varepsilon'\big(\lambda(\rho)^2\big)\lambda\lambda'\partial_r, \quad \text{on } \partial\Sigma, \tag{2.14}$$

$$\begin{aligned}\operatorname{Hess}_\Sigma \psi_\varepsilon(Y,Z) &= 2\big[2f_\varepsilon''\big(\lambda(\rho)^2\big)\lambda^2(\lambda')^2 + f_\varepsilon'\big(\lambda(\rho)^2\big)\lambda\lambda''\big]\langle Y, \partial_r\rangle\langle Z, \partial_r\rangle \\ &\quad + 2f_\varepsilon'\big(\lambda(\rho)^2\big)(\lambda')^2\left\langle Y + \frac{1}{\lambda'}\mathrm{I\!I}(Y)\langle N, X\rangle, Z\right\rangle. \end{aligned} \tag{2.15}$$

When $\varepsilon \in \{0, 1\}$ the first term on the rigth side of equation (2.15) vanishes, while for $\varepsilon = 1$, that term is equal to $2\sinh(r)\cosh(r)\langle Y, \partial_r\rangle\langle Z, \partial_r\rangle$. Thus, on all cases we conclude that $\operatorname{Hess}_\Sigma \psi_\varepsilon \geq 0$ holds when $(\lambda')^{-2}|\mathrm{I\!I}|^2\langle N, X\rangle^2(x) \leq 2$.

Now, define

$$\mathfrak{C} = \Big\{p \in \Sigma : \psi_\varepsilon(p) = \min_{x\in\Sigma}\psi_\varepsilon(x)\Big\}. \tag{2.16}$$

Step 1: Suppose $(\lambda')^{-2}|\mathrm{I\!I}|^2\langle N(x), X\rangle^2 \leq 2$ on Σ. Then

(a) either $\mathfrak{C}$ contains a single point $p \in \Sigma \setminus \partial\Sigma$, in which case Σ must be a totally geodesic disk;
(b) or $\mathfrak{C}$ is a simple closed geodesic in $\Sigma \setminus \partial\Sigma$ and Σ is homeomorphic to an annulus.

We claim that $\mathfrak{C}$ is a totally convex subset of Σ, i.e. any geodesic in Σ joining two points in $\mathfrak{C}$, lies entirely in $\mathfrak{C}$. Indeed, given $p, q \in \mathfrak{C}$, let $\gamma : [0,1] \to \Sigma$ be a geodesic such that $\gamma(0) = p$ and $\gamma(1) = q$. By the discussion above, $(\psi_\varepsilon \circ \gamma)''(t) \geq 0$ for all $t \in [0,1]$. By the definition of $\mathfrak{C}$, as $\psi_\varepsilon \circ \gamma$ attains its minimum at $t = 0$ and $t = 1$ we conclude that $(\psi_\varepsilon \circ \gamma)(t) \equiv \min_\Sigma \psi_\varepsilon$. Therefore $\gamma([0,1]) \subset \mathfrak{C}$ and the claim follows.

Since Σ meets $\partial\mathbb{B}^3_\varepsilon(r)$ orthogonally, the geodesic curvature k_g of $\partial\Sigma$ in Σ, computed with respect to the unit outward pointing conormal, satisfies $k_g \equiv \cot_\varepsilon r$. In particular, $\partial\Sigma$ is strictly convex in Σ. This implies that for all $p, q \in \Sigma$ there exists a minimising geodesic in Σ joining p to q. Thus, the totally convex set $\mathfrak{C}$ is connected. Also, $\mathfrak{C}$ is contained in the interior of Σ, because by equation (2.14), the gradient of ψ_ε is non-zero on $\partial\Sigma$ and points outwards.

Now, let us suppose that $\mathfrak{C}$ contains only one point $p \in \Sigma \setminus \partial\Sigma$. Given $[\alpha] \in \pi_1(\Sigma, p)$, let us assume that $[\alpha]$ is a nontrivial homotopy class. Since $\partial\Sigma$ is strictly convex, we can find a geodesic loop $\gamma : [0,1] \to \Sigma$, $\gamma(0) = \gamma(1) = p$, such that $\gamma \in [\alpha]$. Since $\mathfrak{C}$ is totally convex, $\gamma([0,1]) \subset \mathfrak{C}$. But $\mathfrak{C} = \{p\}$ and $[\alpha]$ is non-trivial, which is a contradiction. Therefore Σ is homeomorphic to a disk. By Theorem 1, Σ is a flat equatorial disk.

Finally, suppose that $\mathfrak{C}$ does not consist of a single point. By Theorem 1, Σ cannot be homeomorphic to a disk. Let $p \in \mathfrak{C}$ and let $[\alpha] \in \pi_1(\Sigma, p)$ be a non-trivial homotopy class. As above, since $\partial\Sigma$ is strictly convex, we can find a geodesic loop $\gamma : [0,1] \to \Sigma$, $\gamma(0) = \gamma(1)$, such that $\gamma \in [\alpha]$. One can use the fact $\mathfrak{C}$ is totally convex to prove that $\gamma'(0) = \gamma'(1)$, $\gamma([0,1])$ is a simple curve and $\mathfrak{C} = \gamma([0,1])$. Since the above argument also shows that any geodesic loop based at p must be contained in $\mathfrak{C}$, we conclude that $\pi_1(\Sigma, p) = \mathbb{Z}$. By the classification of compact surfaces, Σ is homeomorphic to an annulus.

Step 2: If $(\lambda')^{-2}|\mathbb{I}|^2\langle N, X\rangle^2 < 2$ on Σ, then Σ is a totally geodesic disk.

Since $|\mathbb{I}|^2\langle x, N\rangle^2 < 2$, it follows that the function ψ_ε is strictly convex, hence the set of its minima contains a single point. Thus, Step 1 implies that Σ is a totally geodesic disk.

Step 3: If $(\lambda')^{-2}|\mathbb{I}|^2\langle N(x), X\rangle^2 \leq 2$ on Σ, and $(\lambda')^{-2}|\mathbb{I}|^2\langle N, X\rangle^2 = 2$ at some point $p \in \Sigma$, then Σ is a critical rotational annulus.

Since $(\lambda')^{-2}|\mathbb{I}|^2\langle N, X\rangle^2 = 2$, Σ can not be a disk, otherwise it would be totally geodesic, a contradiction. Thus Σ is homeomorphic to an annulus and $\mathcal{C}$ is a simple closed geodesic $\gamma : [0, l] \to \Sigma$ with arc parameter. Note that $\inf_\Sigma \psi_\varepsilon > 0$.

Let $\rho_0 > 0$ satisfy $s_\varepsilon^2(\rho_0) = \inf_\Sigma \psi_\varepsilon$ and let $S_{\rho_0} \subset \mathbb{E}^3_\varepsilon$ be the geodesic sphere of radius ρ_0 centered at the origin. Then $\Sigma \subset \{x \in \mathbb{E}^3_\varepsilon : \rho(x) \geq \rho_0\}$ and $\Sigma \cap S_{\rho_0} = \gamma([0, l])$. Therefore $T_{\gamma(s)}\Sigma = T_{\gamma(s)}S_{\rho_0}$ for all $s \in [0, l]$. Note that γ is a geodesic in Σ. So it is also a geodesic in S_{ρ_0}. That is, γ is a great circle of S_{ρ_0}.

Based on the analysis above, we can assume that ∂_3 is orthogonal to the hyperplane which contains the circle γ and ∂_0. Then $\{\gamma'(s), \partial_3\}$ is an orthonormal basis of $T_{\gamma(s)}\Sigma$ for all $s \in [0, l]$. Define for any three vectors v_1, v_2, v_3 in $(\mathbb{R}^4, g_\varepsilon)$ their g_ε-cross product $v_1 \wedge v_2 \wedge v_3$ as the unique vector such that for any $v \in \mathbb{R}^4$ such that

$$g_\varepsilon(v_1 \wedge v_2 \wedge v_3, v) = \det(v_1, v_2, v_3, v), \quad \forall\, v \in \mathbb{R}^4,$$

where det is the determinant with respect to the canonical basis.

Now define a function $u : \Sigma \to \mathbb{R}$ by

$$u(x) = g_\varepsilon\big(x \wedge \partial_3 \wedge \partial_0, N(x)\big).$$

It is straightforward to verify that

$$\Delta_\Sigma u + (|\mathbb{I}|^2 + 2\varepsilon)u = 0 \text{ on } \Sigma, \quad \frac{\partial u}{\partial \nu} = c_\varepsilon(r) \cdot u \text{ on } \partial\Sigma.$$

Moreover $u \equiv 0$ on Σ if and only if Σ is a rotational surface in $(\mathbb{R}^4, g_\varepsilon)$ around the subspace generated by ∂_3 and ∂_0.

Now on the one hand, $\gamma([0, l])$ is contained in $u^{-1}(0)$, which implies at any point $\gamma(s)$ we have $\langle \nabla u, \gamma'\rangle = \frac{d(u\circ\gamma)}{ds} = 0$. On the other hand, since $\{\gamma'(s), \partial_3\}$ is an orthonormal basis of $T_{\gamma(s)}\Sigma$ and $\gamma'(s)$ is a principal direction of Σ, the vector ∂_3 is the other principal direction at $\gamma(s)$, which implies $\nabla^{g_\varepsilon}_{\partial_3} N$ is parallel to ∂_3. Thus along the circle γ,

$$\langle \nabla_\Sigma u, \partial_3\rangle = g_\varepsilon\big(\partial_3 \wedge \partial_3 \wedge \partial_0, N\big) + g_\varepsilon\Big(x \wedge \partial_3 \wedge \partial_0, \langle \partial_3, N\rangle x + \nabla^{g_\varepsilon}_{\partial_3} N\Big) = 0.$$

Hence, the critical points of u in the nodal set $u^{-1}(0)$ are not isolated and [11, Theorem 2.5] shows $u \equiv 0$ on Σ. As a consequence, Σ is a rotational annulus as claimed. $\square$

3. The Steklov eigenvalue problem with frequency

3.1. A characterization of free-boundary minimal submanifolds in $\mathbb{B}^n_\varepsilon(r)$.

Proposition 1. *Let (Σ^k, g) be a compact Riemannian manifold and let ν be the outward pointing unit g-normal vector of $\partial\Sigma$. Let $\Phi : (\Sigma, g) \to \mathbb{B}^n_\varepsilon(r)$ be an isometric immersion such that $\Phi(\partial\Sigma) \subset \partial\mathbb{B}^n_\varepsilon(r)$. Then Φ is minimal and free boundary if and only if, the coordinate functions $\phi_i = x_i \circ \Phi$ satisfy*

$$\Delta_g \phi_i + \varepsilon k \phi_i = 0, \quad \text{in } \Sigma, \ i = 0, 1, \dots, n, \tag{3.1}$$

$$\frac{\partial \phi_0}{\partial \nu} + \varepsilon^2 (\tan_\varepsilon r) \phi_0 = 0, \quad \text{on } \partial\Sigma, \tag{3.2}$$

$$\frac{\partial \phi_i}{\partial \nu} - \varepsilon^2 (\cot_\varepsilon r) \phi_i = 0, \quad \text{on } \partial\Sigma, \ i = 1, \dots, n. \tag{3.3}$$

Proof. In the following we write $g_\varepsilon = \langle \cdot, \cdot \rangle$ and denote by D the derivative on $\mathbb{R}^{n+1}$. Observe the position vector is a unit normal vector to $\mathbb{E}^n_\varepsilon$ with respect to g_ε.

Fix $v \in \mathbb{R}^{n+1}$ and consider the function $\phi_v \in C^\infty(\Sigma)$ defined by $\phi_v(x) = \langle \Phi(x), v \rangle$. We denote by $\mathrm{I\!I}$ and $\vec{H}$ respectively the second fundamental form and the mean curvature vector of Φ.

Given $p \in \Sigma$, let $\{E_1, \dots, E_k\}$ be a local g-orthonormal frame defined in a neighborhood of Σ such that $\nabla^g_{E_i} E_j(p) = 0$, for all i, j. Then on p we have

$$\begin{aligned}
\Delta_g \phi_v &= \sum_{i=1}^k E_i E_i(\phi_v) = \sum_{i=1}^k E_i \langle E_i, v \rangle \\
&= \sum_{i=1}^k \langle D_{E_i} E_i, v \rangle = \sum_{i=1}^k \left\langle \nabla^{g_\varepsilon}_{E_i} E_i - \varepsilon \langle E_i, E_i \rangle \Phi, v \right\rangle \\
&= \sum_{i=1}^k \left\langle \nabla^g_{E_i} E_i + \mathrm{I\!I}(E_i, E_i) - \varepsilon \langle E_i, E_i \rangle \Phi, v \right\rangle \\
&= \langle \vec{H}, v \rangle - \varepsilon k \phi_v.
\end{aligned}$$

Thus $\vec{H} = 0$ if, and only if, $\Delta_g \phi_v + \varepsilon k \phi_v = 0$, for all $v \in \mathbb{R}^{n+1}$.

On the other hand,

$$\frac{\partial \phi_v}{\partial \nu} = \langle \nu, v \rangle.$$

The outward pointing unit normal to $\partial\mathbb{B}^n_\varepsilon(r)$ is given by

$$N_{\partial\mathbb{B}^n_\varepsilon(r)} = \frac{1}{s_\varepsilon(r)} \big(\varepsilon c_\varepsilon(r) x - \partial_0 \big),$$

and the immersion Φ is free boundary if, and only if, $\nu = N_{\partial\mathbb{B}^n_\varepsilon(r)}$, i.e., if and only if

$$\frac{\partial \phi_v}{\partial \nu} = \frac{1}{s_\varepsilon(r)} \big(\varepsilon c_\varepsilon(r) \langle \Phi, v \rangle - \langle \partial_0, v \rangle \big). \tag{3.4}$$

Choosing $v = \partial_i$, $i = 0, 1, \dots, n$, we obtain (3.2) and (3.3). □

Proposition 2. *Let (Σ^k, g) be a compact Riemannian manifold and $\Phi : (\Sigma, g) \to \mathbb{B}^n_\varepsilon(r)$ be an isometric minimal immersion with free boundary. Then, the Dirichlet problem*

$$\begin{cases} -\Delta_g w - k w = \lambda w, & \text{in } \Sigma, \\ w = 0, & \text{on } \partial\Sigma, \end{cases}$$

has no nontrivial solution for any $\lambda \le 0$. In particular, for any function $u \in C^\infty(\Sigma)$ that satisfies $u = 0$ on $\partial\Sigma$, it holds that

$$\int_\Sigma (|\nabla^g u|^2 - ku^2) dA \ge 0.$$

Proof. The case $\varepsilon \le 0$ is obvious, so let us suppose $\varepsilon = 1$. By Proposition 1, we know $\phi_0 = \langle \Phi, \partial_0 \rangle$ satisfies $\Delta\phi_0 + k\phi_0 = 0$ with $\phi_0 > 0$ in Σ.

Suppose, by contradiction, there exists a nontrivial solution w to the Dirichlet problem above with $\lambda \le 0$. We can write $w = h\phi_0$, for some function h, where $h = 0$ on $\partial\Sigma$. Hence,

$$\begin{aligned} 0 &= \int_\Sigma (|\nabla w|^2 - (k+\lambda)w^2) dA \\ &= \int_\Sigma \big(|\nabla h|^2 \phi_0^2 + 2h\phi_0 \langle \nabla h, \nabla \phi_0 \rangle + h^2 |\nabla \phi_0|^2 - (k+\lambda) h^2 \phi_0^2\big) dA. \end{aligned}$$

Applying the divergence theorem and using that $\Delta\phi_0 + k\phi_0 = 0$ in Σ and $h = 0$ on $\partial\Sigma$, we get

$$\int_\Sigma h\phi_0 \langle \nabla h, \nabla\phi_0 \rangle dA = -\int_\Sigma \big(h\phi_0 \langle \nabla h, \nabla \phi_0\rangle + h^2|\nabla\phi_0|^2 + h^2\phi_0\Delta\phi_0\big)\, dA + \int_{\partial\Sigma} h^2 \phi_0 \frac{\partial \phi_0}{\partial \nu} dL$$

$$\implies 2\int_\Sigma h\phi_0 \langle \nabla h, \nabla\phi_0\rangle dA = -\int_\Sigma h^2 |\nabla\phi_0|^2 dA + k\int_\Sigma h^2\phi_0^2 dA.$$

Then,

$$0 = \int_\Sigma (|\nabla w|^2 - kw^2) dA = \int_\Sigma |\nabla h|^2 \phi_0^2 dA - \lambda \int_\Sigma h^2 \phi_0^2 dA.$$

Since $\lambda \le 0$ and $\phi_0 > 0$, it implies that h is necessarily a constant function. Thus, since $h = 0$ on $\partial\Sigma$, we conclude that $h \equiv 0$ in Σ and, therefore, $w \equiv 0$ in Σ, a contradiction. $\square$

3.2. **Steklov-type eigenvalue problems.** The previous two results motivate the study of an interesting eigenvalue problem.

Let (Σ, g) be a compact Riemannian manifold with boundary, and let ν_g be the outward pointing unit conormal vector of $\partial\Sigma$. Fix $\alpha \in \mathbb{R}$ which is not in the spectrum of $-\Delta_g$ with Dirichlet boundary condition. Then, given $u \in C^\infty(\partial\Sigma)$, there is a unique $\widehat{u} \in C^\infty(\Sigma)$ such that $\widehat{u}|_{\partial\Sigma} = u$ and $\Delta_g \widehat{u} + \alpha\widehat{u} = 0$ in Σ. The *Steklov eigenvalue problem at frequency* α consists of finding $u \neq 0$ and $\sigma \in \mathbb{R}$ such that

$$\frac{\partial \widehat{u}}{\partial \nu} = \sigma u \text{ on } \partial\Sigma.$$

The numbers σ for which there is u satisfying the equation above form a sequence

$$\sigma_0(g,\alpha) < \sigma_1(g,\alpha) \le \ldots \sigma_j(g,\alpha) \le \ldots \to +\infty,$$

and correspond to the spectrum of an operator $\mathcal{D}_\alpha$, which is called the *Dirichlet-to-Neumann operator at frequency* α and it is defined as follows.

- If $u \in H^1(\Sigma, g)$ and $f \in L^2(\Sigma, g)$, we say $\Delta_g u = f$ in the weak sense if
$$-\int_\Sigma g\big(\nabla^g u, \nabla^g v\big)\, dA_g = \int_\Sigma fv\, dA_g, \quad \forall\, v \in H^1_0(\Sigma, g).$$
- Suppose $u \in H^1(\Sigma, g)$ and $\Delta_g u = f$ in the weak sense. We say that $\dfrac{\partial u}{\partial \nu_g} \in L^2(\partial\Sigma, g)$ if there exists $\varphi \in L^2(\partial\Sigma, g)$ such that
$$\int_\Sigma g\big(\nabla^g u, \nabla^g v\big)\, dA_g + \int_\Sigma fv\, dA_g = \int_{\partial\Sigma} \varphi v\, dL_g, \quad \forall\, v \in H^1(\Sigma, g).$$

Then, we define

$$\mathcal{D}_\alpha u = \frac{\partial \widehat{u}}{\partial \nu_g}, \quad \mathrm{dom}(\mathcal{D}_\alpha) = \Big\{ u \in L^2(\partial\Sigma, g);\ \exists\, \widehat{u} \in H^1(\Sigma, g) \text{ such that } \mathrm{Tr}\,\widehat{u} = u,$$
$$\Delta_g \widehat{u} + \alpha \widehat{u} = 0 \text{ in the weak sense, and } \frac{\partial \widehat{u}}{\partial \nu_g} \in L^2(\partial\Sigma, g) \Big\},$$

where $\mathrm{Tr} : H^1(\Sigma, g) \to L^2(\partial\Sigma, g)$ is the trace operator.

It is known that $\mathrm{dom}(\mathcal{D}_\alpha)$ is a dense subspace of $L^2(\partial\Sigma, g)$ and $\mathcal{D}_\alpha$ is a self-adjoint operator which is bounded below and has compact resolvent. In particular, the spectrum of $\mathcal{D}_\alpha$ is discrete and bounded below. For more details about the construction of $\mathcal{D}_\alpha$, see [5, 6] (observe that their Laplacian is the negative of ours).

The eigenvalues $\sigma_0(g, \alpha)$ and $\sigma_1(g, \alpha)$ have a variational characterization. Let Q be the quadratic form on $\mathrm{dom}(\mathcal{D}_\alpha)$ defined by,

$$Q(u, u) = \int_\Sigma |\nabla^g \widehat{u}|_g^2 \, dA_g - \alpha \int_\Sigma \widehat{u}^2 \, dA_g.$$

Then,

$$\sigma_0(g, \alpha) = \inf \left\{ Q(v, v)/||u||_{L^2(\partial\Sigma)};\ u \in \mathrm{dom}(\mathcal{D}_\alpha) \setminus \{0\} \right\}, \tag{3.5}$$

and this eigenvalue is simple. Denote by ϕ_0 a first eigenfunction, which we can choose to be positive. Then, applying the min-max principle to $\mathcal{D}_\alpha$, it follows that

$$\sigma_1(g, \alpha) = \inf \left\{ Q(v, v)/||u||_{L^2(\partial\Sigma)};\ u \in \mathrm{dom}(\mathcal{D}_\alpha) \setminus \{0\} \text{ and } \int_{\partial\Sigma} u\phi_0 \, dL_g = 0 \right\}.$$

Now, we will present an alternative characterization of $\sigma_0(g, \alpha)$ and $\sigma_1(g, \alpha)$ for a special class of metrics. Denote by $\lambda_1^D(g)$ the first eingenvalue of $-\Delta_g$ with Dirichlet boundary condition. Suppose $\alpha > 0$. If $\lambda_1^D(g) > \alpha$, then

$$\sigma_0(g, \alpha) = \inf \left\{ \left(\int_\Sigma |\nabla^g u|_g^2 \, dA_g - \alpha \int_\Sigma u^2 \, dA_g \right) \Big/ \int_{\partial\Sigma} u^2 \, dL_g \,:\, u \in W^{1,2}(\Sigma) \text{ and } \mathrm{Tr}\, u \neq 0 \right\}, \tag{3.6}$$

and

$$\sigma_1(g, \alpha) = \inf \Bigg\{ \left(\int_\Sigma |\nabla^g u|_g^2 \, dA_g - \alpha \int_\Sigma u^2 \, dA_g \right) \Big/ \int_{\partial\Sigma} u^2 \, dL_g \,:\, u \in W^{1,2}(\Sigma), \mathrm{Tr}\, u \neq 0$$
$$\text{and } \int_{\partial\Sigma} u\phi_0 \, dL_g = 0 \Bigg\}. \tag{3.7}$$

Exercise 2. Prove that (3.6) and (3.7) hold.

Remark 2. *For $\alpha > 0$, the quotient $\int_\Sigma \left(|\nabla^g u|_g^2 - \alpha u^2 \right) dA_g / \int_{\partial\Sigma} u^2 \, dL_g$ is bounded from below in the domain $\{u \in W^{1,2}(\Sigma), \mathrm{Tr}\, u \neq 0\}$ if and only if, $\lambda_1^D(g) > \alpha$. So, the previous characterization only holds in the case $\lambda_1^D(g) > \alpha$.*

Observe the hypothesis $\lambda_1^D(g) > \alpha$ is equivalent to the following condition: the Dirichlet eigenvalue problem (3.8) does not admit non-positive eigenvalues λ,

$$\begin{aligned} -\Delta_g u - \alpha u &= \lambda u, \text{ in } \Sigma, \\ u &= 0, \text{ on } \partial\Sigma. \end{aligned} \tag{3.8}$$

Remark 3. *It follows from Propositions 1 and 2 that the coordinates functions of a free boundary minimal immersion $\Phi : (\Sigma, g) \to \mathbb{B}^n_\varepsilon(r)$ are Steklov eigenfunctions with frequency*

2ε. *Moreover, since the eigenfunction* ϕ_0 *associated to the eigenvalue* $\sigma = -\tan_\varepsilon r$ *is positive, we have* $\sigma_0(g, 2\varepsilon) = -\tan_\varepsilon r$.

Now suppose $\alpha = 0$. In this case $\lambda_1^D(g) > \alpha = 0$ always holds and the spectrum of $\mathcal{D}_0$ satisfies

$$0 = \sigma_0(g, 0) < \sigma_1(g, 0) \leq \ldots \sigma_j(g, 0) \leq \ldots \to +\infty.$$

These numbers are simply called Steklov eigenvalues. Moreover the $\sigma_0(g, 0)$-eigenspace is generated by the constant function 1.

3.3. **Multiplicity bounds.** For later applications we will need upper bounds for the multiplicity of $\sigma_j(g, \alpha)$, $j \geq 1$. In order to obtain such bounds we will use the following result concerning nodal domains.

Theorem 4 (Hassannezhad-Sher, [28]). *Let* (Σ, g) *be a compact Riemannian manifold with boundary such that* $\lambda_1^D(g) > \alpha$. *Then, for any* $\sigma_j(g, \alpha)$*-eigenfunction, its number of nodal domains* N_j *in* Σ *satisfies*

$$N_j \leq j + 1.$$

We prove:

Theorem 5. *Let* (Σ, g) *be a compact Riemannian surface of genus* γ *with* ℓ *boundary components such that* $\lambda_1^D(g) > \alpha$. *Then, the multiplicity of* $\sigma_j(g, \alpha)$ *is at most* $4\gamma + 2j + 1$.

Proof. The proof follows the same strategy as in [25, thm 2.3]. Consider a function $\phi \in C^\infty(\Sigma)$ satysfying

$$\begin{aligned} &\Delta_g \phi + \alpha \phi = 0 \text{ in } \Sigma, \\ &\frac{\partial \phi}{\partial \nu_g} - \sigma_j \phi = 0 \text{ on } \partial\Sigma. \end{aligned} \tag{3.9}$$

Claim 1: The set $\mathcal{S}_\phi = \big\{p \in \Sigma;\ \phi(p) = 0, \nabla^g \phi(p) = 0\big\}$ is finite.

The claim for $\big(\Sigma \setminus \partial\Sigma\big) \cap \mathcal{S}_\phi$ is a classical result and it follows as in [11].

Let $p \in \big(\partial\Sigma\big) \cap \mathcal{S}_\phi$ and consider a chart $\Psi : \mathcal{U} \to U \subset \mathbb{R}^2_+$, where U is the half-disk centered at the origin, such that $\Psi(p) = (0, 0)$, the associated coordinates x, y are isothermal, and $\Psi_* \nu_g$ is on the direction of ∂_y. By abuse of notation, we will also denote $\phi \circ \Psi^{-1}$ by ϕ. In these coordinates, we have $g = \mu\delta$, where $\delta = dx_2 + dy^2$ and μ is a positive function. It follows from (3.9) that $v = e^{\sigma_j \mu y}\phi$ satisfies

$$\Delta_\delta\, v + \langle b, \nabla_\delta\, v\rangle + cv = 0 \text{ on } U, \text{ and } \frac{\partial v}{\partial y}(x, 0) = 0,$$

where b and c are smooth and bounded up to the boundary. The remaining of the proof follows exactly as in [25], extending v by reflection on the boundary and using unique continuation results.

Claim 2: Consider $p \in \Sigma \setminus \partial\Sigma$ such that $\phi(p) = 0$. Then, the order of vanishing of ϕ at p is at most $2\gamma + j$.

Again the proof follows as in [25], the only difference being that in [25] the authors make use of the Nodal Domain Theorem for the Steklov problem (the case $\alpha = 0$), and in our case we need to use Theorem 4.

Claim 3: The multiplicity of σ_j is at most $4\gamma + 2j + 1$.

Let m_j be the multiplicity of σ_j and consider a basis $\{\phi_1, \ldots, \phi_{m_j}\}$ of the j-th eigenspace. Suppose that $m_j > 4\gamma + 2j + 1$. The strategy is to show this implies the existence of a σ_j-eigenfunction $\phi = a_1\phi_1 + \ldots + a_{m_j}\phi_{m_j}$ and a point $p \in \Sigma \setminus \partial\Sigma$ such that the order of vanishing of ϕ at p is greater than $2\gamma + j$, which contradicts Claim 2.

Let $p \in \Sigma \setminus \partial\Sigma$ and consider isothermal coordinates x, y around p such that p is mapped to $(0,0)$ and $g = \mu\delta$. Since $\Delta_g \phi_j + \alpha\phi_j = 0$ and $0 \le l \le q$, we have

$$\frac{\partial^q \Delta_\delta \phi_j}{\partial^l x\, \partial^{q-l} y} = L\phi_j,$$

where L is a linear operator of order q. Supposing that all the derivatives of ϕ_j of order $0 \le l \le q+1$ vanish at $(0,0)$ we obtain

$$\frac{\partial^{q+2}\phi_i}{\partial^{l+2} x\, \partial^{q-l} y}(0,0) + \frac{\partial^{q+2}\phi_i}{\partial^{l} x\, \partial^{q-l+2} y}(0,0) = 0.$$

Now we can argue as in [25] to finish the proof. □

Remark 4. *Theorem 5 and Proposition 2 imply that for any free boundary minimal immersion $\Phi : (\Sigma^2, g) \to \mathbb{B}^n_\varepsilon(r)$ the multiplicity of $\sigma_j(g,2)$ is at most $4\gamma + 2j + 1$.*

4. A functional defined via Steklov eigenvalues

Let Σ be a compact surface with boundary. Fraser and Schoen [22, 23, 25] considered the functional (on the space of Riemannian metrics on Σ)

$$g \mapsto \sigma_1(g,0)\mathrm{lenght}_g(\partial\Sigma),$$

and proved that is bounded from above ans its maximizers are realized by branched free-boundary minimal immersions in Euclidean balls. This renewed the interesting on the theory of free-boundary minimal surfaces and led to many developments [40]. The goal of the remaining of these notes is to discuss an analogous functional connected to free-boundary minimal immersions in spherical caps (geodesic balls of the round spheres).

4.1. Eigenvalue bounds and a Weinstock-type result.

Denote by $\mathcal{M}(\Sigma)$ the space of smooth Riemannian metrics g on Σ such that $\lambda_1^D(g) > 2$. For simplicity, on the remainder of the paper we will denote $\sigma_k(g) = \sigma_k(g,2)$, $k \ge 0$.

Fix $r \in (0, \frac{\pi}{2})$. We define the functional

$$\Theta_r(\Sigma, g) = \big[\sigma_0(g)\cos^2 r + \sigma_1(g)\sin^2 r\big]|\partial\Sigma|_g + 2|\Sigma|_g, \tag{4.1}$$

where $|\partial\Sigma|_g$ and $|\Sigma|_g$ denote respectively the lenght of $\partial\Sigma$ and the area of Σ, both with respect to g.

The goal of this section is to prove the existence of an upper bound for the functional Θ_r that is sharp in the case of a disk. This analogous to a result due to Weinstock [62].

Exercise 3. Prove that for any $0 < r < \frac{\pi}{2}$, $\mathbb{B}^2_1(r)$ satisfies $\sigma_0 = -\tan r$ and $\sigma_1 = \cot r$.

Theorem A. *Let Σ be a compact orientable surface of genus γ and ℓ boundary components. Then, for any $g \in \mathcal{M}(\Sigma)$, we have*

$$\Theta_r(\Sigma, g) \le 4\pi(1 - \cos r)(\gamma + \ell). \tag{4.2}$$

Moreover, if Σ is a disk, then the equality in (4.2) holds if and only if, (Σ, g) is isometric to $\mathbb{B}^2_1(r)$.

Proof. Let Σ be a compact orientable surface of genus γ and ℓ boundary components. By a result of Gabard [27], there exists a proper conformal branched cover $w : \Sigma \to \mathbb{D}^2$ of degree at most $\gamma + \ell$, where $\mathbb{D}^2 \subset \mathbb{R}^2$ is the closed unit disk with center at the origin.

Fix $r \in (0, \frac{\pi}{2})$. Recall that the area of $\mathbb{B}^2_1(r)$ with respect to the round metric is $2\pi(1-\cos r)$. Since $\mathbb{D}^2$ is conformally equivalent to $\mathbb{B}^2_1(r)$, there is also a proper conformal branched cover $u : (\Sigma, g) \to \mathbb{B}^2_1(r)$ of the same degree as w.

Denote by x_i and ∂_i, $i = 0, 1, 2$, the canonical coordinates of $\mathbb{R}^3$ and the corresponding coordinate vector fields. Define $u_j = x_j \circ u$. In particular, along $\partial\Sigma$ it holds

$$u_0 = \cos r, \quad u_1^2 + u_2^2 = \sin^2 r. \tag{4.3}$$

Moreover, by using conformal diffeomorphisms of $\mathbb{B}^2_1(r)$, we can also assume (see [41])

$$\int_{\partial\Sigma} u_j \phi_0 \, dL = 0, \quad j = 1, 2,$$

where ϕ_0 is a positive eigenfunction associated to σ_0.

Since $u : \Sigma \to \mathbb{B}^2_1(r)$ is conformal, there is a positive function $f \in C^\infty(\Sigma)$ such that, for all $X, Y \in \mathfrak{X}(\Sigma)$, we have

$$g_{\mathbb{B}^2_1(r)}\big(u_* X, u_* Y\big) = f^2 g(X, Y),$$

which implies

$$\sum_{j=0}^{2} |\nabla^g u_j|_g^2 = 2f^2. \tag{4.4}$$

Thus,

$$\sum_{j=0}^{2} \int_\Sigma |\nabla^g u_j|_g^2 \, dA_g = 2\int_\Sigma f^2 dA_g = 4\deg(u)\pi(1-\cos r) \le 4\pi(1-\cos r)(\gamma+\ell). \tag{4.5}$$

By the variational characterization of the eigenvalues, we have

$$\sigma_0(g)\int_{\partial\Sigma} u_0^2 \, dL_g \le \int_\Sigma |\nabla^g u_0|_g^2 \, dA_g - 2\int_\Sigma u_0^2 \, dA_g, \tag{4.6}$$

and for $j = 1, 2$,

$$\sigma_1(g)\int_{\partial\Sigma} u_j^2 \, dL_g \le \int_\Sigma |\nabla^g u_j|_g^2 \, dA_g - 2\int_\Sigma u_j^2 \, dA_g. \tag{4.7}$$

Summing these inequalities and using (4.3) and (4.5), we obtain

$$\big(\sigma_0(g)\cos^2 r + \sigma_1(g)\sin^2 r\big)\, |\partial\Sigma|_g + 2|\Sigma|_g \le 4\pi(1-\cos r)(\gamma+\ell).$$

Now we will show that in the case the surface is a disk the equality holds if and only if, the surface is isometric to the spherical cap. First let us suppose $\Sigma = \mathbb{B}^2_1(r)$. Then, by Exercise 3, we have

$$\sigma_0(g) = -\tan r, \quad \sigma_1(g) = \cot r.$$

Hence, $\big(\sigma_0(g)\cos^2 r + \sigma_1(g)\sin^2 r\big)\, |\partial\Sigma|_g + 2|\Sigma|_g = 4\pi(1-\cos r)$.

Now suppose the equality occurs in (4.2) and $\gamma = 0$ and $\ell = 1$. Then, in particular, we have equality in (4.6) and (4.7), which implies that u_0 is an eigenfunction with eigenvalue σ_0, while u_1 and u_2 are eigenfunctions with eigenvalue σ_1. So, using $u_0^2 + u_1^2 + u_2^2 = 1$, we conclude

$$0 = \Delta_g\Big(\sum_{i=0}^{2} u_i^2\Big) = 2\sum_{i=0}^{2} u_i \Delta_g\, u_i + 2\sum_{i=0}^{2} |\nabla^g u_i|_g^2 = -4 + 2\sum_{i=0}^{2} |\nabla^g u_i|_g^2,$$

$$0 = \frac{\partial}{\partial\nu}\Big(\sum_{i=0}^{2} u_i^2\Big) = 2\sigma_0(g)\cos^2 r + 2\sigma_1(g)\sin^2 r.$$

Thus,

$$\sum_{i=0}^{2} |\nabla^g u_i|_g^2 = 2 \quad \text{and} \quad \sigma_0(g)\cos^2 r + \sigma_1(g)\sin^2 r = 0. \tag{4.8}$$

Combining equations (4.4) and (4.8), it follows that $f \equiv 1$. Hence, using the fact that $\gamma = 0$ and $\ell = 1$ in (4.5), we conclude that $\deg(u) = 1$. Therefore, $(du)^* g_{\mathbb{B}_1^2(r)} = g$. $\square$

4.2. **A maximization problem.** Let Σ be a compact orientable surface with boundary and fix $r \in (0, \frac{\pi}{2})$. Define

$$\Theta_r^*(\Sigma) = \sup_{g\in\mathcal{M}(\Sigma)} \Theta_r(\Sigma, g),$$

where $\Theta_r(\Sigma, g)$ was defined in (4.1). For simplicity, given $u \in C^\infty(\partial\Sigma)$ and $g \in \mathcal{M}(\Sigma)$, we will also denote by u its extension to Σ satisfying $\Delta_g u + 2u = 0$.

We want to characterize the metrics realizing $\Theta_r^*(\Sigma)$. The following result shows that a smooth maximizer should not be attained at the boundary of $\mathcal{M}(\Sigma)$ (for the proof see [46]).

Proposition 3. *Let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\Sigma)$ converging smoothly to g, such that $\lambda_1^D(g) = 2$. Then, up to a subsequence, $\sigma_0(g_i) \to -\infty$.*

Now we will study the regularity of $\Theta_r(\Sigma, g)$ with respect to g. First we need the following result (for the proof see [46]).

Lemma 1. *Let $g \in \mathcal{M}(\Sigma)$ and consider a smooth path of metrics $t \mapsto g(t)$ such that $g(0) = g$ and $g(t) \in \mathcal{M}(\Sigma)$ for all $t \in [-\eta, \eta]$. Then $t \mapsto \sigma_0\big(g(t)\big)$ is smooth in $[-\eta, \eta]$, while $t \mapsto \sigma_1\big(g(t)\big)$ is Lipschitz in $[-\eta, \eta]$.*

Fix $g \in \mathcal{M}(\Sigma)$ and consider a smooth path of metrics $g(t)$ as in the conclusion of the previous lemma. Choose a smooth path of functions $u_i(t)$, $i = 0, 1$, such that $u_0(t)$ is an eigenfunction associated to $\sigma_0(t)$, $u_1(0) = u_1$ is an eigenfunction associated to $\sigma_1(0)$, and $u_1(t)$ is orthogonal to $u_0(t)$ for all t. Denote $h = \frac{dg}{dt}\Big|_{t=0}$, $v_i = u_i'(0)$, $i = 0, 1$, and define

$$f_i(t) = \int_\Sigma \big|\nabla^{g(t)} u_i(t)\big|_{g(t)}^2 \, dA_{g(t)} - 2\int_\Sigma u_i^2(t)\, dA_{g(t)} - \sigma_1(t)\int_{\partial\Sigma} u_i^2(t)\, dL_{g(t)}. \tag{4.9}$$

We choose $u_i(0)$ such that $||u_i||_{L^2(\partial\Sigma, g)} = 1$

Observe that by the definition of the eigenvalues, we have $f_i(t) \geq 0$ for any t, and $f_i(0) = 0$. Moreover $\sigma_0(t)$ is differentiable. So, if we assume that $\Theta_r(\Sigma, g(t))$ is differentiable at $t = 0$, then, in particular, $\sigma_1(t)$ is also differentiable at $t = 0$. It follows that f_i has a critical point at $t = 0$.

Exercise 4. Prove that

$$(\sigma_i)'(0) = \int_\Sigma \Big\langle \frac{1}{2}\Big(\big|\nabla^g u_i\big|_g^2 - 2u_i^2\Big) g - du_i \otimes du_i, h \Big\rangle \, dA_g - \frac{1}{2}\,\sigma_i(g)\int_{\partial\Sigma} u_1^2 h(T, T)\, dL_g. \tag{4.10}$$

Hint: Use the facts $f_i'(0) = 0$, $u_i(0)$ is a eigenfunction and the following formulas

$$\frac{d}{dt}\Big|_{t=0} dA_{g(t)} = \frac{1}{2}\langle g, h\rangle\, dA_g, \quad \frac{d}{dt}\Big|_{t=0} dL_{g(t)} = \frac{1}{2}h(T,T)\, dL_g, \tag{4.11}$$

where T is a g-unit tangent vector field to $\partial\Sigma$.

Therefore, if $\Theta_r(t) := \Theta_r\big(\Sigma, g(t)\big)$ is differentiable at $t = 0$, using (4.11) and (4.10) we obtain

$$\begin{aligned}\Theta_r'(0) = Q_h(u_1) := &\int_\Sigma \Big\langle \tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u_1\big) + g, h\Big\rangle\, dA_g \\ &+ \int_{\partial\Sigma} F\big(\cos r\, u_0, \sin r\, u_1\big) h(T,T)\, dL_g,\end{aligned}$$

where

$$\begin{aligned}\tau(v) &= \frac{|\partial\Sigma|_g}{2}\big(|\nabla^g v|_g^2 - 2v^2\big)g - |\partial\Sigma|_g\, dv \otimes dv, \\ F\big(v_0, v_1\big) &= \frac{\sigma_0(g)}{2}\big(\cos^2 r - |\partial\Sigma|_g v_0^2\big) + \frac{\sigma_1(g)}{2}\big(\sin^2 r - |\partial\Sigma|_g v_1^2\big).\end{aligned}$$

Let $V_k(g)$ be the eigenspace associated to $\sigma_k(g)$, $k = 0, 1$. Consider the Hilbert space

$$\mathcal{H}_g := L^2\big(S^2(\Sigma), g\big) \times L^2(\partial\Sigma, g),$$

where $S^2(\Sigma)$ is the space of symmetric $(0,2)$-tensor fields on Σ. The next lemma and its proof follow the ideas of [25], Lemma 5.3.

Lemma 2. *Suppose $g \in \mathcal{M}(\Sigma)$ satisfies $\Theta_r(\Sigma, g) = \Theta_r^*(\Sigma)$. Then, for any $(h,\psi) \in \mathcal{H}_g$, there exists $u_1 \in V_1(g)$ such that $||u_1||_{L^2(\partial\Sigma,g)} = 1$ and*

$$\Big\langle (h,\psi), \Big(\tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u_1\big) + g, F\big(\cos r\, u_0, \sin r\, u_1\big)\Big)\Big\rangle_{\mathcal{H}_g} = 0.$$

Proof. Consider $(h,\psi) \in \mathcal{H}_g$. Since $C^\infty\big(S^2(\Sigma)\big) \times C^\infty(\partial\Sigma)$ is dense in $\mathcal{H}_g$, for each $k \in \mathbb{N}$ there exists a smooth pair $(\tilde{h}_k, \tilde{\psi}_k)$ such that $(\tilde{h}_k, \tilde{\psi}_k) \to (h,\psi)$ in L^2. Moreover, for each k, we may redefine $\tilde{h}_k$ in a neighborhood of the boundary to a smooth tensor field h_k with $h_k(T,T)|_{\partial\Sigma} = \psi$, and such that the change in the L^2 norm is arbitrarily small. Thus, we obtain a smooth sequence h_k such that $\big(h_k, h_k(T,T)\big) \to (h,\psi)$ in L^2.

Let $g(t) = g + th_k$. Then $g_0 = g$ and $\frac{dg}{dt}\Big|_{t=0} = h_k$. Since g maximizes Θ_r, given any $\varepsilon > 0$, we have

$$\int_{-\varepsilon}^0 \Theta_r'(t)\, dt = \Theta_r(0) - \Theta_r(-\epsilon) \geq 0,$$

Therefore, there is $t \in (-\varepsilon, 0)$ such that $\Theta_r'(t)$ exists and is nonnegative. Let t_j be such a sequence of points with $t_j < 0$ and $t_j \to 0$ such that $\Theta_r'(t_j) \geq 0$. Choose $u_j \in V_1\big(g(t_j)\big)$ with $||u_j||_{L^2(\partial\Sigma)} = 1$. We can use standard elliptic boundary value estimates to obtain bounds on u_j and its derivatives up to the boundary. Thus, after passing to a subsequence, u_j converges in $C^2(\Sigma)$ to an eigenfunction $u_k^- \in V_1(g)$ with $||u_k^-||_{L^2(\partial\Sigma,g)} = 1$ and, consequently, it follows that $Q_{h_k}(u_k^-) \geq 0$. By a similar argument, taking a limit from the right, there exists $u_k^+ \in V_1(g)$ with $||u_k^+||_{L^2(\partial\Sigma,g)} = 1$ such that $Q_{h_k}(u_k^+) \leq 0$.

As above, after passing to subsequences, $u_k^+ \to u_+$ and $u_k^- \to u_-$ in $C^2(\Sigma)$, and

$$\Big\langle (h,\psi), \Big(\tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u^\pm\big) + g, F\big(\cos r\, u_0, \sin r\, u^\pm\big)\Big)\Big\rangle_{\mathcal{H}_g} = \lim_{i\to\infty} Q_{h_k}(u_k^\pm) \leq 0$$

Therefore, the lemma follows. ∎

Theorem 6 (Lima-Menezes [46])**.** *Let Σ be a compact surface with boundary.*

(i) *Suppose $g \in \mathcal{M}(\Sigma)$ satisfies $\Theta_r(\Sigma, g) = \Theta_r^*(\Sigma)$. Then, there exist independent eigenfunctions $v_0 \in V_0(g)$ and $v_1, \dots, v_n \in V_1(g)$ which induce a free boundary minimal isometric immersion $v = (v_0, v_1, \dots, v_n) : (\Sigma, g) \to \mathbb{B}^n(r)$.*
(ii) *Suppose $g \in \mathcal{M}(\Sigma) \cap \mathfrak{C}$ satisfies $\Theta_r(\Sigma, g) = \Theta_r(\Sigma, \mathfrak{C})$. Then, there exist independent eigenfunctions $v_0 \in V_0(g)$ and $v_1, \dots, v_n \in V_1(g)$ which induce a free boundary harmonic map $v = (v_0, v_1, \dots, v_n) : (\Sigma, g) \to \mathbb{B}^n(r)$.*

Proof. We will give the proof of (i). Define $\mathcal{K}$ as the convex hull in $\mathcal{H}_g$ of the subset

$$\left\{\Big(\tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u_1\big) + \, g, F\big(\cos r\, u_0, \sin r\, u_1\big)\Big);\ u_1 \in V_1(g), ||u_1||_{L^2(\partial\Sigma, g)} = 1\right\}.$$

We claim that $(0,0) \in \mathcal{K}$. Suppose this is not true. Since $\mathcal{K}$ is a closed convex subset of $\mathcal{H}_g$, by the geometric form of the Hahn-Banach Theorem, there exists $(h, \psi) \in \mathcal{H}_g$ such that for any $u_1 \in V_1(g)$ with $||u_1||_{L^2(\partial\Sigma, g)} = 1$, we have

$$\left\langle (h, \psi)\, , \Big(\tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u_1\big) + \, g, F\big(\cos r\, u_0, \sin r\, u_1\big)\Big)\right\rangle_{\mathcal{H}_g} < 0.$$

However, this last inequality contradicts the previous lemma.

Thus, there exist $u_1, \dots, u_n \in V_1(g)$ with $||u_i||_{L^2(\partial\Sigma, g)} = 1$, and $t_1, \dots, t_n \in \mathbb{R}_+$ with $\sum_{j=1}^{n} t_j = 1$ such that

$$0 = \sum_{j=1}^{n} t_j \Big[\tau\big(\cos r\, u_0\big) + \tau\big(\sin r\, u_j\big) + \, g\Big], \text{ in } \Sigma, \tag{4.12}$$

$$0 = \sum_{j=1}^{n} t_j F\big(\cos r\, u_0, \sin r\, u_j\big), \text{ on } \partial\Sigma. \tag{4.13}$$

Denote $v_0 = |\partial\Sigma|^{\frac{1}{2}}(\cos r)u_0$ and $v_j = (t_j|\partial\Sigma|)^{\frac{1}{2}}\,(\sin r)u_j$, $j = 1, \dots, n$. Hence, by the definition of τ, we get

$$\sum_{j=0}^{n} \left[dv_j \otimes dv_j - \frac{1}{2}\Big(\left|\nabla^g v_j\right|^2 - 2\, v_j^2\Big) g - g\right] = 0. \tag{4.14}$$

Since $\operatorname{tr}_g g = 2$ and $\operatorname{tr}_g(dv_j \otimes dv_j) = \left|\nabla^g v_j\right|^2$, taking the trace in the above equality yields

$$\sum_{j=0}^{n} v_j^2 = 1. \tag{4.15}$$

In particular, using this fact back into (4.14), we conclude that

$$\sum_{j=0}^{n} dv_j \otimes dv_j = \frac{1}{2}\left(\sum_{j=0}^{n} \left|\nabla^g v_j\right|^2\right) g. \tag{4.16}$$

Using that $\Delta_g\, v_j + 2v_j = 0$ together with (4.15), we obtain

$$0 = \Delta_g\left(\sum_{j=0}^{n} v_j^2\right) = -4\sum_{j=0}^{n} v_j^2 + 2\sum_{j=0}^{n} |\nabla^g v_j|^2 \quad \Longrightarrow \quad \sum_{j=0}^{n} |\nabla^g v_j|^2 = 2.$$

Therefore, by (4.16), we conclude that

$$\sum_{j=0}^{n} dv_j \otimes dv_j = g, \tag{4.17}$$

and $v = (v_0, v_1, \ldots, v_n)$ is an isometric minimal immersion of Σ into $\mathbb{S}^n$.

Now, we will prove that $v(\Sigma) \subset \mathbb{B}^n_\varepsilon(r)$. Observe that by (4.13) along $\partial\Sigma$ we have

$$\begin{aligned}
0 &= \sum_{j=1}^{n} t_j F\left(|\partial\Sigma|^{-\frac{1}{2}} v_0, (t_j|\partial\Sigma|)^{-\frac{1}{2}}\, v_j\right) = \frac{\sigma_0}{2}(\cos^2 r - v_0^2) + \frac{\sigma_1}{2}\left(\sin^2 r - \sum_{j=1}^{n} v_j^2\right) \\
&= \frac{1}{2}\sigma_0 \cos^2 r + \frac{1}{2}\sigma_1 \sin^2 r - \frac{1}{2}\left[\sum_{j=0}^{n} v_j \frac{\partial v_j}{\partial \nu}\right] = \frac{1}{2}\sigma_0 \cos^2 r + \frac{1}{2}\sigma_1 \sin^2 r - \frac{1}{4}\frac{\partial}{\partial \nu}\left(\sum_{j=0}^{n} v_j^2\right) \\
&= \frac{1}{2}\sigma_0 \cos^2 r + \frac{1}{2}\sigma_1 \sin^2 r.
\end{aligned}$$

Hence,

$$\sigma_1 = -(\cot^2 r)\sigma_0. \tag{4.18}$$

By (3.5), we have $\sigma_0 < 0$. On the other hand,

$$0 = \frac{1}{2}\frac{\partial}{\partial \nu}\left(\sum_{j=0}^{n} v_j^2\right) = \sigma_0 v_0^2 + \sigma_1\left(\sum_{j=1}^{n} v_j^2\right) = \sigma_0 v_0^2 + \sigma_1(1 - v_0^2),$$

from where it follows that

$$\sigma_1 = (\sigma_1 - \sigma_0)v_0^2 \text{ on } \partial\Sigma. \tag{4.19}$$

Thus, combining (4.18) and (4.19), and using the fact that v_0 is positive, we conclude

$$v_0 = \cos r \text{ on } \partial\Sigma.$$

Since v_0 is positive and $\Delta_g v_0 + 2v_0 = 0$, its minimum is attained on the boundary; in particular,

$$v_0 \geq \cos r \text{ in } \Sigma.$$

Therefore, $v(\Sigma) \subset \mathbb{B}^n(r)$.

It remains to check the free boundary condition. Applying the tensor fields in (4.17) to (ν, ν), we obtain that on $\partial\Sigma$ it holds

$$1 = \sum_{j=0}^{n}\left(\frac{\partial v_j}{\partial \nu}\right)^2 = \sigma_0^2 v_0^2 + \sigma_1^2\left(\sum_{j=1}^{n} v_j^2\right) = \sigma_0^2 \cos^2 r + \sigma_0^2 \cot^4 r(1 - \cos^2 r),$$

from where it follows that $\sigma_0 = -\tan r$ (since $\sigma_0 < 0$) and, consequently, $\sigma_1 = \cot r$. Therefore, by Proposition 1, the immersion is free boundary. $\square$

Exercise 5. Prove item (ii) of Theorem 6.

Remark 5. *Inspired by our work on [46], Medvedev defined the functionals*

$$\Theta_{r,k}(\Sigma, g) = \left(\sigma_0(g,2)\cos^2 r + \sigma_k(g,2)\sin^2 r\right)|\partial\Sigma|_g + 2|\Sigma|_g,\ k \geqslant 1,$$
$$\Omega_{r,k}(\Sigma, g) = \left(-\sigma_0(g,-2)\cosh^2 r + \sigma_k(g,-2)\sinh^2 r\right)|\partial\Sigma|_g + 2|\Sigma|_g,\ k \geqslant 1,$$

and explored the connection between critical points of $\Theta_{r,k}(\Sigma, g)$ and $\Omega_{r,k}(\Sigma, g)$ with free-boundary minimal immersions in geodesic balls of $\mathbb{S}^n_+$ and $\mathbb{H}^n$ respectively, see [48, Theorem 1.2]. He alsos proves $\Theta_{r,k}$ is bounded from above, but not bounded from below, while $\Omega_{r,k}$ is non-negative, but not bounded from above. Observe $\Theta_{r,1}(\Sigma, g)$ coincides with our functional.

5. Steklov eigenfunctions and uniqueness of free boundary minimal annuli

Exercise 6. Let ϕ_i, $i = 0, 1, 2, 3$, denote the coordinates of the immersion (2.10). Prove that ϕ_0 is a $\sigma_0(g)$-eigenfunction, while ϕ_1, ϕ_2 and ϕ_3 are $\sigma_1(g)$-eigenfunctions.

Fraser and Schoen proved the following uniqueness result for free-boundary minimal annuli on Euclidean balls.

Theorem 7 (Fraser-Schoen [25])**.** *Let Σ be an annulus and consider $\Phi = (\phi_1, \ldots, \phi_n) : \Sigma \to \mathbb{B}^n \subset \mathbb{R}^n$ a free boundary minimal immersion. Suppose ϕ_j is a first Steklov eigenfunction, for $j = 1, \ldots, n$. Then $n = 3$ and $\Phi(\Sigma)$ is a critical catenoid.*

The goal of this section is to prove the following analogue on the case the ambient manifold is a spherical cap.

Theorem 8 (Lima-Menezes [46])**.** *Let Σ be an annulus and consider $\Phi = (\phi_0, \ldots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n_1(r) \subset \mathbb{S}^n$ a free boundary isometric minimal immersion. Suppose ϕ_j is a $\sigma_1(g)$-eigenfunction, for $j = 1, \ldots, n$. Then $n = 3$ and $\Phi(\Sigma)$ is a critical rotational annulus.*

If Σ is an annulus, there is $T > 0$ such that (Σ, g) is conformal to $[-T, T] \times \mathbb{S}^1$ endowed with the cylindrical metric $g_{\text{cyl}} = ds^2 + g_{\mathbb{S}^1}$, i.e., $g = \lambda g_{\text{cyl}}$ for some positive function λ. Let ∂_θ be the (globally defined) vector field associated to any angular coordinate. Observe that ∂_θ spans $T_p\partial\Sigma$, $\forall\, p \in \partial\Sigma$, and the outward pointing unit g-normal of $\partial\Sigma$ is $\pm\lambda^{-1/2}\partial_s$. The proof of Theorem 8 is divided into two cases:

Case 1 : $\displaystyle\int_{\partial\Sigma} \frac{\partial\Phi}{\partial\theta} dA = 0;$

Case 2 : $\displaystyle\int_{\partial\Sigma} \frac{\partial\Phi}{\partial\theta} dA \neq 0.$

Remark 6. *Theorem 8 is also analogous to results of Montiel-Ros [49] and El Soufi-Ilias [15] which characterize the Clifford torus and the flat equilateral torus as the only minimal tori on round spheres immersed by first Laplacian eigenfunctions.*

5.1. Case 1.

Let Σ^k be a compact manifold and let $\Phi : (\Sigma, g) \to \mathbb{B}^n_1(r)$ be a free boundary isometric minimal immersion. We have $\Phi^*T\mathbb{S}^n = \Phi_*(T\Sigma) \oplus N\Sigma$, where $N\Sigma$ is the normal bundle of Φ. So, given $V \in \Gamma(\Phi^*T\mathbb{S}^n)$, $\forall\, p \in \Sigma$ there exist unique $V_p^\top \in \Phi_*(T_p\Sigma)$ and $V_p^\perp \in (N\Sigma)_p$ such that

$$V = V^\top + V^\perp.$$

The pullback connection ∇ is defined as follows: consider $V \in \Gamma(\Phi^*T\mathbb{S}^n)$, $X \in \Gamma(T\Sigma)$ and $p \in \Sigma$; there exist a neighborhood $\mathcal{U}$ of $\Phi(p)$ in $\mathbb{S}^n$ and a section $\widehat{V} \in \Gamma(T\mathcal{U})$, such that $V|_{\mathcal{U}} = \widehat{V} \circ \Phi|_{\mathcal{U}}$; then

$$\nabla_X V(p) := \nabla^{\mathbb{S}^n}_{\Phi_* X} \widehat{V}\big(\Phi(p)\big).$$

For $V, W \in \Gamma(\Phi^*T\mathbb{S}^n)$ such that $V_p, W_p \in T_{\Phi(p)}\partial\mathbb{B}^n_\varepsilon(r)$, $\forall\, p \in \partial\Sigma$, we consider the bilinear form

$$Q(V, W) = \int_\Sigma \big(\langle \nabla V, \nabla W\rangle - 2\langle V, W\rangle + \langle V^\top, W^\top\rangle\big) dA - \cot(r) \int_{\partial\Sigma} \langle V, W\rangle \, dL.$$

Observe that

$$Q(V, V) = \delta^2\mathcal{E}(V),$$

where $\delta^2\mathcal{E}$ denotes the second variation of the energy of the harmonic map Φ.

Lemma 3. *Let $\Phi = (\phi_0, \dots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n(r)$ be a free boundary minimal immersion of an annulus. Then, for any $V \in \Gamma(\Phi^* T\mathbb{S}^n)$ such that $V_p \in T_p\partial\mathbb{B}^n_\varepsilon(r)$, $\forall\, p \in \partial\Sigma$, we have*

$$Q(V, \Phi_\theta) = 0.$$

Proof. Since $\Delta_g = \lambda^{-1}\Delta_{\mathrm{cyl}}$, using Proposition 1 we have

$$\Delta_{\mathrm{cyl}}\,\phi_i + 2\lambda\phi_i = 0 \quad \Rightarrow \quad \partial_\theta\big(\Delta_{\mathrm{cyl}}\,\phi_i + 2\lambda\phi_i\big) = 0$$

$$\Rightarrow \quad \Delta_{\mathrm{cyl}}\frac{\partial\phi_i}{\partial\theta} + 2\frac{\partial\lambda}{\partial\theta}\phi_i + 2\lambda\frac{\partial\phi_i}{\partial\theta} = 0 \quad \Rightarrow \quad \left\langle V, \Delta_g\frac{\partial\Phi}{\partial\theta} + 2\lambda^{-1}\frac{\partial\lambda}{\partial\theta}\Phi + 2\frac{\partial\Phi}{\partial\theta}\right\rangle = 0.$$

Since $\langle V, \Phi\rangle = 0$ we conclude

$$\int_\Sigma \left\langle V, \Delta_g\frac{\partial\Phi}{\partial\theta}\right\rangle dA + 2\int_\Sigma \left\langle V, \frac{\partial\Phi}{\partial\theta}\right\rangle dA = 0. \tag{5.1}$$

Write $V = \displaystyle\sum_{j=0}^{n} V_j\partial_j$. Applying the Divergence thm to the vector fields $V_j\nabla^g\frac{\partial\phi_j}{\partial\theta}$ we obtain

$$\int_\Sigma \left\langle V, \Delta_g\frac{\partial\Phi}{\partial\theta}\right\rangle dA = -\sum_{j=0}^{n}\int_\Sigma \left\langle \nabla^g V_j, \nabla^g\frac{\partial\phi_j}{\partial\theta}\right\rangle dA + \sum_{j=0}^{n}\int_{\partial\Sigma} V_j\left\langle \nu_g, \nabla^g\frac{\partial\phi_j}{\partial\theta}\right\rangle dL. \tag{5.2}$$

Since $\nu_g = \lambda^{-1/2}\nu_{\mathrm{cyl}}$ and ϕ_j satisfies (3.3), for $j = 1, \dots, n$, we get on $\partial\Sigma$

$$\left\langle \nu_g, \nabla^g\frac{\partial\phi_j}{\partial\theta}\right\rangle == \lambda^{-1/2}\nu_{\mathrm{cyl}}\left(\frac{\partial\phi_j}{\partial\theta}\right) = \cot(r)\frac{\partial\phi_j}{\partial\theta} + \frac{\partial\lambda^{1/2}}{\partial\theta}\lambda^{-1/2}\cot(r)\phi_j.$$

Also, notice that $\dfrac{\partial\phi_0}{\partial\theta}\big|_{\partial\Sigma} = 0$, since ϕ_0 is constant along $\partial\Sigma$. So,

$$\left\langle \nu_g, \nabla^g\frac{\partial\phi_0}{\partial\theta}\right\rangle = -\frac{\partial\lambda^{1/2}}{\partial\theta}\lambda^{-1/2}\tan(r)\phi_0 = -\frac{\partial\lambda^{1/2}}{\partial\theta}\lambda^{-1/2}\sin(r), \quad \text{on } \partial\Sigma.$$

Hence,

$$\begin{aligned}\sum_{j=0}^{n}\int_{\partial\Sigma} V_j\left\langle \nu_g, \nabla^g\frac{\partial\phi_j}{\partial\theta}\right\rangle dL &= \sum_{j=0}^{n}\int_{\partial\Sigma} V_j\left[\cot(r)\frac{\partial\phi_j}{\partial\theta} + \frac{\partial\lambda^{1/2}}{\partial\theta}\lambda^{-1/2}\cot(r)\phi_j\right] dL\\ &= \cot(r)\int_{\partial\Sigma}\left\langle V, \frac{\partial\Phi}{\partial\theta}\right\rangle dL,\end{aligned} \tag{5.3}$$

where in the first equality we used the fact that $V_0 = 0$ on $\partial\Sigma$, and in the last equality we used that $\langle V, \Phi\rangle = 0$.

One can check that

$$\sum_j \langle\nabla^g V_j, \nabla^g W_j\rangle = \big\langle\nabla V, \nabla W\big\rangle + \langle V^\top, W^\top\rangle. \tag{5.4}$$

Using (5.2), (5.3) and (5.4) in (5.1), we get

$$Q(V, \Phi_\theta) = \int_\Sigma\left[\Big\langle\nabla V, \nabla\frac{\partial\Phi}{\partial\theta}\Big\rangle + \Big\langle V^\top, \Big(\frac{\partial\Phi}{\partial\theta}\Big)^\top\Big\rangle - 2\Big\langle V, \frac{\partial\Phi}{\partial\theta}\Big\rangle\right] dA - \cot(r)\int_{\partial\Sigma}\Big\langle V, \frac{\partial\Phi}{\partial\theta}\Big\rangle dL = 0.$$

□

Exercise 7. Prove that equation (5.4) holds.

Proof of Case 1 of Theorem 8.

Recall that $\phi_i = \langle\Phi, \partial_i\rangle$ satisfies the equations (3.1), (3.2), (3.3), and we are assuming that $\sigma_0 = -\tan(r)$ and $\sigma_1 = \cot(r)$. The multiplicity bound on σ_1 (see Theorem 5 and

Remark 4) implies that $n=3$ necessarily. Also $\frac{\partial \Phi}{\partial \theta} \in \Gamma(\Phi^* T\mathbb{S}^3)$ and, since the annulus is free boundary, we know $\frac{\partial \Phi}{\partial \theta} \in T\partial\mathbb{B}^3(r)$ on $\partial\Sigma$.

The calculations in the proof of Lemma 3 imply that

$$\Delta_g \frac{\partial \Phi}{\partial \theta} + 2\lambda^{-1}\frac{\partial \lambda}{\partial \theta}\Phi + 2\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{in } \Sigma \tag{5.5}$$

and (taking $V = \frac{\partial \Phi}{\partial \theta}$)

$$\sum_{j=0}^{3}\int_\Sigma \left|\nabla^g \frac{\partial \phi_j}{\partial \theta}\right|^2 dA - \sum_{j=0}^{3}\int_{\partial\Sigma} \frac{\partial \phi_j}{\partial \theta}\,\nu_g\left(\frac{\partial \phi_j}{\partial \theta}\right) dL - 2\int_\Sigma \left|\frac{\partial \Phi}{\partial \theta}\right|^2 dA = 0, \tag{5.6}$$

where

$$\nu_g\left(\frac{\partial \phi_i}{\partial \theta}\right) = \sigma_1 \frac{\partial \phi_i}{\partial \theta} + \frac{\partial \lambda^{1/2}}{\partial \theta}\lambda^{-1/2}\sigma_1\phi_i, \quad \text{on } \partial\Sigma \quad \text{for } i=1,2,3.$$

Since $\frac{\partial \phi_0}{\partial \theta}\Big|_{\partial\Sigma} = 0$ and $\left\langle \Phi, \frac{\partial \Phi}{\partial \theta}\right\rangle = 0$, we can rewrite (5.6) as

$$\begin{aligned}\sum_{i=1}^{3}\left[\int_\Sigma \left(\left|\nabla^g \frac{\partial \phi_i}{\partial \theta}\right|^2 - 2\left|\frac{\partial \phi_i}{\partial \theta}\right|^2\right) dA - \sigma_1\int_{\partial\Sigma}\left|\frac{\partial \phi_i}{\partial \theta}\right|^2 dL\right] + \\ \int_\Sigma \left(\left|\nabla^g \frac{\partial \phi_0}{\partial \theta}\right|^2 - 2\left|\frac{\partial \phi_0}{\partial \theta}\right|^2\right) dA - \sigma_0\int_{\partial\Sigma}\left|\frac{\partial \phi_0}{\partial \theta}\right|^2 dL = 0.\end{aligned} \tag{5.7}$$

First observe that since $\frac{\partial \phi_0}{\partial \theta}\Big|_{\partial\Sigma} = 0$, Proposition 2 implies that the second to last term in the above expression is nonnegative.

Moreover, since $\phi_0\big|_{\partial\Sigma}$ is constant and we are assuming ϕ_0 is a first eigenfunction, the condition $\int_{\partial\Sigma}\frac{\partial \Phi}{\partial \theta} dA = 0$ implies that, if $\frac{\partial \phi_i}{\partial \theta}\big|_{\partial\Sigma} \neq 0$, then $\frac{\partial \phi_i}{\partial \theta}$ is a test function for the variational characterization of σ_1, in particular, we have

$$\int_\Sigma \left(\left|\nabla^g \frac{\partial \phi_i}{\partial \theta}\right|^2 - 2\left|\frac{\partial \phi_i}{\partial \theta}\right|^2\right) dA - \sigma_1\int_{\partial\Sigma}\left|\frac{\partial \phi_i}{\partial \theta}\right|^2 dL \geq 0.$$

For the case where $\frac{\partial \phi_i}{\partial \theta}\big|_{\partial\Sigma} = 0$, we get the same conclusion above using Proposition 2.

Therefore, (5.7) implies that all inequalities above are equalities. Hence, if $\frac{\partial \phi_i}{\partial \theta}\Big|_{\partial\Sigma} \neq 0$, then $\frac{\partial \phi_i}{\partial \theta}$ is a (nontrivial) σ_1-eigenfunction. In any case, it holds

$$\Delta_g \frac{\partial \phi_i}{\partial \theta} + 2\frac{\partial \phi_i}{\partial \theta} = 0 \text{ in } \Sigma \text{ for } i=1,2,3.$$

Hence, by (5.5) we get

$$\lambda^{-1}\frac{\partial \lambda}{\partial \theta}\phi_i = 0 \text{ in } \Sigma \text{ for } i=1,2,3.$$

Since $\sum_{i=1}^{3}\phi_i^2 \neq 0$ in $\Sigma \setminus \{(1,0,0,0)\}$ we conclude that $\frac{\partial \lambda}{\partial \theta} \equiv 0$ in Σ, that is, the metric is intrisically rotational.

Now we are going to show that this implies the surface Σ is rotational.

In the proof of Cases 1 and 2 above, we concluded that for $i = 1, 2, 3$, either $\dfrac{\partial \phi_i}{\partial \theta}$ is identically zero or is a σ_1-eigenfunction. Since the multiplicity of σ_1 is three and $\{\phi_1, \phi_2, \phi_3\}$ form a basis for its eigenspace, there are constants a_{ij} such that $\dfrac{\partial \phi_i}{\partial \theta} = \displaystyle\sum_{j=1}^{3} a_{ij}\phi_j$. Also, observe that $\dfrac{\partial \phi_0}{\partial \theta}\Big|_{\partial\Sigma} = 0$ and, since $\dfrac{\partial \lambda}{\partial \theta} = 0$, $\Delta_g \dfrac{\partial \phi_0}{\partial \theta} + 2\dfrac{\partial \phi_0}{\partial \theta} = 0$ in Σ. Then, since 2 is not a Dirichlet eigenvalue, we conclude that $\dfrac{\partial \phi_0}{\partial \theta} = 0$ on Σ.

Let us denote by A the 4×4 matrix with constant entries defined by

$$\frac{\partial \Phi}{\partial \theta}(s, \theta) = A\Phi(s, \theta). \tag{5.8}$$

Our goal is to show that A is antisymmetric. This is equivalent to show that the symmetric constant matrix B defined below is equal to 0;

$$\langle Bv, w\rangle = \langle Av, w\rangle + \langle v, Aw\rangle \text{ for any } v, w \in \mathbb{R}^4.$$

Observe that $\langle A\Phi, \Phi\rangle = \langle \Phi_\theta, \Phi\rangle = 0$. Hence, differentiating the left-hand side of this equality, we conclude that

$$\langle A\Phi(s, \theta), v\rangle + \langle \Phi(s, \theta), Av\rangle = 0,$$

for any $v \in T_{\Phi(s,\theta)}\Phi(\Sigma)$. So, $\langle B\Phi, \Phi\rangle = 0$ and $\langle B\Phi, Y\rangle = 0$ for any $Y \in T_{\Phi(s,\theta)}\Phi(\Sigma)$. Thus, there is a function $a = a(s, \theta)$ such that $B\Phi = aN$, where $N = N(s, \theta)$ is the unit normal to Σ at $\Phi(s, \theta)$.

One can show that

$$\begin{aligned} \left\langle \nabla^{\mathbb{S}^3}_{\Phi_s} N, \Phi_s \right\rangle &= \langle B\Phi_s, \Phi_s\rangle = 0, \\ a\left\langle \nabla^{\mathbb{S}^3}_{\Phi_\theta} N, \Phi_\theta \right\rangle &= \langle B\Phi_\theta, \Phi_\theta\rangle = 0, \\ a\left\langle \nabla^{\mathbb{S}^3}_{\Phi_s} N, \Phi_\theta \right\rangle &= \langle B\Phi_s, \Phi_\theta\rangle = 0. \end{aligned} \tag{5.9}$$

If $a(s, \theta) \neq 0$ for some (s, θ), then there would be an open neighborhood of (s, θ) such that $a(p) \neq 0$ for all $p \in U$. Hence, by (5.9) we would have

$$\left\langle \nabla^{\mathbb{S}^3}_{\Phi_s} N, \Phi_\theta \right\rangle = \left\langle \nabla^{\mathbb{S}^3}_{\Phi_s} N, \Phi_s \right\rangle = \left\langle \nabla^{\mathbb{S}^3}_{\Phi_\theta} N, \Phi_\theta \right\rangle = 0 \quad \text{in } U,$$

which would imply that Σ is totally geodesic, a contradiction, since Σ is an annulus. Therefore, $a \equiv 0$ and we conclude that $Bx = 0$ for all $x \in \Sigma$.

Suppose B is not the zero matrix. Then Σ is contained in the hyperplane $\mathrm{Ker}(B)$ which implies that Σ is totally geodesic, a contradiction. Therefore, B is the zero matrix and A is antisymmetric.

Observe that by the definition of A, we know that $Ae_0 = 0$; hence, 0 is an eigenvalue of A and, since it is a 4×4 antisymmetric matrix, it has multiplicity at least 2. Then, there exists a unit vector v orthogonal to e_0 such that $Av = 0$. Now consider the space $W = (\mathrm{span}\{e_0, v\})^\perp$. Since A is antisymmetric, this space is invariant by A, i.e., $A(W) \subset W$.

Hence, if we take $\{w_1, w_2\}$ an orthonormal basis of W, then $\{e_0, v, w_1, w_2\}$ is an orthonormal basis for $\mathbb{R}^4$ and the matrix A is written with respect to this basis as:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & \mu & 0 \end{pmatrix} \text{ for some } \mu \neq 0.$$

Now, for any t, consider

$$R_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}.$$

Observe that $\frac{d}{dt}(R_t) = \mu^{-1}A(R_t)$, i.e., the one-parameter subgroup generated by A is given by rotations. Also, equation (5.8) implies $Ax \in T_x\Sigma$ for any $x \in \Sigma$ and $Ay \in T_y\partial\Sigma$ for any $y \in \partial\Sigma$. Uniqueness of solutions to ODE's, allows us to conclude $R_t(\partial\Sigma) \subset \partial\Sigma$ and $R_t(\Sigma) \subset \Sigma$. By the result of do Carmo and Dajczer [13] (see also [39]), Σ is one of the rotational examples given by (2.10). $\square$

5.2. **Case 2.** We say that $V \in \Gamma(\Phi^*T\mathbb{S}^n)$ is *conformal* if there exists a function $u \in C^\infty(\Sigma)$ such that for any $X, Y \in \Gamma(T\Sigma)$ we have

$$\langle \nabla_X V, \Phi_* Y\rangle + \langle \nabla_Y V, \Phi_* X\rangle = 2u\langle X, Y\rangle.$$

Lemma 4. *Let $\Phi : (\Sigma^2, g) \to \mathbb{B}^n_1(r)$ be a free boundary isometric minimal immersion, where Σ is a compact orientable surface. If $V \in \Gamma(\Phi^*T\mathbb{S}^n)$ is conformal and $V_p \in T_p\partial\mathbb{B}^n_1(r)$, $\forall\, p \in \partial\Sigma$, then,*

$$Q(V,V) = S\big(V^\perp, V^\perp\big). \tag{5.10}$$

Proof. The author proved (see the formula at the end of the proof of Theorem 5 in [43]) that

$$(\delta^2\mathcal{E})(V) = (\delta^2\mathcal{A})(V^\perp) + 8\int_\Sigma |\zeta|^2 dA,$$

where $\delta^2\mathcal{A}$ is the second variation of area and, with respect to a local isothermal complex coordinate $z = x + iy$, we have

$$\zeta = \Phi_*\big(\nabla^g_{\partial_z}(f\partial_{\bar z})\big) + \big(\nabla_{\partial_z}V^\perp\big)^\top,$$

where $V^\top = \bar f\partial_z + f\partial_{\bar z}$, and we extended the connections to $\Phi^*(T\mathbb{S}^3)\otimes_{\mathbb{R}}\mathbb{C}$ complex bilinearly. However, the condition $\zeta = 0$ is equivalent to the equation (see section 2 of [14])

$$\langle \nabla_{\partial_z} V, \Phi_*\partial_z\rangle = 0,$$

which can be rewritten as

$$\left\langle \nabla_{\partial_x} V, \frac{\partial\Phi}{\partial x}\right\rangle = \left\langle \nabla_{\partial_y} V, \frac{\partial\Phi}{\partial y}\right\rangle \quad \text{and} \quad \left\langle \nabla_{\partial_x} V, \frac{\partial\Phi}{\partial y}\right\rangle = -\left\langle \nabla_{\partial_y} V, \frac{\partial\Phi}{\partial x}\right\rangle,$$

and this is equivalent to V being conformal. Hence, we have $(\delta^2\mathcal{E})(V) = (\delta^2\mathcal{A})(V^\perp)$, and we obtain (5.10). $\square$

Now let us specialize to the case of codimension one and suppose Σ is orientable. We know the bundle $N\Sigma$ is trivial, and there is a globally defined unit normal vector N in Σ. The index form associated to the second variation of area is given by

$$\begin{aligned} S(\psi,\phi) &= \int_\Sigma \big[\langle \nabla^g\psi, \nabla^g\phi\rangle - (n-1+|\mathrm{II}|^2)\psi\phi\big]\,dA - \cot(r)\int_{\partial\Sigma}\psi\phi\,dL, \\ &= -\int_\Sigma \big(\Delta_g\psi + (n-1)\,\psi + |\mathrm{II}|^2\psi\big)\phi\,dA + \int_{\partial\Sigma}\left(\frac{\partial\psi}{\partial\nu} - \cot(r)\psi\right)\phi\,dL, \end{aligned}$$

where II is the second fundamental of Σ in $\mathbb{B}^n_\varepsilon(r)$. Observe that if $\Phi(\Sigma)$ is not contained in a hyperplane containing the origin, then the space of functions $\mathcal{C} = \{\langle\Phi, v\rangle; v \in \mathbb{R}^{n+1}\}$

has dimension $n+1$. The *Morse index* of Φ is defined is the maximum dimension of a subspace of $C^\infty(\Sigma)$ where S is negative definite.

Lemma 5. *Let $\Phi : \Sigma^{n-1} \to \mathbb{B}^n(r)$ be as above. Then, for any $\psi \in \mathcal{C}$, we have*

$$S(\psi,\psi) = -\int_\Sigma |\mathrm{I\!I}|^2\psi^2 dA - \langle v, \partial_0\rangle^2 \cot(r)|\partial\Sigma|,$$

where v is the vector that defines ψ, i.e., $\psi = \langle \Phi, v\rangle$. Hence, if Σ^{n-1} is not contained in a hyperplane containing the origin, then its Morse index is at least $n+1$.

Exercise 8. Prove Lemma 5

Lemma 6. *Suppose that $\Phi : (\Sigma, g) \to \mathbb{B}^3(r)$ is a free boundary isometric minimal immersion of an annulus. Then, there is a subspace $\mathcal{C}_1$ of $\mathcal{C}$ of dimension at least three such that for all $\psi \in \mathcal{C}_1$ there is $Y^\top \in \Gamma(\Phi_* T\Sigma)$ such that $Y^\top$ is tangential to $\partial\mathbb{B}^3(r)$ along $\partial\Sigma$ and $Y(\psi) = Y^\top + \psi N$ is conformal. Moreover, for any two such vector fields $Y_1^\top$ and $Y_2^\top$, there exist a constant $c \in \mathbb{R}$ such that*

$$Y_1^\top - Y_2^\top = c\,\frac{\partial\Phi}{\partial\theta}.$$

Proof. We need to find $Y^\top \in \Gamma(\Phi_* T\Sigma)$ such that $Y(\psi) = Y^\top + \psi N$ is conformal, that is,

$$\left\langle \nabla_{\partial_\theta} Y, \frac{\partial\Phi}{\partial s}\right\rangle + \left\langle \nabla_{\partial_s} Y, \frac{\partial\Phi}{\partial\theta}\right\rangle = 0 \quad \text{and} \quad \left\langle \nabla_{\partial_s} Y, \frac{\partial\Phi}{\partial s}\right\rangle = \left\langle \nabla_{\partial_\theta} Y, \frac{\partial\Phi}{\partial\theta}\right\rangle.$$

These two equations are equivalent respectively to

$$\begin{aligned}
\langle \nabla^g_{\partial_\theta} Y^\top, \partial_s\rangle + \langle \nabla^g_{\partial_s} Y^\top, \partial_\theta\rangle - 2\psi\mathrm{I\!I}(\partial_\theta, \partial_s) &= 0,\\
\langle \nabla^g_{\partial_s} Y^\top, \partial_s\rangle - \langle \nabla^g_{\partial_\theta} Y^\top, \partial_\theta\rangle - 2\psi\mathrm{I\!I}(\partial_s, \partial_s) &= 0.
\end{aligned}$$

Recall $\{\partial_s, \partial_\theta\}$ form a (globally defined) basis for $T\Sigma$ and $|\partial_s|_g^2 = |\partial_\theta|_g^2 = \lambda$. Hence, we can write $Y^\top = u\Phi_s + v\Phi_\theta$ for some functions u, v. Using the expressions for $\nabla^g_{\partial_s} Y^\top$ and $\nabla^g_{\partial_\theta} Y^\top$ obtained from the Koszul formula and putting them into the previous system, we get

$$\begin{cases} \partial_\theta u + \partial_s v &= 2\lambda^{-1}\psi\mathrm{I\!I}(\partial_\theta, \partial_s),\\ \partial_s u - \partial_\theta v &= 2\lambda^{-1}\psi\mathrm{I\!I}(\partial_s, \partial_s).\end{cases}$$

Thus, if we set $f = u + iv$ and $z = s + i\theta$, these equations become

$$\partial_{\bar z}(f) = h, \tag{5.11}$$

where $h = \lambda^{-1}\psi\mathrm{I\!I}(\partial_s, \partial_s) + i\lambda^{-1}\psi\mathrm{I\!I}(\partial_\theta, \partial_s)$. Moreover, the boundary condition that $Y^\top$ is tangential to $\partial\mathbb{B}^3(r)$ along $\partial\Sigma$ means that we need $Y^\top$ parallel to ∂_θ along $\partial\Sigma$, that is, $u = 0$ on $\partial\Sigma$. So, we are imposing

$$\mathfrak{Re} f = 0 \text{ on } \partial\Sigma.$$

Hence, we need to solve the following boundary linear problem

$$\begin{cases} \partial_{\bar z}(f) &= h, \text{ in } \Sigma\\ \mathfrak{Re} f &= 0, \text{ on } \partial\Sigma.\end{cases} \tag{5.12}$$

By the Fredholm alternative, it has a solution if and only if, h is $L^2(\Sigma, g_{\mathrm{cyl}})$-orthogonal to the kernel of the adjoint of the operator $\partial_{\bar z}$ (defined on the space of complex differentiable functions which are real on the boundary). This kernel consists of the complex differentiable functions satisfying the following:

$$\begin{cases} -\partial_z \phi &= 0, \text{ in } \Sigma \\ \mathfrak{Im}\,\phi &= 0, \text{ on } \partial\Sigma. \end{cases}$$

The solutions of this boundary value problem are $\phi \equiv c$, $c \in \mathbb{R}$.

Defining

$$\mathcal{C}_1 = \left\{ \psi \in \mathcal{C};\, \mathfrak{Re}\left(\int_\Sigma \left[\lambda^{-1}\psi \mathrm{I\!I}(\partial_\theta, \partial_s) + i\lambda^{-1}\psi \mathrm{I\!I}(\partial_s, \partial_s) \right] dA \right) = 0 \right\},$$

it follows from previous analysis that $\dim \mathcal{C}_1 \geq \dim \mathcal{C} - 1 = 3$.

To prove the uniqueness statement, observe that, if there are two solutions f_1, f_2 to (5.12), then,

$$\begin{cases} \frac{\partial(f_1 - f_2)}{\partial \bar{z}} &= 0, \text{ in } \Sigma \\ \mathfrak{Re}(f_1 - f_2) &= 0, \text{ on } \partial\Sigma. \end{cases}$$

So, the maximum principle implies that $f_1 - f_2 = ic$ in Σ for some real constant c, and the statement follows. ☐

Exercise 9. Fill in the details of the calculations on the previous proof.

Proof of Case 2 of Theorem 8. By Lemma 6, for any $\psi \in \mathcal{C}_1$, there is a conformal vector field whose normal component is ψN, and which is unique up to addition of a real multiple of Φ_θ. We denote by $Y(\psi) = Y^\top + \psi N$ the unique conformal vector field such that $\left\langle \int_{\partial\Sigma} Y(\psi) dL, \int_{\partial\Sigma} \Phi_\theta dL \right\rangle = 0$.

Also, by Lemma 6, we have that $Y^\top$ is tangential to $\partial\mathbb{B}^3(r)$. Hence, in particular, since the unit normal vector to $\partial\mathbb{B}^3(r)$ at a point x is given by $(\sin r)^{-1}\big(\langle \partial_0, x\rangle x - \partial_0\big)$, we conclude that $\langle Y(\psi), \partial_0 \rangle = 0$.

By construction, the map $\psi \to Y(\psi)$ is linear. Consider the vector space

$$\mathcal{V} = \{ Y(\psi) + c\,\Phi_\theta; \psi \in \mathcal{C}_1, c \in \mathbb{R} \}.$$

Notice that $\mathcal{V}$ has dimension at least 4, since $\dim(\mathcal{C}_1) \geq 3$ and any nonzero vector field $Y(\psi)$ has a nontrivial normal component, while Φ_θ is tangential to Σ. Also, every element in $\mathcal{V}$ has first coordinate zero, i.e., $\langle Y(\psi) + c\,\Phi_\theta, \partial_0 \rangle = 0$, for any $\psi \in \mathcal{C}_1, c \in \mathbb{R}$.

Define a linear transformation $T : \mathcal{V} \to \{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4$ by

$$T\left(Y(\psi) + c\,\Phi_\theta\right) = \int_{\partial\Sigma} \left(Y(\psi) + c\,\Phi_\theta\right) dL.$$

Since $\dim(V) \geq 4$, this linear transformation has a nontrivial nullspace, i.e., there exist $\psi \in \mathcal{C}_1$ and $c \in \mathbb{R}$ such that $\int_{\partial\Sigma} \left(Y(\psi) + c\,\Phi_\theta\right) dL = 0$ and $Y(\psi) + c\,\Phi_\theta \neq 0$.

We have

$$Q\left(Y(\psi) + c\,\Phi_\theta, Y(\psi) + c\,\Phi_\theta\right) = Q\big(Y(\psi), Y(\psi)\big) + 2c\,Q\left(Y(\psi), \Phi_\theta\right) + c^2 Q\left(\Phi_\theta, \Phi_\theta\right).$$

By Lemma 3, we know

$$Q\left(Y(\psi), \Phi_\theta\right) = Q\left(\Phi_\theta, \Phi_\theta\right) = 0,$$

and, since $Y(\psi)$ is conformal, Lemmas 5 and 4 give that

$$Q(Y(\psi), Y(\psi)) = S(\psi, \psi) = -\int_\Sigma |\mathrm{I\!I}|^2 \psi^2 dA - \langle v, \partial_0 \rangle^2 \cot(r) |\partial\Sigma|.$$

Hence,

$$Q\left(Y(\psi)+c\,\Phi_\theta, Y(\psi)+c\,\Phi_\theta\right) = -\int_\Sigma |\mathrm{I\!I}|^2\psi^2 dA - \langle v, \partial_0\rangle^2 \cot(r)|\partial\Sigma| \le 0. \tag{5.13}$$

Denote

$$f_i = \left(Y(\psi)+c\,\Phi_\theta\right)_i = \langle Y(\psi)+c\,\Phi_\theta, \partial_i\rangle\,,\ i=0,1,2,3.$$

Since $f_0\big|_{\partial\Sigma}=0$, we know by Proposition 2 that

$$\int_\Sigma \left(|\nabla^g f_0|^2 - 2f_0^2\right) dA \ge 0.$$

On the other hand, since $\int_{\partial\Sigma}\left(Y(\psi)+c\,\Phi_\theta\right) dL = 0$, for $i=1,2,3$, either the function f_i is a test function for the variational characterization of σ_1, which implies that

$$\int_\Sigma \left(|\nabla^g f_i|^2 - 2f_i^2\right) dA - \cot(r)\int_{\partial\Sigma} f_i^2 dL \ge 0;$$

or $f_i|_{\partial\Sigma}=0$ and, by Proposition 2, we have

$$\int_\Sigma \left(|\nabla^g f_i|^2 - 2f_i^2\right) dA \ge 0.$$

Then,

$$\begin{aligned} Q\left(Y(\psi)+c\,\Phi_\theta, Y(\psi)+c\,\Phi_\theta\right) = & \sum_{i=0}^{3}\left(\int_\Sigma \left(|\nabla^g f_i|^2 - 2f_i^2\right) dA - \cot(r)\int_{\partial\Sigma} f_i^2 dL\right) \\ & + \int_\Sigma \left|\left(Y(\psi)+c\,\Phi_\theta\right)^\top\right|^2 dA \ge 0. \end{aligned} \tag{5.14}$$

Thus, (5.13) and (5.14) yield that

$$-\int_\Sigma |\mathrm{I\!I}|^2\psi^2 dA = 0,$$

which implies $|\mathrm{I\!I}|^2\psi^2=0$. If $\psi>0$ at some point p, then the same holds in a neighborhood of p, which implies $|\mathrm{I\!I}|\equiv 0$ in Σ (by analytic continuation). However, this contradicts the fact that Σ is an annulus. Thus, $\psi\equiv 0$ and, hence, $Y(\psi)=0$. Therefore, $\int_{\partial\Sigma}\Phi_\theta dA = 0$ and we are in the situation of our previous case, where we can conclude that the surface is rotational. □

Inspired by the previous results Medvedev [48] studied the uniqueness of free-boundary minimal annuli on geodesic balls of hyperbolic spaces.

Theorem 9 (Medvedev [48]). *Let Σ be an annulus and consider $\Phi = (\phi_0,\ldots,\phi_n) : (\Sigma, g) \to \mathbb{B}^n_{-1}(r) \subset \mathbb{H}^n$ a free boundary isometric minimal immersion. Suppose ϕ_j is a $\sigma_1(g,-2)$-eigenfunction, for $j=1,\ldots,n$. Then $n=3$ and $\Phi(\Sigma)$ is is a critical rotational annulus.*

The proof also follows the strategy of Fraser-Schoen [25, Theorem 6.6] and can be adapted from the proof of Theorem 8.

6. Further results and open problems

6.1. **The uniqueness of the critical catenoid.** We begin this section stating one of the most important open problems on the theory of free-boundary minimal surfaces, which can be seen as an analogue of the *Lawson Conjecture* proved by Brendle [2].

Critical Catenoid Conjecture. *Let $\Sigma \subset \mathbb{B}^3 \subset \mathbb{R}^3$ be an embedded free-boundary minimal annulus. Then Σ is a critical catenoid.*

This conjecture dates back to the work of Nitsche [52] and its was explicitely stated by Fraser-Li in [20]. Its is important to remark that the statement is false without the assumption of embeddeness, as counterexamples were constructed in [17] and [30]. Observe Theorems 2 and 7 are partial results towards the conjecture. Other partial results were obtained in [**?**, 17, 26, 37, 47, 56, 61] under different assumptions. Among them we would like to highlight the following one.

Theorem 10 (Kusker-McGrath [37])**.** *Let $\Sigma \subset \mathbb{B}^3 \subset \mathbb{R}^3$ be an embedded free boundary minimal annulus, which is invariant by the antipodal map. Then Σ is a critical catenoid.*

The main tool in the proof is the *two-piece property* for free-boundary minimal surfaces proved by the author in joint work with A. Menezes:

Theorem 11 (Lima-Menezes [44,45])**.** *Let $\Sigma \subset \mathbb{B}^n$ be a compact, embedded, free-boundary minimal hypersurface. Then for every hyperplane $\Pi \subset \mathbb{R}^n$, either $\Sigma = \Pi \cap \mathbb{B}^n$, or Π divide Σ in exactly two connected components.*

Kusner and McGrath use the two-piece property together with the other assumptions to show the first Steklov eigenvalue of Σ is 1. Since the coordinates functions are Steklov eigenfunctions with eigenvalue 1, it follows from Theorem 7 that Σ is a critical catenoid.

It is worth highliting that the two-piece is implied by the following conjecture.

Conjecture (Fraser-Li [20])**.** *Let $\Sigma \subset \mathbb{B}^n$ be a compact, embedded, free-boundary minimal hypersurface. Then the first steklov eigenvalue of Σ is equal to* 1*.*

Our proofs in [44, 45] are independent of the statement of this conjecture, so our result gives evidence to it.

Remark 7. *Theorem 11 was inspired by an analogous result for closed minimal surfaces in $\mathbb{S}^n$, due to Ros [55]. His result gives evidence to Yau's Conjecture on the first Laplace eigenvalue of minimal surfaces of the sphere (which inspired the previous conjecture).*

One can wonder about the analogous problems on $\mathbb{B}^n_\varepsilon(r)$. In view of Theorems 8 and 9 is reasonable to state the following.

Conjecture. *Let $\Sigma \subset \mathbb{B}^3_\varepsilon(r)$ be an embedded free-boundary minimal annulus. Then Σ is congruent to a critical rotational annulus.*

Again, the statement is false without the assumption of embeddeness, as counterexamples were constructed in [9, 10].

The two-piece property was extended to the case of $\mathbb{B}^3_\varepsilon(r)$ by Naff and Zhu. Actually, they proved a stronger statement.

Theorem 12 (Naff-Zhu [51])**.** *If $\Sigma_0, \Sigma_1 \subset \mathbb{B}^n_\varepsilon(r)$ are free boundary embedded minimal hypersurfaces in a closed geodesic ball of radius $r \in (0, \mathrm{diam}(\mathbb{E}^n_\varepsilon))$ then*

$$\mathbb{B}^n_{\varepsilon,+}(r) \cap \Sigma_0 \cap \Sigma_1 \neq \emptyset,$$

where $\mathbb{B}^3_{\varepsilon,+}(r) \subset \mathbb{B}^3_\varepsilon(r)$ is a closed half-ball.

This implies:

Corollary 2 (Two-piece property for $\mathbb{B}^n_\varepsilon(r)$)**.** *Let Σ be a compact, embedded, free boundary minimal hypersurface in $\mathbb{B}^n_\varepsilon(r)$. If $v \in \mathcal{C}$ then*

$$\Sigma \cap H_+(v) \quad \textit{and} \quad \Sigma \cap H_-(v)$$

are connect sets, where $H_+(v) = \{w \in \mathbb{R}^{n+1} : g_\varepsilon(v,w) > 0\}$ and $H_-(v) = \{w \in \mathbb{R}^{n+1} : g_\varepsilon(v,w) < 0\}$.

Inspired by these developments and following the ideas of [37], César Lima obtained a version of Theorem 10 for $\mathbb{B}^3_\varepsilon(r)$.

Theorem 13 (C. Lima [42])**.** *Let $\Sigma \subset \mathbb{B}^3_\varepsilon(r)$ be an embedded free boundary minimal annulus, which is invariant by the antipodal map. Then Σ is congruent to a critical rotational annulus.*

Remark 8. *We define the antipodal map on $\mathbb{B}^3_\varepsilon(r)$ by*

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, -x_1, \dots, -x_n).$$

6.2. **Free-boundary minimal Möbius bands.** Fraser and Schoen constructed an embedded free-boundary minimal surface in $\mathbb{B}^4$ with the topology of the Möbius band [25] and called this example the *critical Möbius band*. This is the image of the map

$$(t,\theta) \in [-T_0, T_0] \times \mathbb{S}^1 \mapsto \big(2\sinh t\cos\theta, 2\sinh t\sin\theta, \cosh(2t)\cos(2\theta), \cosh(2t)\sin(2\theta)\big) \in \mathbb{R}^4,$$

where T_0 is the unique positive solution of $\coth t = 2\tanh(2t)$. The idea is to use separation of variables to compute the eigenvalues and eigenfunctions of the Dirichlet-to-Neumann map for rotationally symmetric metrics on the Möbius band.

Moreover Fraser and Schoen proved the following uniqueness result.

Theorem 14 (Fraser-Schoen [25])**.** *Let Σ be a Möbius band and consider $\Phi = (\phi_1, \dots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n \subset \mathbb{R}^n$ a free boundary minimal immersion. Suppose ϕ_j is a first Steklov eigenfunction, for $j = 1, \dots, n$. Then $n = 4$ and $\Phi(\Sigma)$ is congruent to the critical Möbius band.*

In view of this we have two natural questions:

(1) Is there a free-boundary minimal Möbius band in $\mathbb{B}^3$?
(2) Is there a free-boundary minimal Möbius band in $\mathbb{B}^n_\varepsilon(r)$, for some n?

Question (1) was answered on the negative by Carlos Toro.

Theorem 15 (Toro [59,60])**.** *There is no free boundary minimal immersion of the Möbius band in $\mathbb{B}^3_\varepsilon(r)$. In the Euclidean ball the result holds even if we allow branch points.*

Concerning question 2, Mateus Spezia proved the following result.

Theorem 16 (Spezia [58])**.** *Let Σ be a Möbius band. Then there exists a free boundary minimal immersion $\Phi = (\phi_0, \dots, \phi_4) : (\Sigma, g) \to \mathbb{B}^4_1(r)$, such that ϕ_j is a $\sigma_1(g)$-eigenfunction, for $j = 1, \dots, 4$.*

The idea for the construction is similar to that of the critical Möbius band, however this case is harder because is not clear how to find explicit solutions of the differential equations involved. Spezia also proved the following analogue of Theorem 14.

Theorem 17 (Spezia [58])**.** *Let Σ be a Möbius band and consider $\Phi = (\phi_1, \dots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n \subset \mathbb{R}^n$ a free boundary minimal immersion. Suppose ϕ_j is a first Steklov eigenfunction, for $j = 1, \dots, n$. Then (Σ, g) is intrinsically rotationally symmetric.*

6.3. **Existence of free-boundary minimal surfaces in $\mathbb{B}^n_\varepsilon(r)$.** In the last years many authors constructed examples of free-boundary minimal surfaces in $\mathbb{B}^3$ [**?**, 7, 18, 19, 31, 36]. This culminated in the result by Kusner-Karpukhin-MacGrath-Stern [32], which states that for every compact orientable surface with boundary, there is a minimal embedding $\Phi : \Sigma \to \mathbb{B}^3$. They proved this exploring the connection between free-boundary minimal surfaces in Euclidean balls and critical points of the functional

$$g \mapsto \sigma_1(g, 0)|\partial\Sigma|_g. \tag{6.1}$$

Another interesting question is wether there exist maximizers for (6.1). The best general results are due to Petrides.

Theorem 18 (Petrides [54]). *Let Σ be a compact surface with boundary and let be $[h]$ be a conformal class of Σ. If*

$$\sup_{g\in[h]} \sigma_1(g, 0)|\partial\Sigma|_g > 2\pi,$$

then there is a metric $\widehat{g} \in [h]$ (possibly with a finite set of conical singularities) such that

$$\sup_{g\in[h]} \sigma_1(g, 0)|\partial\Sigma|_g = \sigma_1(\widehat{g}, 0)|\partial\Sigma|_{\widehat{g}}.$$

Moreover, this maximizing metric is the pull-back of the Euclidean metric by a free boundary harmonic map $\Phi : \Sigma \to \mathbb{B}^n$, for some n

Theorem 19 (Petrides [54]). *Let $\mathrm{S}_{\gamma,m}$ denote the compact orientable surface of genus γ and $m \geq 1$ boundary connected components and define*

$$\sigma_1(\gamma, m) = \sup \sigma_1(g, 0)|\partial\Sigma|_g,$$

where the supremum is taken over all Riemannian metrics on $\mathrm{S}_{\gamma,m}$. Suppose

$$\sigma_1(\gamma, m) > \sigma_1(\gamma, m-1) \text{ or } m = 1; \text{ and}$$
$$\sigma_1(\gamma, m) > \sigma_1(\gamma-1, m+1) \text{ or } \gamma = 0.$$

Then there exists a metric $\widehat{g}$ on Σ (possibly with a finite set of conical singularities) such that

$$\sigma_1(\gamma, m) = \sigma_1(\widehat{g}, 0)|\partial\Sigma|_{\widehat{g}}.$$

Moreover, up to scaling, this maximizing metric is the pull-back of the Euclidean metric by a conformal free boundary minimal immersion in $\Phi : \Sigma \to \mathbb{B}^n$, for some n.

Combining these results with the works of Fraser-Schoen [25] and Karpukhin-Stern [33, 34] it follows that for every m, $\sigma_1(0, m)$ is realized by the induced metric of an embedded free-boundary minimal surface $\Sigma_m \subset \mathbb{B}^3$. Moreover, as $m \to \infty$, $\{\Sigma_m\}_{m\in\mathbb{N}}$ converges (in the sense of varifolds) to $\mathbb{S}^2 = \partial\mathbb{B}^3$.

In [32] the authors also address que question of existence of maximizers for (6.1).

Theorem 20 (Kusner-Karpukhin-MacGrath-Stern [32]). *Each compact oriented surface with boundary whose genus is one admits a metric maximizing* (6.1).

In view of this one can ask the following questions?

(i) Is there an immersion of $\mathrm{S}_{\gamma,m}$ in $\mathbb{B}^n_\varepsilon(r)$? What about an embedding?
(ii) Is there a metric $\widehat{g}$ such that

$$\sup_{g\in\mathcal{M}(\Sigma)} \Theta_r(\mathrm{S}_{\gamma,m}, g) = \Theta_r(\mathrm{S}_{\gamma,m}, \widehat{g})?$$

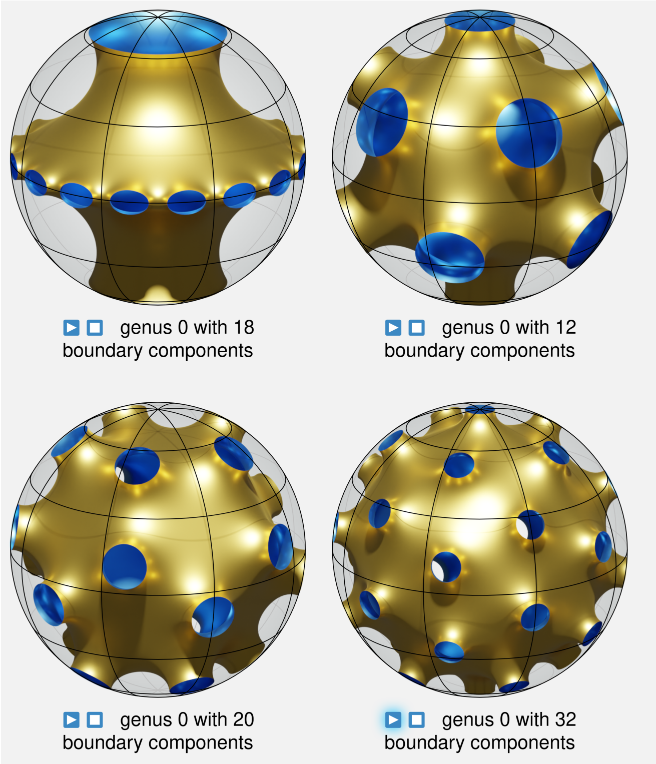

FIGURE 4. The surfaces Σ_m. Picture by Mario Schulz.

Observe that in the case of an annulus, if (ii) is true, then by Theorems 6 and 8, $\widehat{g}$ is the metric of a critical rotational annulus.

Recall that in the case $\varepsilon = 1$ we assumed $r < \pi/2$. In the case $r = \pi/2$, Kusner-Karpukhin-MacGrath-Stern [32] answered question (i) affirmatively. Their idea is to maximize the functional

$$g \mapsto \min\{\lambda_0^D(g), \lambda_0^N(g)\}|\Sigma|_g, \tag{6.2}$$

where $\lambda_0^N(g)$ is the first eigenvalue of the Laplacian with Neumann boundary condition. It would be interesting to connect (6.2) with the functional Θ_r as $r \to \pi/2$.

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