DUALIZABLE CATEGORIES AND LOCALIZING MOTIVES

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Contents

1. Talk 1

Consider a qcqs scheme X. There are two categories associated with X: the category $\mathrm{Perf}(X) = D_{\mathrm{qc}}(X)^{\omega}$ of perfect complexes and $D_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$.

There are two invariants we might consider:

$$K(X) = K(\mathrm{Perf}(X))$$
$$\mathrm{HH}(X) = \mathrm{HH}(\mathrm{Perf}(X)),$$
$$K(D_{\mathrm{qc}}(X)) = 0.$$

The idea is to understand $K(X)$ in terms of $D_{\mathrm{qc}}(X)$.

1.1. Compactly assembled categories.

1.1. Compactly assembled categories. Compactly assembled categories were first developed by Lurie, Joyal, Johnstone etc.

If $\mathcal{C} = \mathrm{Ind}(\mathcal{A})$, then for all $x \in \mathcal{C}$ we have an ind-system $(x_i)_i$ with $x_i \in \mathcal{C}^{\omega}$ such that $x = \varinjlim_i x_i$. Observe that the assignment $x \mapsto \text{“}\varinjlim\text{”}_i\, x_i$ is gives a well-defined functor

$$\widehat{Y} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C}),$$

which is left adjoint to $\mathrm{colim} \colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

Definition 1.1. Let $\mathcal{C}$ be an accessible ∞-category with filtered colimits. Then $\mathcal{C}$ is *compactly assembled* if there exists $\widehat{Y} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ which is left adjoint to colim.

Example 1.2. (1) $\mathrm{Ind}(\mathcal{A})$ is compactly assembled.
(2) $(\mathbb{R} \cup \{+\infty\}, \leq)$ is compactly assembled with $\widehat{Y}(a) = \text{“}\varinjlim\text{”}_{b<a}\, b$.

(3) Let X be a locally compact Hausdorff space. Then $\mathrm{Open}(X)$ is compactly assembled with $\widehat{Y}(U) = \text{“}\varinjlim\text{”}_{V \Subset U} V$. Also, $\mathcal{K}(X)^{\mathrm{op}}$ is compactly assembled with $\widehat{Y}(Z) = \text{“}\varinjlim\text{”}_{Z' \Supset Z} Z'$.

(4) **Exercise:** Let $\mathrm{Seminorm}_1$ be the category of $\mathbb{R}$-vector spaces with a seminorm $\|\cdot\|$ and contractible maps (i.e., $\|f(x)\| \leq \|x\|$). If $\dim V < \infty$ and $(V, \|\cdot\|)$ is normed, then

$$\widehat{Y}(V, \|\cdot\|) = \text{“}\varinjlim_{c>1}\text{”}(V, c\|\cdot\|)$$

and $\widehat{Y}(\mathbb{R}, 0) = \text{“}\varinjlim\text{”}_{\varepsilon>0}(\mathbb{R}, \varepsilon|\cdot|)$.

(5) $\mathrm{Shv}(X, \underline{\mathcal{C}})$ is compactly assembled, where X is a locally compact Hausdorff space and $\underline{\mathcal{C}}$ is a presheaf of dualizable categories.

(6) The categories $\mathrm{Nuc}(R_{\widehat{I}})$ and $\widetilde{\mathrm{Nuc}}(R_{\widehat{I}})$ of nuclear modules are compactly assembled, and $\mathrm{Nuc}(R_{\widehat{I}})^{\omega} = \mathrm{Perf}(R_{\widehat{I}})$.

(7) Let R be an associative ring and $J \subseteq R$ an ideal which is flat as a right module and satisfying $J^2 = J$. Put

$$\mathrm{Mod}_a(R) = \mathrm{Mod}(R)/\mathrm{Mod}(R/J)$$

Then $D(\mathrm{Mod}_a(R))$ is compactly assembled.

We introduce the following notation:

- $\mathrm{Cat}^{\mathrm{idem}}$ is the category of small idempotent complete categories,
- $\mathrm{Cat}^{\mathrm{perf}}$ is the category of small idempotent complete stable categories, and
- CompAss is the category of compactly assembled categories, where the 1-morphisms are the strongly continuous functors, i.e., $F\colon \mathcal{C} \to \mathcal{D}$ such that F commutes with filtered colimits and the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \widehat{Y}}\downarrow & & \downarrow{\scriptstyle \widehat{Y}} \\ \mathrm{Ind}(\mathcal{C}) & \xrightarrow[\mathrm{Ind}(F)]{} & \mathrm{Ind}(\mathcal{D}). \end{array}$$

- Denote $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \subseteq \mathrm{CompAss}$ the subcaetgory of compactly assembled stable categories with strongly continuous exact functors. Note that both $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ and CompAss are cocomplete.

Proposition 1.3 (Lurie)**.** *(1)* $\mathrm{Cat}^{\mathrm{idem}}$ *is generated under colimits by* $[1]$*, which is compact.*
(2) $\mathrm{Cat}^{\mathrm{perf}}$ *is generated under colimits by* Sp^{ω}*, which is compact.*

Sketch. Let $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Fun}([1], \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}([1], \mathcal{D})$. Then F is an equivalence. $\square$

`Efimov:Urysohn`

Theorem 1.4 (Urysohn's lemma)**.** *(1)* CompAss *is generated under colimits by* $\mathbb{R} \cup \{\infty\}$*, which is* ω_1*-compact.*
(2) $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ *is generated under colimits by* $\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}; \mathrm{Sp})$*.*
In particular, CompAss *and* $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ *are* ω_1*-presentable.*[1]

Remark 1.5. The usual Urysohn's lemma for compact Hausdorff spaces says that $\mathrm{CompHaus}^{\mathrm{op}}$ is generated under colimits by $[0, 1]$, which is ω_1-compact.

[1]The last statement ist due to Ramzi.

Fact 1.6. We have an equivalence $\mathrm{CompAss} \xrightarrow{\sim} \mathrm{Cat}^{\mathrm{dual}}$, where $\mathrm{Cat}^{\mathrm{dual}}$ is the category of dualizable objects in Pr^L whose 1-morphisms are those functors $F\colon \mathcal{C} \to \mathcal{D}$ for which the right adjoint F^R is colimit-preserving. The equivalence takes

$$\mathrm{Ind}(\mathcal{A}) \leftrightarrow \mathrm{PSh}(\mathcal{A}; \mathrm{Ani}),$$
$$\mathcal{C} \mapsto \left\{ F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ani} \,\middle|\, \begin{array}{l} F \text{ commutes with} \\ \text{cofiltered limits} \end{array} \right\},$$
$$\left\{ G\colon \mathcal{D}^{\vee} \to \mathrm{Ani} \,\middle|\, \begin{array}{l} G \text{ is colimit preserving} \\ \text{and left exact} \end{array} \right\} \leftarrow\!\shortmid \mathcal{D}.$$

Why is CompAss generated by $\mathbb{R} \cup \{\infty\}$? It suffices to show that if $F\colon \mathcal{C} \to \mathcal{D}$ is strongly continuous such that

$$\mathrm{Fun}^{\mathrm{str.cont.}}(\mathbb{R} \cup \{\infty\}, \mathcal{C})^{\simeq} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{str.cont.}}(\mathbb{R} \cup \{\infty\}, \mathcal{D})^{\simeq},$$

then F is an equivalence.

Proposition 1.7. *Any compactly assembled category is ω_1-accessible.*

Sketch. We have $\widehat{Y}(x) = \text{“}\varinjlim_{i \in I}\text{”}\, x_i$, where I is a directed poset. Then $x \simeq \varinjlim_{f\colon \mathbb{N} \to I} \varinjlim_n x_{f(n)}$, where $x_{f(n)}$ is ω_1-compact. □

Definition 1.8. In a compactly assembled category, a map $f\colon x \to y$ is compact if it factors as

$$\begin{array}{ccc} Y(x) & \xrightarrow{Y(f)} & Y(y) \\ & \searrow & \uparrow \\ & & \widehat{Y}(y). \end{array}$$

Lemma 1.9. *If $\mathcal{C}$ is compactly assembled, and $f\colon x \to y$ in $\mathcal{C}$ is compact, then $f = g \circ h$ such that g, h are compact.*

Sketch. Write $\widehat{Y}(y) = \text{“}\varinjlim\text{”}\text{“}\varinjlim\text{”}_i\, y_i$. Then $\widehat{Y}(y) = \varinjlim_i \widehat{Y}(y_i)$, so we have a factorization

$$\begin{array}{ccccc} x & & \xrightarrow{\quad f \quad} & & y \\ & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle h} & \\ & & y_i, & & \end{array}$$

where g and h are compact. □

Lemma 1.10. *The functor*

$$\mathrm{Fun}^{\mathrm{str.cont.}}(\mathbb{R} \cup \{\infty\}, \mathcal{C}) \to \mathcal{C}^{\omega_1},$$
$$G \mapsto G(\infty)$$

is essentially surjective.

Proof. Let $x \in \mathcal{C}^{\omega_1}$. Then $\widehat{Y}(x) = $ "$\varinjlim$"$(x_0 \to x_1 \to \cdots)$ such that each $x_n \to x_{n+1}$ is compact. Define $G_0 \colon \mathbb{Z} \cup \{\infty\} \to \mathcal{C}$ by

$$G(m) = \begin{cases} x, & \text{if } h = \infty, \\ x_m, & \text{if } m \geq 0, \\ 0, & \text{if } m < 0. \end{cases}$$

Inductively, define compatible $G_n \colon \frac{1}{2^n}\mathbb{Z} \cup \{\infty\} \to \mathcal{C}$ such that all transition maps are compact. We obtain $G \colon \mathbb{Z}[1/2] \cup \{\infty\} \to \mathcal{C}$ and put

$$G' \colon \mathbb{Z}[1/2] \cup \{\infty\} \to \mathcal{C}, \qquad G'(r) = \varinjlim_{b<r} G(b).$$

Take a left Kan extension of G' to $\mathbb{R} \cup \{\infty\}$ to get a strongly continuous functor $H \colon \mathbb{R} \cup \{\infty\} \to \mathcal{C}$ such that $H(\infty) = x$. □

For F as above, we already know that $F^{\omega_1} \colon \mathcal{C}^{\omega_1} \to \mathcal{D}^{\omega_1}$ is essentially surjective. In order to finish the proof of Theorem 1.4.(1), we need to show that it is fully faithful. Consider the pullback diagram

$$\begin{array}{ccc} \varprojlim_{a>0} \operatorname{Map}_{\mathcal{C}}(G(0), H(a)) & \longrightarrow & \operatorname{Fun}^{\text{str.cont.}}(\mathbb{R} \cup \{\infty\}, \mathcal{C})^{\simeq} \\ \downarrow & & \downarrow \\ * & \xrightarrow[(G,H)]{} & \operatorname{Fun}^{\text{str.cont.}}(\mathbb{R}_{\leq 0}, \mathcal{C})^{\simeq} \times \operatorname{Fun}^{\text{str.cont.}}(\mathbb{R}_{>0} \cup \{\infty\}, \mathcal{C})^{\simeq}. \end{array}$$

It follows that $\varprojlim_{a>0} \operatorname{Map}_{\mathcal{C}}(G(0), H(a)) \xrightarrow{\sim} \varprojlim_{a>0} \operatorname{Map}_{\mathcal{D}}(F(G(0)), F(H(a)))$.

Given strongly continuous functors $G, H \colon \mathbb{R} \cup \{\infty\} \to \mathcal{C}$, we get

$$\begin{aligned} \operatorname{Map}_{\mathcal{C}}(G(\infty), H(\infty)) &= \varprojlim_{a<\infty} \varinjlim_{b<\infty} \operatorname{Map}_{\mathcal{C}}(G(a), H(b)) \\ &= \varprojlim_{a<\infty} \varinjlim_{b<\infty} \varprojlim_{c>b} \operatorname{Map}_{\mathcal{C}}(G(a), H(c)) \\ &\xrightarrow{\sim} \varprojlim_{a<\infty} \varinjlim_{b<\infty} \varprojlim_{c>b} \operatorname{Map}_{\mathcal{D}}(F(G(a)), F(H(c))) \\ &\xrightarrow{\sim} \cdots \xrightarrow{\sim} \operatorname{Map}_{\mathcal{D}}(F(G(\infty)), F(H(\infty))). \end{aligned}$$

Hence, F^{ω_1} is fully faithful.

The functor $\iota \colon \operatorname{Cat}_{\text{st}}^{\text{dual}} \to \text{CompAss}$ is conservative and commutes with filtered colimits. It has a left adjoint

$$\begin{aligned} \operatorname{Stab}^{\text{cont}} \colon \text{CompAss} &\to \operatorname{Cat}_{\text{st}}^{\text{dual}}, \\ \mathcal{C} &\mapsto \{F \colon \mathcal{C}^{\text{op}} \to \operatorname{Sp} \mid F \text{ commutes with filtered colmilits}\}. \end{aligned}$$

It follows that $\operatorname{Stab}^{\text{cont}}(\mathbb{R} \cup \{\infty\}) = \operatorname{Shv}_{\mathbb{R} \times \mathbb{R}_{\geq 0}}(\mathbb{R}, \operatorname{Sp})$ generates $\operatorname{Cat}_{\text{st}}^{\text{dual}}$

Proposition 1.11. *For a sheaf $\mathcal{F} \in \operatorname{Shv}(\mathbb{R}, \operatorname{Sp})$, the following are equivalent:*

(1) The singular support (microsupport) $\operatorname{SS}(\mathcal{F})$ is a subset of $\mathbb{R} \times \mathbb{R}_{\geq 0}$.
(2) For all $a < b$, $\mathcal{F}((-\infty, b)) \xrightarrow{\sim} \mathcal{F}((a, b))$.

Corollary 1.12. *We have equivalences*

$$\begin{aligned}\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R},\mathrm{Sp}) &\simeq \left\{ F\colon (\mathbb{R}_{\leq})^{\mathrm{op}} \to \mathrm{Sp} \;\middle|\; \forall a\in\mathbb{R}, F(a) \xrightarrow{\sim} \varprojlim_{b<a} F(b) \right\}\\ &\simeq \mathrm{Stab}^{\mathrm{cont}}(\mathbb{R}\cup\{\infty\}).\end{aligned}$$

2. Talk 2

Definition 2.1. Let $\mathcal{C}$ be a presentable stable category. Then $\mathcal{C}$ is called *flat* if $\mathcal{C}\otimes -\colon \mathrm{Pr}^L_{\mathrm{st}} \to \mathrm{Pr}^L_{\mathrm{st}}$ preserves fully faithful functors.

Question: Is every $\mathcal{C}\in\mathrm{Pr}^L_{\mathrm{st}}$ flat?

Efimov:flat

Theorem 2.2. *$\mathcal{C}$ is flat if and only if $\mathcal{C}$ is dualizable.*

Proof. If $\mathcal{C}$ is dualizable, then $\mathcal{C}$ is obviously flat, since $\mathcal{C}\otimes - = \mathrm{Fun}^L(\mathcal{C}^\vee, -)$. The other direction will be proved below. □

Notation. Let $\mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}}$ be the $(\infty,2)$-category of presentable stable categories and accessible exact functors.

Proposition 2.3. *(a) For any $\mathcal{C}\in\mathrm{Pr}^L_{\mathrm{st}}$, there exists a natural oplax 2-functor $\mathcal{C}\otimes -\colon \mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}} \to \mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}}$ (meaning that there are 2-morphisms $\mathcal{C}\otimes(F\circ G)\to(\mathcal{C}\otimes F)\circ(\mathcal{C}\otimes G)$ which need not be invertible) which extends the usual 2-functor $\mathcal{C}\otimes -\colon \mathrm{Pr}^L_{\mathrm{st}}\to\mathrm{Pr}^L_{\mathrm{st}}$.*
(b) If $\mathcal{C}$ is flat, then $\mathcal{C}\otimes -$ is an honest 2-functor.

Assume that $\mathcal{C}$ is κ-presentable and $F\colon \mathcal{D}\to\mathcal{E}$ is an accessible functor. Then

$$\mathcal{C}\otimes F\colon \mathcal{C}\otimes\mathcal{D} \simeq \mathrm{Fun}^{\kappa\text{-lex}}((\mathcal{C}^\kappa)^{\mathrm{op}},\mathcal{D}) \xrightarrow{F\circ -} \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}^{\kappa\text{-lex}}((\mathcal{C}^\kappa)^{\mathrm{op}},\mathcal{E}) \cong \mathcal{C}\otimes\mathcal{E},$$

where the second map is given by the left adjoint to the inclusion.

Observation: For presentable stable $\mathcal{D}$, $\mathcal{E}$, we have an equivalence of categories

$$\begin{aligned}\mathrm{Fun}^{\mathrm{acc}}(\mathcal{D},\mathcal{E}) \simeq \mathrm{Corr}(\mathcal{D},\mathcal{E}) &= \left\{ (\mathcal{T}, i_0, i_1) \;\middle|\; \begin{array}{l} i_0\colon \mathcal{D}\to\mathcal{T} \text{ and } i_1\colon\mathcal{E}\to\mathcal{T} \text{ are fully faithful} \\ \text{continuous, and } \mathcal{T} = \langle i_1(\mathcal{E}), i_0(\mathcal{D})\rangle \end{array} \right\}\\ i_1^R\circ i_0 &\mapsfrom (\mathcal{T}, i_0, i_1),\\ F &\mapsto \mathcal{E}\oplus_F\mathcal{D} = \{(x\in\mathcal{E}, y\in\mathcal{D}, x\to F(y))\} \quad \text{the oplax limit.}\end{aligned}$$

We obtain a functor

$$\begin{aligned}\mathcal{C}\otimes -\colon\ &\mathrm{Corr}(\mathcal{D},\mathcal{E}) \to \mathrm{Corr}(\mathcal{C}\otimes\mathcal{D}, \mathcal{C}\otimes\mathcal{E}),\\ &(\mathcal{T}, i_0, i_1) \mapsto (\mathcal{C}\otimes\mathcal{T}, \mathcal{C}\otimes i_0, \mathcal{C}\otimes i_1).\end{aligned}$$

We want to understand the composition

$$\begin{aligned}&\mathrm{Corr}(\mathcal{D}_1,\mathcal{D}_2)\times\mathrm{Corr}(\mathcal{D}_0,\mathcal{D}_1)\to\mathrm{Corr}(\mathcal{D}_0,\mathcal{D}_2),\\ &(\mathcal{T}_{12},\mathcal{T}_{01}) \mapsto \mathcal{T}_{02}\subseteq\mathcal{T}_{012} = \mathcal{T}_{01}\sqcup_{\mathcal{D}_1}\mathcal{T}_{12},\end{aligned}$$

where $\mathcal{T}_{02}$ is generated by the images of $\mathcal{D}_0$ and $\mathcal{D}_2$.

Note that

$$\begin{array}{ccc} (\mathcal{C}\otimes\mathcal{T}_{01})\coprod_{\mathcal{C}\otimes\mathcal{D}_1}(\mathcal{C}\otimes\mathcal{T}_{12}) & \xrightarrow{\sim} & \mathcal{C}\otimes(\mathcal{T}_{01}\sqcup_{\mathcal{D}_1}\mathcal{T}_{12}) \\ & & \uparrow \\ \mathcal{C}\otimes\mathcal{T}_{02} & \longrightarrow & (\mathcal{C}\otimes\mathcal{T}_{01})\circ(\mathcal{C}\otimes\mathcal{T}_{12}). \end{array}$$

The bottom map need not be an equivalence in general, because $\mathcal{C}\otimes\mathcal{T}_{02}\to\mathcal{C}\otimes\mathcal{T}_{012}$ is not fully faithful in general. But if $\mathcal{C}$ is flat, then it is fully faithful, hence we obtain the functor $\mathcal{C}\otimes-\colon \mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}}\to\mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}}$.

We have a fibration $\mathrm{Corr}\to\Delta$ given by

$$\mathrm{Corr}_n=\coprod_{(\mathcal{D}_0,\dots,\mathcal{D}_n)}\mathrm{Corr}(\mathcal{D}_0,\mathcal{D}_1,\dots,\mathcal{D}_n),$$

$$\mathrm{Corr}(\mathcal{D}_0,\mathcal{D}_1,\dots,\mathcal{D}_n)=\left\{(\mathcal{T},i_0,\dots,i_n)\;\middle|\;\begin{array}{l} i_k\colon\mathcal{D}_k\to\mathcal{T}\text{ are fully faithful, continuous}\\ \mathcal{T}=\langle i_n(\mathcal{D}_n),\dots,i_0(\mathcal{D}_0)\rangle\\ \text{If }\mathcal{T}_{k,k+1}\subset\mathcal{T}\text{ is generated by }\mathcal{D}_k,\mathcal{D}_{k+1}\text{, then}\\ \mathcal{T}\simeq\mathcal{T}_{01}\sqcup_{\mathcal{D}_1}\mathcal{T}_{12}\sqcup_{\mathcal{D}_2}\cdots\sqcup_{\mathcal{D}_n}\mathcal{T}_{n-1,n}\end{array}\right\}.$$

We have a functor $\mathcal{C}\otimes-\colon\ \mathrm{Corr}(\mathcal{D}_0,\mathcal{D}_1,\dots,\mathcal{D}_n)\to\mathrm{Corr}(\mathcal{C}\otimes\mathcal{D}_0,\dots,\mathcal{C}\otimes\mathcal{D}_n)$ and hence we get a map

$$\mathcal{C}\otimes-\colon\ \mathrm{Corr}\to\mathrm{Corr}$$

of fibrations over Δ.

Proof of Theorem 2.2. We now prove that if $\mathcal{C}$ is flat, then $\mathcal{C}$ is dualizable. Recall that $\mathcal{C}$ is dualizable if and only if (AB6) holds in $\mathcal{C}$: that is, for all directed posets J_i, $i\in I$, and functors $J_i\to\mathcal{C}$, $j_i\mapsto x_{j_i}$, then the map

$$\varinjlim_{(j_i)_i\in\prod_i J_i}\prod_i x_{j_i}\xrightarrow{\sim}\prod_i\varinjlim_{j_i}x_{j_i}.$$

We have a commutative square

{Efimov:AB6}
$$\begin{array}{ccc} \prod_i\mathrm{Fun}(J_i,\mathcal{C}) & \xrightarrow{T=\prod_i\mathrm{colim}} & \prod_i\mathcal{C} \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle V=\mathrm{diag}} \\ \mathrm{Fun}(\prod_i J_i,\mathcal{C}) & \xrightarrow[W=\mathrm{colim}]{} & \mathcal{C} \end{array}\tag{1}$$

where U is the left Kan extension. Then (AB6) means that the dual Beck–Chevalley condition holds, i.e., $W\circ U^R\xrightarrow{\sim}V^R\circ T$.

Observe that (1) $\simeq\mathcal{C}\otimes$(same square for Sp). Using that $\mathcal{C}\otimes-$ is a 2-functor on $\mathrm{Pr}^{\mathrm{acc}}_{\mathrm{st}}$ and using (AB6) for Sp, then it follows that (AB6) holds for $\mathcal{C}$. □

Suppose we have a short exact sequence

$$0\to\mathrm{Ind}(\mathcal{A})\xrightarrow{F}\mathcal{C}\xrightarrow{G}\mathrm{Ind}(\mathcal{B})\to 0$$

with F and G strongly continuous.

Question: Is it true that $\mathcal{C}$ is dualizable?

Consider the sequence of adjunctions $G\dashv G^R\dashv G^{RR}$. We have

$$\mathcal{C}=\langle G^R(\mathrm{Ind}(\mathcal{B})),F(\mathrm{Ind}(\mathcal{A}))\rangle\simeq\mathrm{Ind}(B)\oplus_\Phi\mathrm{Ind}(\mathcal{A}),$$

where $\Phi := G^{RR} \circ F$ is the gluing datum, imposing that $\mathrm{Hom}_{\mathcal{C}}(x,y) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{B})}(x, \Phi(y))$ for any $x \in G^R(\mathrm{Ind}(\mathcal{B}))$ and $y \in F(\mathrm{Ind}(\mathcal{A}))$.

Observe (almost tautologically) that

$$\mathrm{Fun}^{\mathrm{acc}}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{B})) = \mathrm{Fun}(\mathcal{B}, \mathrm{Pro}(\mathrm{Ind}(\mathcal{A})))^{\mathrm{op}}$$
$$\Phi \mapsto \Psi.$$

Efimov:Tate

Proposition 2.4. *With the above notation, the following are equivalent:*

(1) $\mathcal{C}$ is dualizable.
(2) $\mathcal{C}$ is compactly generated.
(3) $\mathrm{Im}(\Psi) \subseteq \mathrm{Tate}(\mathcal{A})$, the idempotent-complete stable subcategory of $\mathrm{Pro}(\mathrm{Ind}(\mathcal{A}))$ generated by $\mathrm{Pro}(\mathcal{A})$ and $\mathrm{Ind}(\mathcal{A})$.

Corollary 2.5. *We have*

$$\mathrm{Ext}^1(\mathcal{B}, \mathcal{A}) = \{0 \to \mathcal{A} \to \mathcal{D} \to \mathcal{B} \to 0\}$$
$$\simeq \mathrm{Fun}(\mathcal{B}, \mathrm{Tate}(\mathcal{A}))^{\simeq}.$$

Proof of Proposition 2.4. We have $\mathcal{A} \subseteq \mathcal{C}^\omega$. So if $\mathcal{C}$ is dualizable, then

$$(\mathcal{C}/\,\mathrm{Ind}(\mathcal{A}))^\omega = (\mathcal{C}^\omega/\mathcal{A})^{\mathrm{idem}},$$

hence for all $x \in (\mathcal{C}/\,\mathrm{Ind}(\mathcal{A}))^\omega$, there exists $y \in \mathcal{C}^\omega$ such that $y \mapsto x \oplus x[1]$. We deduce (1) $\iff$ (2).

For the equivalence (2) $\iff$ (3) we use that for $x \in \mathcal{B}$ the following are equivalent:

(i) There exists a lift $y \in \mathcal{C}^\omega$ of $x \oplus x[1]$.
(ii) There is a fiber/cofiber sequence $U \to \Psi(x) \oplus \Psi(x)[1] \to V$ such that $U \in \mathrm{Pro}(\mathcal{A})$ and $V \in \mathrm{Ind}(\mathcal{A})$.

Equivalently, $\Psi(x) \oplus \Psi(x)[1] \in \mathrm{Tate}_{\mathrm{el}}(\mathcal{A})$. By Thomason–Trobaugh's theorem, this is equivalent to $\Psi(x) \in \mathrm{Tate}(\mathcal{A})$.

□

Example 2.6. Consider the sequence

$$0 \to \mathrm{Perf}_{p\text{-tors}}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Z}[p^{-1}]) \to 0.$$

The corresponding $\mathbb{Z}$-linear functor $\mathrm{Perf}(\mathbb{Z}[p^{-1}]) \to \mathrm{Tate}(\mathrm{Perf}_{p\text{-tors}}(\mathbb{Z}))$ sends $\mathbb{Z}[p^{-1}] \mapsto \mathbb{Q}_p$.

Proposition 2.7. *Let k be a field and let $\mathcal{C} = \{(V, W \in D(k); \varphi\colon \bigoplus_{\mathbb{N}} V \to \bigoplus_{\mathbb{N}} W)\}$. A direct computation shows*

$$\Psi(k) = \text{“}\varprojlim_{f\colon \mathbb{N}\to\mathbb{N}}\text{”}\, X_f, \qquad \text{where } X_f := \bigoplus_{n\in\mathbb{N}} k^{f(n)},$$

which is not a Tate object. We need to show that $(\overline{X}_f)_f$ in $\mathrm{Pro}(\mathrm{Calk}(k))$ is not pro-constant. This uses that for $f \le g$ the transition map $X_g \to X_f$ is a split epimorphism. If $(\overline{X}_f)_f$ were pro-constant, then $(\overline{X}_f)_f$ would be eventually constant, which is false since $\mathrm{fib}(X_{f+1} \to X_f) = \bigoplus_{\mathbb{N}} k$.

Let R be a commutative noetherian ring. Then Neeman showed

$$\left\{\begin{matrix}\text{localizing subcategories}\\ \text{of } D(R)\end{matrix}\right\} \cong \{\text{subsets of } \mathrm{Spec}(R)\},$$
$$D_S(R) = \langle \kappa(p) \mid p \in S\rangle \leftarrow\!\shortmid S$$

Neeman shows that $D_S(R) \to D(R)$ is strongly continuous if and only if S is closed under specialization.

Theorem 2.8. *The following are equivalent for $S \subseteq \mathrm{Spec}(R)$.*

(i) $D_S(R)$ *is dualizable.*
(ii) S *is convex, i.e., if* $x \rightsquigarrow y \rightsquigarrow z$ *with* $x, z \in S$*, then* $y \in S$*.*
(iii) $D_S(R)$ *is compactly generated.*

Example 2.9. Let k be a field, take $\mathcal{C} = \langle M \mid M[x^{-1}]/yM[x^{-1}] = 0\rangle \subseteq D(k[x,y])$. Then $\mathcal{C}$ is not dualizable.

Intuitively, $\mathcal{C} = \mathrm{QCoh}(\mathbb{A}^2/\{x = 0, y \neq 0\})$.

3. Talk 3: Localizing invariants of categories of sheaves

Recall the category

$$\begin{aligned}\mathrm{Shv}_{\geq 0}(\mathbb{R};\mathrm{Sp}) &\simeq \mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R};\mathrm{Sp})\\ &\simeq \mathrm{Stab}^{\mathrm{cont}}(\mathbb{R}\cup\{\infty\}) = \left\{F\colon \mathbb{R}_{\leq}^{\mathrm{op}} \to \mathrm{Sp} \,\middle|\, \forall a \in \mathbb{R}, F(a) = \varinjlim_{b<a} F(b)\right\}.\end{aligned}$$

Proposition 3.1. *Take any accessible localizing invariant* $\Phi\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$*, where* $\mathcal{E}$ *is a stable accessible category. Then*

$$\Phi^{\mathrm{cont}}(\mathrm{Shv}_{\geq 0}(\mathbb{R};\mathrm{Sp})) = 0.$$

Applications 3.2. (a) $K\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ commutes with small products.
(b) Computation of $F^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C}))$, where X is a finite CW complex and $\mathcal{C}$ is a dualizable category.

Step 1: K_0 commutes with small products.

Proposition 3.3 (Heller's criterion)**.** *If* $\mathcal{T}$ *is a small triangulated category, and* $x, y \in \mathcal{T}$*, then the following are equivalent:*

(i) $[x] = [y]$ *in* $K_0(\mathcal{T})$*.*
(ii) There exist $z, u, v \in \mathcal{T}$ *and distinguished triangles*

$$\begin{aligned}u \to x \oplus z \to v\\ u \to y \oplus z \to v.\end{aligned}$$

Corollary 3.4. $K_0(\prod_i \mathcal{T}_i) \xrightarrow{\sim} \prod_i K_0(\mathcal{T}_i)$.

Step 2: We have a short exact sequence

$$0 \to \mathrm{Shv}_{\geq 0}(\mathbb{R};\mathrm{Sp}) \to \mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathrm{op}},\mathrm{Sp}) \xrightarrow{F} \prod_{\mathbb{Q}} \mathrm{Sp} \to 0,$$

where F is given by

$$F(G)_a = \mathrm{Cone}\big(\varinjlim_{b>a} G(b) \to G(a)\big).$$

Note that F^R is fully faithful, i.e., $F \circ F^R = \mathrm{id}$. It follows that

$$\mathrm{Shv}_{\geq 0}(\mathbb{R};\mathrm{Sp}) \simeq F^R\Big(\prod_{\mathbb{Q}} \mathrm{Sp}\Big)^{\perp} \simeq {}^{\perp}F^R\Big(\prod_{\mathbb{Q}} \mathrm{Sp}\Big) = \mathrm{Ker}(F).$$

Note also that the categories $\mathcal{A} := \mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathrm{op}},\mathrm{Sp})$ and $\mathcal{B} := \prod_{\mathbb{Q}} \mathrm{Sp}$ are compactly generated.

We need to show that $F^\omega\colon \mathcal{A}^\omega \to \mathcal{B}^\omega$ is a K-equivalence, i.e., there exists $G\colon \mathcal{B}^\omega \to \mathcal{A}^\omega$ such that $[F^\omega \circ G] = [\mathrm{id}]$ in $K_0(\mathrm{Fun}(\mathcal{B}^\omega, \mathcal{B}^\omega))$ and $[G \circ F^\omega] = [\mathrm{id}]$ in $K_0(\mathrm{Fun}(\mathcal{A}^\omega, \mathcal{A}^\omega))$.

Here, we take $G\colon \mathcal{B}^\omega = \bigoplus_{\mathbb{Q}} \mathrm{Sp}^\omega \to \mathcal{A}^\omega$ corresponding to $(h_a)_{a\in\mathbb{Q}}$, where

$$h_a(b) = \begin{cases} \mathbb{S}, & \text{if } b \leq a, \\ 0, & \text{if } b > a. \end{cases}$$

With this definition, $F^\omega \circ G = \mathrm{id}$. It remains to show $[G \circ F^\omega] = [\mathrm{id}]$ in $K_0(\mathrm{Fun}(\mathcal{A}^\omega, \mathcal{A}^\omega))$.

Proposition 3.5. $K_0(\mathrm{Fun}(\mathcal{A}^\omega, \mathcal{A}^\omega)) = \mathrm{End}(\bigoplus_{\mathbb{Q}} \mathbb{Z})$.

Step 3:

Proposition 3.6. *Let $\mathcal{C}$ be a small idempotent-complete stable category with semi-orthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_0, \mathcal{A}_1, \dots \rangle,$$

meaning that $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ *for $i > j$, and the $\mathcal{A}_i$ generate $\mathcal{C}$. Write $\mathcal{B}_n \coloneqq \langle \mathcal{A}_0, \dots, \mathcal{A}_n \rangle \subseteq \mathcal{C}$. Then we have functors $\mathcal{B}_{n+1} \to \mathcal{B}_n$ which are right adjoint to the inclusion. Consider the composite*

$$\varprojlim_n \mathcal{B}_n \to \mathcal{B}_k \xrightarrow{p_k} \mathcal{A}_k,$$

where p_k is the right adjoint to the inclusion.

Then the functor

$$\varprojlim_n \mathcal{B}_n \to \prod_{n\in\mathbb{N}} \mathcal{A}_n$$

is a K-equivalence.

Sketch: Write $\mathcal{B} \coloneqq \varprojlim_n \mathcal{B}_n$ and write $\pi_n\colon \mathcal{B} \to \mathcal{A}_n$. Consider the inclusions $\iota_n\colon \mathcal{A}_n \to \mathcal{B}$, given by compatible functors $\mathcal{A}_n \to \mathcal{B}_n \hookrightarrow \mathcal{B}_k$. Write $\iota\colon \prod_n \mathcal{A}_n \to \mathcal{B}$.

Then $\pi \circ \iota = \mathrm{id}$ of $\prod_n \mathcal{A}_n$.

Claim. $[\iota \circ \pi] = [\mathrm{id}]$ in $K_0(\mathrm{Fun}(\mathcal{B}, \mathcal{B}))$.

Consider the functor

$$\psi_n\colon \mathcal{B} \to \mathcal{B}_{n-1}^{\perp} \to \mathcal{B},$$

where each $\mathcal{B}_n \hookrightarrow \mathcal{B}$ (so that the right orthogonal makes sense) and we put $\mathcal{B}_{-1} = 0$. Now observe: consider the exact sequence

$$\bigoplus_{n\geq 1} \psi_n \to \bigoplus_{n\geq 0} \psi_n \twoheadrightarrow \iota \circ \pi,$$

where the first map is induced by the maps $\psi_{n+1} \to \psi_n$. We compute

$$[\mathrm{id}] = [\psi_0] = \Big[\bigoplus_{n\geq 0} \psi_n\Big] - \Big[\bigoplus_{n\geq 1} \psi_n\Big] = [\iota \circ \pi].$$

□

Corollary 3.7. *Let $\mathcal{B}_0 \leftarrow \mathcal{B}_1 \leftarrow \mathcal{B}_2 \leftarrow \cdots$ be an inverse system in* $\mathrm{Cat}^{\mathrm{perf}}$ *such that $\mathcal{B}_{n+1} \to \mathcal{B}_n$ has a fully faithful right adjoint. Then*

$$K_0(\varprojlim_n \mathcal{B}_n) = \varprojlim_n K_0(\mathcal{B}_n).$$

Proof. Apply the above to $\varinjlim_n \mathcal{B}_n$ with respect to the right adjoints. Denote $\mathcal{A}_n = \mathrm{Ker}(\mathcal{B}_{n+1} \to \mathcal{B})$. Then

$$K_0\big(\varprojlim_n \mathcal{B}_n\big) \simeq K_0\Big(\prod_n \mathcal{A}_n\Big) = \prod_n K_0(\mathcal{A}_n) = \varprojlim_n K_0(\mathcal{B}_n).$$

□

Corollary 3.8. $K_0(\mathrm{Fun}(\mathcal{A}^\omega, \mathcal{A}^\omega)) = \mathrm{End}\Big(\bigoplus_{\mathbb{Q}} \mathbb{Z}\Big)$.

Proof. Choose a bijection $\mathbb{N} \xrightarrow{\sim} \mathbb{Q}$, $n \mapsto a_n$. Let $\mathcal{C}_n \subseteq \mathcal{A}^\omega$ be a stable subcategory generated by the representable presheaves $h_{a_0}, \dots, h_{a_n}$. Then

$$\begin{aligned} K_0(\mathrm{Fun}(\mathcal{A}^\omega, \mathcal{A}^\omega)) &= \varprojlim_n K_0(\mathrm{Fun}(\mathcal{C}_n, \mathcal{A}^\omega)) \simeq \varprojlim_n K_0(\mathrm{Fun}([n], \mathcal{A}^\omega)) \\ &= \varprojlim_n \prod_{i=0}^{n} K_0(\mathcal{A}^\omega) = \mathrm{End}\Big(\bigoplus_{\mathbb{Q}} \mathbb{Z}\Big), \end{aligned}$$

since we know that

$$K_0(\mathcal{A}^\omega) = \bigoplus_{\mathbb{Q}} K_0(\mathrm{Sp}^\omega) = \bigoplus_{\mathbb{Q}} \mathbb{Z}.$$

□

Recall the functor $G\colon \mathcal{B}^\omega \to \mathcal{A}^\omega$ from the above. Then $(G \circ F^\omega)(h_a) = h_a$, and this finally implies $[G \circ F^\omega] = [\mathrm{id}]$.

Theorem 3.9. *(1) Let $F, G\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be accessible localizing invariants, let $\varphi\colon F \to G$ be map, and suppose that $\mathcal{E}$ has a non-degenerate t-structure. Suppose moreover that the induced map $\pi_0\varphi\colon \pi_0 F \to \pi_0 G$ on connected components is an isomorphism.*
Then $\varphi\colon F \xrightarrow{\sim} G$ is an isomorphism.
(2) $K\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ commutes with small products.

Proof. For the proof that (1) $\implies$ (2) we need to show: for any set I, the map

$$K\Big(\prod_I \mathcal{C}\Big) \xrightarrow{\varphi} \prod_I K(\mathcal{C})$$

is an isomorphism. Observe that the source and target are localizing invariants in $\mathcal{C}$ and $\pi_0\varphi$ is an isomorphism.

It remains to prove (1). The fact that $\pi_0\varphi$ is an isomorphism implies that $\pi_n\varphi$ is an isomorphism for $n \leq 0$. Now, consider the resolution

$$0 \to \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Calk}_{\omega_1}(\mathcal{C}) \to 0$$

and proceed by induction. □

We also deduce from the proof that for all dualizable categories $\mathcal{D}$, $\pi_n\varphi^{\mathrm{cont}}_{\mathcal{D}}$ is an isomorphism for $n \leq -1$. Then we use

$$0 \to \mathrm{Shv}_{>0}(\mathbb{R}; \mathrm{Sp}) \to \mathrm{Shv}_{\geq 0}(\mathbb{R}; \mathrm{Sp}) \xrightarrow{\Gamma_c(-)[1]} \mathrm{Sp} \to 0.$$

An inductive argument shows that $\pi_n\varphi$ is an isomorphism for all $n \in \mathbb{Z}$.

Corollary 3.10. *For any accessible localizing invariant $F\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ we have*

$$F^{\mathrm{cont}}\big(\mathrm{Shv}(\mathbb{R} \cup \{\infty\}; \mathrm{Sp})\big) = 0.$$

Proof. Observe the exact sequence

$$0 \to \mathrm{Shv}_{\geq 0}(\mathbb{R};\mathrm{Sp}) \xrightarrow{\overline{\varphi}_\gamma^*} \mathrm{Shv}(\mathbb{R}\cup\{\infty\};\mathrm{Sp}) \xrightarrow{\alpha} \mathrm{Shv}_{\leq 0}(\mathbb{R};\mathrm{Sp}),$$

where α is the left adjoint to $j_!\varphi_{-\gamma}^*$: Let $\gamma = \mathbb{R}_{\leq 0}$ and write $\mathbb{R}_\gamma$ for $\mathbb{R}$ equipped with the γ-topology ($U \subseteq \mathbb{R}_\gamma$ is open if $U + \gamma = U$ and $U \subseteq \mathbb{R}$ is open). Similarly, define $\mathbb{R}_{-\gamma}$.

Then $\varphi_\gamma\colon \mathbb{R} \to \mathbb{R}_\gamma$, $\varphi_{-\gamma}\colon \mathbb{R} \to \mathbb{R}_{-\gamma}$ and $\varphi_\gamma\colon \mathbb{R}\cup\{\infty\} \to \overline{\mathbb{R}}_\gamma$. □

Theorem 3.11. *Let X be a finite CW complex and let $\mathcal{C}$ be a dualizable category. Let $F\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be an accessible localizing invariant.*

Then $F^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C})) = F^{\mathrm{cont}}(\mathcal{C})^X$, where the right hand side is the X-∞-groupoid.

Observe that for all finite CW complexes Y we have an isomorphism

$$F^{\mathrm{cont}}(\mathrm{Shv}(Y\times[0,1];\mathcal{C})) \xrightarrow{\sim} F^{\mathrm{cont}}(\mathrm{Shv}(Y;\mathcal{C})).$$

We get a functor $G\colon (S^{\mathrm{fin}})^{\mathrm{op}} \to \mathcal{E}$ such that $G(X) = F^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C}))$ for any finite CW complex X. We know $G(*) = F^{\mathrm{cont}}(\mathcal{C})$ and $G(\varnothing) = 0$.

We need to show that G commutes with pullbacks. Consider a cellular embedding $X \hookrightarrow Y$ of finite CW complexes, and let $X \to Z$ be some continuous map of finite CW complexes. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}(Y \sqcup_X Z;\mathrm{Sp}) & \longrightarrow & \mathrm{Shv}(Y;\mathrm{Sp}) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(Z;\mathrm{Sp}) & \longrightarrow & \mathrm{Shv}(X;\mathrm{Sp}) \end{array}$$

which is a pullback square both in $\mathrm{Pr}^L_{\mathrm{st}}$ and in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ (because the right vertical map is a quotient functor).

We deduce that F^{cont} commutes with such pullbacks:

Lemma 3.12. *Let*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

be a pullback square in $\mathrm{Pr}^L_{\mathrm{st}}$. Assume that $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are dualizable, all functors are strongly continuous and the right vertical map is a localization.

Then the left vertical map is a localization, $\mathcal{A} = \mathcal{B}\times^{\mathrm{dual}}_{\mathcal{D}}\mathcal{C}$ and

$$F^{\mathrm{cont}}(\mathcal{A}) = F^{\mathrm{cont}}(\mathcal{B}) \times_{F^{\mathrm{cont}}(\mathcal{D})} F^{\mathrm{cont}}(\mathcal{C}).$$

4. Talk 4

Let X be a locally compact Hausdorff space and let $\underline{\mathcal{C}}$ be a presheaf with values in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ such that $\underline{\mathcal{C}}(\varnothing) = 0$.

Theorem 4.1. *We have*

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathrm{Shv}(X;\underline{\mathcal{C}})) = \Gamma_c(X;(\mathcal{U}_{\mathrm{loc}}(\underline{\mathcal{C}}))^\sharp),$$

where $\mathcal{U}_{\mathrm{loc}}$ is the universal localizing invariant. The same holds for K-theory.

Let $\mathcal{F} \in \mathrm{Shv}(X;\underline{\mathcal{C}})$. Recall the restriction map $\mathrm{res}_{U,V}\colon \underline{\mathcal{C}}(U) \to \underline{\mathcal{C}}(V)$ for $V \subseteq U$. Being a sheaf requires that

(1) $\mathcal{F}(\varnothing) = 0$.
(2) For open subsets $U, V \subseteq X$ there is a pullback

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathrm{res}^R_{U\cup V, U} \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathrm{res}^R_{U\cup V, V} \mathcal{F}(V) & \longrightarrow & \mathrm{res}^R_{U\cup V, U\cap V} \mathcal{F}(U \cap V) \end{array}$$

in $\underline{\mathcal{C}}(U \cup V)$.

Efimov:sheaf-3

(3) $\mathcal{F}(U) = \varprojlim_{V \Subset U} \mathrm{res}^R_{U,V} \mathcal{F}(V)$.

Proposition 4.2. *Let $\mathcal{F} \in \mathrm{Shv}(X; \underline{\mathcal{C}})$. The following are equivalent:*

(i) $\mathcal{F}$ *is compact.*
(ii) $\mathrm{Supp}(\mathcal{F})$ *is compact and* $\mathcal{F}$ *can be covered by* U *such that* $\mathcal{F}\big|_U \simeq P_U$, *where* $P \in \underline{\mathcal{C}}(U)^\omega$.

Proof. The implication (ii) $\Longrightarrow$ (i) is an easy exercise.

Let us prove (i) $\Longrightarrow$ (ii):

Step 1: Suppose that $\mathcal{F}_x = 0$ in $\underline{\mathcal{C}}_x$. Then there exists $U \ni x$ such that $\mathcal{F}\big|_U = 0$. Indeed, we have

$$\mathcal{F} = \varinjlim_{V \Subset X \setminus \{x\}} j_{V!} j_V^* \mathcal{F}.$$

Hence, $\mathcal{F}$ is a summand of some $j_{V!} j_V^* \mathcal{F}$ and thus $\mathcal{F}\big|_{X \setminus \overline{V}} = 0$.

Exercise: $(-)^\omega \colon \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}$ commutes with filtered colimits.

Step 2: Suppose $\mathcal{F}_x \neq 0$. Then $\mathcal{F}_x \in \underline{\mathcal{C}}_x^\omega$. It follows that there exists $U \ni x$ such that $\mathcal{F}_x$ lifts to $P \in \underline{\mathcal{C}}(U)^\omega$. Shrinking U if necessary, we get a map $\varphi \colon P_U \to \mathcal{F}$. Then, choosing $x \in V \Subset U$, we get $\mathrm{Cone}(\varphi)\big|_{\overline{V}} \in \mathrm{Shv}(\overline{V}; \underline{\mathcal{C}}\big|_V)^\omega$. Hence, $\mathrm{Cone}(\varphi)\big|_W = 0$ for some $W \ni x$.

Step 3: $\mathcal{F} = \varinjlim_{U \Subset X} j_{U!} j_U^* \mathcal{F}$ implies that $\mathrm{Supp}(\mathcal{F})$ is compact.

Remark 4.3. Some proofs use the following fact: There exists a conservative continuous functor $\mathrm{Mot}^{\mathrm{loc}} \to \mathrm{Sp}$. This follows from rigidity (hence dualizability) of $\mathrm{Mot}^{\mathrm{loc}}$.

Note that

$$\begin{array}{ccc} \mathrm{Shv}(X; \underline{\mathcal{C}}) & \subset & \mathrm{PSh}^{\mathrm{cont}}(X; \underline{\mathcal{C}}) \\ \downarrow & & \downarrow \simeq \\ \mathrm{Shv}_{\mathcal{K}}(X; \underline{\mathcal{C}}) & \subset & \mathrm{PSh}^{\mathrm{cont}}_{\mathcal{K}}(X; \underline{\mathcal{C}}), \end{array}$$

where $\mathrm{PSh}^{\mathrm{cont}}$ denotes those presheaves which satisfy (3), and $\mathrm{PSh}^{\mathrm{cont}}_{\mathcal{K}}$ denotes those presheaves defined on compact subsets such that $\mathcal{F}(Y) = \varinjlim_{Z \Supset Y} \mathrm{res}_{Z,Y} \mathcal{F}(Z)$.

Note that for compact $Y \subseteq X$ we have $\underline{\mathcal{C}}(Y) = \varinjlim_{U \supset Y} \underline{\mathcal{C}}(U)$.

Exercise: We have that

$$\mathcal{U}_{\mathrm{loc}}(\mathrm{PSh}^{\mathrm{cont}}_{\mathcal{K}}(X; \underline{\mathcal{C}})) = \bigoplus_{\substack{V \subseteq X \\ \text{open, compact}}} \mathcal{U}_{\mathrm{loc}}(\underline{\mathcal{C}}(V)).$$

Hint: use the semi-orthogonal decomposition $\mathrm{PSh}_{\mathcal{K}}(X; \underline{\mathcal{C}}) = \langle \underline{\mathcal{C}}(Y), Y \subset X \text{ compact} \rangle$ and

$$\mathrm{PSh}_{\mathcal{K}}(X; \underline{\mathcal{C}}) / \mathrm{PSh}^{\mathrm{cont}}_{\mathcal{K}}(X; \underline{\mathcal{C}}) = \langle \underline{\mathcal{C}}(Y), Y \subset X \text{ compact but not open} \rangle$$

Assuming that X is compact, we have $\mathrm{Shv}(X; \underline{\mathcal{C}}) \simeq \mathrm{Shv}(X \cup \{\infty\}; j_! \underline{\mathcal{C}})$.

Step 4: Approximate $\mathrm{Shv}_{\mathcal{K}}(X;\underline{\mathcal{C}})$ by finite limits of $\mathrm{PSh}_{\mathcal{K}}(Y;\underline{\mathcal{C}})$. If $X = Y_1 \sqcup \cdots \sqcup Y_n$, denoting $Y_I = \bigcap_{i\in I} Y_i$ for $I \neq \varnothing$, then we have the approximation

$$\varprojlim_{I\neq\varnothing} \mathrm{PSh}^{\mathrm{cont}}_{\mathcal{K}}(Y_I, \underline{\mathcal{C}}|_{Y_I}).$$

"Categorify Čech cohomology" ☐

4.1. **The internal Hom in** $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$**.**

Theorem 4.4. *Let $\mathcal{C}$ be a rigid symmetric monoidal category (in particular, $\mathcal{C}$ is dualizable). Let $\mathcal{A}$ and $\mathcal{B}$ be dualizable categories over $\mathcal{C}$.*

(1) Suppose that $\mathcal{A}$ is proper and ω_1-compact.
Then for all uncountable regular cardinals κ, we have

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{A},\mathcal{B})) \simeq \underline{\mathrm{Hom}}(\mathcal{U}_{\mathrm{loc},\kappa}(\mathcal{A}), \mathcal{U}_{\mathrm{loc},\kappa}(\mathcal{B}))$$

in $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C},\kappa}$.

(2) If additionally $\mathcal{C}$ is compactly generated, then

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}\big(\underline{\mathrm{Hom}}^{\mathrm{dual}}_{\mathcal{C}}(\mathcal{A},\mathcal{B})\big) \simeq \underline{\mathrm{Hom}}\big(\mathcal{U}_{\mathrm{loc}}(\mathcal{A}), \mathcal{U}_{\mathrm{loc}}(\mathcal{B})\big)$$

in $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C}}$.

Definition 4.5. Recall that $\mathcal{A}$ is *proper* if the evaluation functor $\mathcal{A}\otimes_{\mathcal{C}}\mathcal{A}^\vee \to \mathcal{C}$ is strongly continuous.

We say that $\mathcal{A}$ is ω_1-*compact* if $\mathrm{coev}(\mathbf{1}) \in (\mathcal{A}\otimes_{\mathcal{C}}\mathcal{A}^\vee)^{\omega_1}$.

Corollary 4.6. *Let R be a noetherian commutative ring and $I \subseteq R$ an ideal. Then*

$$K^{\mathrm{cont}}(\widetilde{\mathrm{Nuc}}(R_{\widehat{I}})) \simeq \varprojlim_n K(R/I^n).$$

Corollary 4.7. *If X is a $\mathcal{C}^0$-manifold and countable at ∞, then*

$$K^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X;\mathcal{C})) \simeq \mathrm{H}^{\mathrm{BM}}(X; K^{\mathrm{cont}}(\mathcal{C})).$$

Suppose that $\mathcal{C} = \mathrm{Mod}(k)$, where k is an $\mathbb{E}_\infty$-ring. Let R, S be $\mathbb{E}_1$-algebras over k. Suppose that $\mathcal{A} = \mathrm{Mod}(R)$ and $\mathcal{B} = \mathrm{Mod}(S)$.

Then $\mathcal{A}$ is proper if and only if $R \in \mathrm{Perf}(k)$.

Question 4.8. What is $\underline{\mathrm{Hom}}^{\mathrm{dual}}_k(\mathrm{Mod}(R), \mathrm{Mod}(S))$?

We use the adjunction

$$\begin{aligned} \mathrm{Cat}^{\mathrm{dual}}_k &\underset{\longleftarrow}{\xrightarrow{\mathrm{incl}}} \mathrm{Pr}^L_{k,\omega_1}, \\ \mathrm{Ind}(\mathcal{E}^{\omega_1}) &\longleftarrow\!\!\shortmid\ \mathcal{E}. \end{aligned}$$

Corollary 4.9. *We have*

$$\underline{\mathrm{Hom}}^{\mathrm{dual}}_k(\mathrm{Mod}(R), \mathrm{Mod}(S)) = \mathrm{Ker}^{\mathrm{dual}}\big(\mathrm{Ind}(\mathrm{BiMod}(R,S)^{\omega_1}) \to \mathrm{Ind}(\mathrm{Rep}(R, \mathrm{Calk}_{\omega_1}(S)))\big),$$

noting that $\mathrm{BiMod}(R,S) = \mathrm{Rep}(R, \mathrm{Mod}(S)^{\omega_1})$.

Crucial Fact 4.10. The functor

$$\mathrm{Rep}_k(R, \mathrm{Mod}(S)^{\omega_1}) \to \mathrm{Rep}_k(R, \mathrm{Calk}_{\omega_1}(S))$$

is a homological epi.

Recall: to show that a functor $\mathcal{D} \to \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{perf}}$ is a homological epi, we need that for all $x, y \in \mathcal{E}$ it holds that

$$\mathrm{Hom}_{\mathcal{E}}(F(-), y) \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}(x, F(-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(x, y).$$

The left hand side is considered as an object of $\mathrm{Ind}(\mathcal{D}) \otimes_k \mathrm{Ind}(\mathcal{D}^{\mathrm{op}}) \xrightarrow{\mathrm{ev}} \mathrm{Mod}(k)$.

Efimov:fact

Fact 4.11. (1) Suppose that $\mathcal{D}$ is a compactly assembled presentable category (or least assume that strong (AB5) and (AB6) for countable products hold). Let I be a directed poset and $F\colon \mathbb{N}^{\mathrm{op}} \times I \to \mathcal{D}$ be a functor.

Then

$$\varinjlim_{\varphi\colon \mathbb{N}\to I} \varprojlim_{n\leq m} F(n, \varphi(m)) \xrightarrow{\sim} \varprojlim_{n} \varinjlim_{i} F(n, i),$$

where $(n \leq m) \in \mathbb{N}^{\mathrm{op}} \times \mathbb{N}$.

(2) Let $\mathcal{D}$ be as above, and let I, J be directed posets. Let $F\colon \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \times I \times J \to \mathcal{D}$. Assume that the following conditions are satisfied:

(i) $\varprojlim_n \varinjlim_i \varprojlim_k \varinjlim_j F(n, k, i, j) \xrightarrow{\sim} \varprojlim_n \varprojlim_k \varinjlim_i \varinjlim_j F(n, k, i, j)$.

(ii) Same with $n \leftrightarrow k$ and $i \leftrightarrow j$.

Then we have that

$$\varinjlim_{\varphi\colon \mathbb{N}\to I} \varinjlim_{\psi\colon \mathbb{N}\to J} \varprojlim_{n\leq m} \varprojlim_{k\leq l} F(n, k, \varphi(m), \psi(l)) \xrightarrow{\sim} \varprojlim_{n} \varinjlim_{i} \varinjlim_{j} F(n, n, i, j)$$

Exercise 4.12. (a) Prove a version of Fact 4.11.(1), where $\mathbb{N}^{\mathrm{op}} \times I$ is replaced with a cocartesian fibration $\mathcal{E} \to \mathbb{N}^{\mathrm{op}}$, with directed fibers and cofinal transition maps.

(b) Let $I, J, \mathcal{D}$ and F be as in Fact 4.11.(2), but without assumptions (i) and (ii). Then we have

$$\varinjlim_{\varphi\colon \mathbb{N}\to I} \varinjlim_{\psi\colon \mathbb{N}\to J} \varprojlim_{n\leq m\leq k\leq l} F(n, k, \varphi(m), \psi(l)) \xrightarrow{\sim} \varprojlim_{n} \varinjlim_{i} \varprojlim_{k} \varinjlim_{j} F(n, k, i, j),$$

where $(n, m, k, l) \in \mathbb{N}^{\mathrm{op}} \times \mathbb{N} \times \mathbb{N}^{\mathrm{op}} \times \mathbb{N}$.

5. Talk 5: Internal homs and inverse limits in $\mathrm{Cat}^{\mathrm{dual}}$

Theorem 5.1. *Let $\mathcal{C}$ be a rigid base category. Let $\mathcal{A}, \mathcal{B}$ be dualizable categories over $\mathcal{C}$ such that $\mathcal{A}$ is proper and ω_1-compact in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{C}}$.*

Then $\underline{\mathrm{Hom}}^{\mathrm{dual}}_{\mathcal{C}}(\mathcal{A}, -)$ preserves short exact sequences, i.e., $\mathcal{A}$ is internally projective.

Corollary 5.2. *For a regular cardinal $\kappa > \omega_1$ and $\mathcal{A}$ as above, we have*

$$\mathcal{U}_{\mathrm{loc},\kappa}\big(\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{A}, \mathcal{B})\big) = \underline{\mathrm{Hom}}\big(\mathcal{U}_{\mathrm{loc},\kappa}(\mathcal{A}), \mathcal{U}_{\mathrm{loc},\kappa}(\mathcal{B})\big).$$

Example 5.3 (Non-example)**.** Let $\mathcal{C} = \mathrm{Sp}$; then $\mathcal{A} = \prod_{x\in[0,1]} \mathrm{Sp}$ is not ω_1-compact.

Let $\mathcal{C} = \mathrm{Mod}(k)$ and $\mathcal{A} = \mathrm{Mod}(R)$, where $R \in \mathrm{Perf}(k)$. Let $\mathcal{B} = \mathrm{Mod}(S)$.

Key Statement 5.4. The functor $F\colon \mathrm{Rep}(R, \mathrm{Mod}(S)^{\omega_1}) \to \mathrm{Rep}(R, \mathrm{Calk}_{\omega_1}(S))$ is a homological epi.

Proof. Let $M, N \in \mathcal{D} := \mathrm{BiMod}(R, S)^{\omega_1} \cong \mathrm{Rep}(R, \mathrm{Mod}(S)^{\omega_1})$ and put $\mathcal{E} = \mathrm{Rep}(R, \mathrm{Calk}_{\omega_1}(S))$. We need to show that

{Efimov:key}

$$\mathrm{Hom}_{\mathcal{E}}(F(-), F(N)) \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}(F(M), F(-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(F(M), F(N)) \tag{2}$$

is an isomorphism and that F is essentially surjective up to retracts. Now, (2) reduces to

$$\mathrm{THC}^*(R/k, \mathrm{Hom}_S(-, S) \otimes_S N) \otimes_{\mathcal{D}} \mathrm{THC}^*(R/k, \mathrm{Hom}_S(M, S) \otimes_S -) \xrightarrow{\sim} \mathrm{THC}^*(R/k, \mathrm{Hom}_S(M, S) \otimes_S N).$$

Replace $\mathrm{Hom}_S(M,S)$ by an abstract $L \in \mathrm{BiMod}(S,R)$. We thus want to show that

$$\mathrm{THC}^*(R/k, \mathrm{Hom}_S(-,S)\otimes_S N)\otimes_{\mathcal{D}} \mathrm{THC}^*(R/k, L\otimes_S -) \xrightarrow{\sim} \mathrm{THC}^*(R/k, L\otimes_S N)$$

is an isomorphism. Choose an approximation $N = \varinjlim_j N_j$ and $L = \varinjlim_i L_i$, where L_i, N_j are compact bimodules. Furthermore, choose an approximation $R \simeq \varinjlim_n X_n$ in $\mathrm{BiMod}(R,R)$ with $X_n \in \mathrm{Perf}(R\otimes R^{\mathrm{op}})$.

Apply Fact 4.11.(1) and then Fact 4.11.(2) to the functor

$$\begin{aligned}&\mathbb{N}^{\mathrm{op}}\times\mathbb{N}^{\mathrm{op}}\times I\times J \to \mathrm{Mod}(k),\\ &\qquad (n,k,i,j)\mapsto \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(X_n\otimes_R X_k, L_i\otimes N_j).\end{aligned}$$

□

Remark 5.5. The dual $(-)^\vee\colon \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}\to\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ is a (covariant!) equivalence; it takes $F\colon\mathcal{C}\to\mathcal{D}$ to $F^\vee\colon\mathcal{C}^\vee\to\mathcal{D}^\vee$.

The proof shows the following: for a map $S\to S'$ of $\mathbb{E}_1$-algebras, we have a commutative square

$$\begin{array}{ccc}\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Mod}(R),\mathrm{Mod}(S'))^\vee & \longrightarrow & \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Mod}(R),\mathrm{Mod}(S))^\vee\\ \downarrow & & \downarrow\\ \mathrm{Ind}(\mathrm{Rep}(R,\mathrm{Mod}(S')^{\omega_1})^{\mathrm{op}}) & \longrightarrow & \mathrm{Ind}(\mathrm{Rep}(R,\mathrm{Mod}(S)^{\omega_1})^{\mathrm{op}}).\end{array}$$

The upper right category is generated by objects of the form $\mathrm{THC}^*(R/k, L\otimes_S -)$.

Corollary 5.6. *If $S\to S'$ is a homological epi, then*

$$\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Mod}(R),\mathrm{Mod}(S))\to\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Mod}(R),\mathrm{Mod}(S'))$$

is a quotient functor.

Example 5.7. Consider a noetherian commutative ring R with an ideal $I\subset R$. We work over R. Define

$$\widetilde{\mathrm{Nuc}}(R_{\widehat{I}}) := \underline{\mathrm{Hom}}_R^{\mathrm{dual}}(D_{I\text{-tors}}(R), D(R)) = D_{I\text{-tors}}(R)^{\mathrm{rig}}.$$

If $I=(f_1,\dots,f_n)$, then $D_{I\text{-tors}}(R)=\mathrm{Mod}(A)$, where $A=\mathrm{End}_R(\mathrm{Kos}(R,f_1,\dots,f_n))$.

Remark 5.8. Let $\mathcal{C}$ be a locally rigid category. Consider its one-point compactification $\mathcal{C}_+\subset\mathrm{Ind}(\mathcal{C})$, which is generated under colimits by $\widehat{Y}(\mathcal{C})$ and $\widehat{Y}(\mathbf{1}_{\mathcal{C}})$. For example, $D_{I\text{-tors}}(R)_+ = D(R_{\widehat{I}})$.

Then $\mathcal{C}^{\mathrm{rig}} = \underline{\mathrm{Hom}}_{\mathcal{C}_+}^{\mathrm{dual}}(\mathcal{C},\mathcal{C}_+)$.

Consider the category

$$\underline{\mathrm{Hom}}_k^{\mathrm{dual}}(\mathrm{Ind}(\mathcal{A}),\mathrm{Ind}(\mathcal{C})),$$

where $\mathcal{A}$ is proper and ω_1-compact in $\mathrm{Cat}_k^{\mathrm{perf}}$. Write $\mathcal{A}=\varinjlim_n\mathcal{B}_n$, where each $\mathcal{B}_n$ is a finitely presented (= compact) category over k.

Proposition 5.9. *We have that*

$$\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Ind}(\mathcal{A}),\mathrm{Ind}(\mathcal{C})) = \varprojlim_n{}^{\mathrm{dual}}(\mathrm{Ind}(\mathrm{Fun}(\mathcal{B}_n,\mathcal{C}))).$$

As a general fact, we have

$$\varprojlim_i{}^{\mathrm{dual}}\mathcal{D}_i = \mathrm{Ker}^{\mathrm{dual}}\big(\mathrm{Ind}(\varprojlim_i\mathcal{D}_i^{\omega_1})\to\mathrm{Ind}(\varprojlim_i\mathrm{Calk}_{\omega_1}^{\mathrm{cont}}(\mathcal{D}_i))\big).$$

Exercise 5.10. $\mathrm{Ring}_{\mathbb{E}_\infty} \to \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$, $R \mapsto \mathrm{Mod}(R)$ is a sheaf in the descendable topology (in the sense of Akhil Mathew).

Theorem 5.11. *We have that*

$$K^{\mathrm{cont}}\big(\varprojlim_n^{\mathrm{dual}} \mathrm{Ind}(\mathrm{Fun}(\mathcal{B}_n, \mathcal{C}))\big) \xrightarrow{\sim} \varprojlim_n K(\mathrm{Fun}(\mathcal{B}_n, \mathcal{C})).$$

Proposition 5.12. *For all n, there exists $m > n$ such that $\mathcal{B}_n \to \mathcal{B}_m$ is trace class in $\mathrm{Cat}_k^{\mathrm{perf}}$.*

Together with the theorem, the proposition implies that

$$\mathrm{Hom}(\mathcal{U}_{\mathrm{loc}}(\mathcal{A}), \mathcal{U}_{\mathrm{loc}}(\mathcal{B})) = KK(\mathcal{A}, \mathcal{C}) = \varprojlim_n K(\mathrm{Fun}(\mathcal{B}_n, \mathcal{C})).$$

Example 5.13. Let $k = \mathbb{Z}[x]$ and $\mathcal{A} = \mathrm{Perf}_{x\text{-tors}}(\mathbb{Z}[x])$, which is proper over $\mathbb{Z}[x]$. Then $\mathcal{A} = \varinjlim_n D^b_{\mathrm{coh}}(\mathbb{Z}[x]/x^n)$.

Exercise 5.14. The map

$$D^b_{\mathrm{coh}}(\mathbb{Z}[x]/x^n) \to D^b_{\mathrm{coh}}(\mathbb{Z}[x]/x^{2n})$$

is trace class in $\mathrm{Cat}_{\mathbb{Z}[x]}^{\mathrm{perf}}$, and

$$\text{“}\varinjlim_n\text{”}\, D^b_{\mathrm{coh}}(\mathbb{Z}[x]/x^n) \xrightarrow{\sim} \text{“}\varinjlim_n\text{”}\, \mathcal{B}_n,$$

where $\mathcal{B}_n$ is finitely presented.

Assuming $I = (f) \subset R$, we get

$$\begin{aligned} K^{\mathrm{cont}}(\widetilde{\mathrm{Nuc}}(R_{\widehat{I}})) &= KK_{\mathbb{Z}[x]}(\mathrm{Perf}_{x\text{-tors}}(\mathbb{Z}[x]), \mathrm{Perf}(R)) \\ &\xrightarrow{\sim} \varprojlim_n K\big(\mathrm{Fun}(D^b_{\mathrm{coh}}(\mathbb{Z}[x]/x^n), \mathrm{Perf}(R)) \simeq \varprojlim_n K(R/f^n). \end{aligned}$$

Theorem 5.15. *Let $\mathcal{D}_1 \leftarrow \mathcal{D}_2 \leftarrow \cdots$ be an inverse system in $\mathrm{Cat}^{\mathrm{perf}}$ such that*

(∗) $\varprojlim_n \mathcal{D}_n^{\omega_1} \to \varprojlim_n \mathrm{Calk}_{\omega_1}(\mathcal{D}_n)$ is a homological epi.

Then we have

$$K^{\mathrm{cont}}(\varprojlim_n^{\mathrm{dual}} \mathrm{Ind}(\mathcal{D}_n)) \xrightarrow{\sim} \varprojlim_n K(\mathcal{D}_n).$$

Idea. Categorify the fiber sequence

$$\varprojlim_n K(\mathcal{D}_n) \to \prod_n K(\mathcal{D}_n) \to \prod_n K(\mathcal{D}_n).$$

Step 1: Prove

$$K^{\mathrm{cont}}(\varprojlim_n^{\mathrm{dual}} \mathrm{Ind}(\mathcal{D}_n)) \cong \Omega K(\varprojlim_n \mathrm{Calk}_{\omega_1}(\mathcal{D}_n)).$$

Step 2: Use

$$\begin{aligned} K\big(\varprojlim_n^{\mathrm{oplax}} \mathrm{Calk}_{\omega_1}(\mathcal{D}_n)\big) &\cong K\Big(\prod_n \mathrm{Calk}_{\omega_1}(\mathcal{D}_n)\Big) \\ &= \prod_n K(\mathrm{Calk}_{\omega_1}(\mathcal{D}_n)). \end{aligned}$$

Step 3: Prove that the functor

$$\varprojlim_n^{\mathrm{oplax}} \mathrm{Ind}(\mathcal{D}_n)^{\omega_1} / \varprojlim_n^{\mathrm{oplax}} \mathcal{D}_n \hookrightarrow \varprojlim_n^{\mathrm{oplax}} \mathrm{Calk}_{\omega_1}(\mathcal{D}_n)$$

is fully faithful.

It remains to show:

$$\varprojlim_n{}^{\mathrm{oplax}}\mathcal{D}_n \Big\backslash \varprojlim_n{}^{\mathrm{oplax}} \mathrm{Ind}(\mathcal{D}_n)^{\omega_1} \Big/ \varprojlim_n \mathrm{Ind}(\mathcal{D}_n)^{\omega_1} \xrightarrow{\text{K-equiv.}} \prod_n \mathrm{Calk}_{\omega_1}(\mathcal{D}_n).$$

□

6. Talk 6: Rigidity of $\mathrm{Mot}^{\mathrm{loc}}$

Theorem 6.1. *Let $\mathcal{C}$ be a rigid symmetric monoidal category. Then $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C}}$, i.e., the target of the universal localizing invariant $\mathcal{U}_{\mathrm{loc}}\colon \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{C}} \to \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C}}$ commuting with filtered colimits, is rigid.*

Corollary 6.2. *The category $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C}}$ is dualizable (but in general not compactly generated).*

Expectation: If $\mathcal{C} \neq 0$, then $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{C}}$ is not compactly generated. This is known to hold for $\mathcal{C} = D(\mathbb{Q}[x])$.

Consider the case $\mathcal{C} = \mathrm{Mod}(k)$ for some $\mathbb{E}_\infty$-ring k. Recall the following definitions:

Definition 6.3 (Kontsevich)**.** Let $\mathcal{D}$ be a dualizable k-linear category.

(1) $\mathcal{D}$ is called *proper* over k if $\mathrm{ev}\colon \mathcal{D}\otimes_k \mathcal{D}^\vee \to \mathrm{Mod}(k)$ is strongly continuous.
(2) $\mathcal{D}$ is called *smooth* over k if $\mathrm{coev}\colon \mathrm{Mod}(k) \to \mathcal{D}\otimes_k \mathcal{D}^\vee$ is strongly continuous, i.e., $\mathrm{coev}(k)$ is compact.

A category $\mathcal{A} \in \mathrm{Cat}^{\mathrm{perf}}_k$ is called *smooth* or *proper* if $\mathrm{Ind}(\mathcal{A})$ is.

Note that $\mathrm{Cat}^{\mathrm{perf}}_k$ is compactly generated.

Proposition 6.4 (TV)**.** *If $\mathcal{A}$ is a finitely presented (compact) category, then $\mathcal{A}$ is smooth.*
If $\mathcal{A}$ is smooth and proper, then $\mathcal{A}$ is finitely presented.

Let $\mathcal{A} = \mathrm{Perf}(A)$, where A is a finitely presented object of $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(k))$. Then for all ind-systems $(M_i)_i$ in $\mathrm{BiMod}(A, A)$, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{BiMod}(A,A)}(\Omega_{A/k}, \varinjlim_i M_i) &\simeq \mathrm{Mod}_{\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(A))}(A, A\oplus \varinjlim_i M_i)\\ &\simeq \varinjlim_i \mathrm{Map}_{\mathrm{Alg}_{/A}}(A, A\oplus M_i)\\ &\simeq \varinjlim_i \mathrm{Map}_{\mathrm{BiMod}(A,A)}(\Omega_{A/k}, M_i).\end{aligned}$$

This implies $\Omega_{A/k} \coloneqq \mathrm{fib}(A\otimes A \to A) \in \mathrm{Perf}(A\otimes A^{\mathrm{op}})$.

Definition 6.5. A k-linear category $\mathcal{B} \in \mathrm{Cat}^{\mathrm{perf}}_k$ is called *nuclear* if for all finitely presented categories $\mathcal{A}$ the natural map

$$\mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k \mathcal{B} \xrightarrow{\sim} \mathrm{Fun}_k(\mathcal{A}, \mathcal{B})$$

is an isomorphism.

Exercise 6.6. If $\mathcal{A}$ is smooth and $\mathcal{B}$ is proper, then

$$\mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k \mathcal{B} \xrightarrow{\sim} \mathrm{Fun}_k(\mathcal{A}, \mathcal{B}).$$

Corollary 6.7. *Any proper category is nuclear.*

To prove rigidity of $\mathrm{Mot}^{\mathrm{loc}}_k$, we need:

(i) $\mathcal{U}_{\mathrm{loc}}(k)$ is compact.[2] In fact, one can show $\mathrm{Map}(\mathcal{U}_{\mathrm{loc}}(k), \mathcal{U}_{\mathrm{loc}}(\mathcal{A})) = K(\mathcal{A})$.
(ii) The category $\mathrm{Mot}_k^{\mathrm{loc}}$ is generated by objects of the form $\varinjlim(x_1 \to x_2 \to \cdots)$ such that the transition maps $x_n \to x_{n+1}$ are trace class.

Suppose that $\mathcal{B}$ is a nuclear object of $(\mathrm{Cat}_k^{\mathrm{perf}})^{\omega_1}$. Then $\mathcal{B} \simeq \varinjlim(\mathcal{A}_1 \to \mathcal{A}_2 \to \cdots)$, where each $\mathcal{A}_n$ is finitely presented and the transition maps $\mathcal{A}_n \to \mathcal{A}_{n+1}$ are trace class. Moreover,

$$\mathcal{U}_{\mathrm{loc}}(\mathcal{B}) = \varinjlim_n \mathcal{U}_{\mathrm{loc}}(\mathcal{A}_n),$$

and the transition maps $\mathcal{U}_{\mathrm{loc}}(\mathcal{A}_n) \to \mathcal{U}_{\mathrm{loc}}(\mathcal{A}_{n+1})$ are trace class. We only need to show that $\mathrm{Mot}_k^{\mathrm{loc}}$ is generated under colimits by $\mathcal{U}_{\mathrm{loc}}(\mathcal{B})$, where $\mathcal{B}$ is nuclear and ω_1-compact.

Lemma 6.8. *If $\mathcal{A}$ is smooth (e.g., finitely presented) and $\mathcal{B} \in \mathrm{Cat}_k^{\mathrm{perf}}$, then*

$$\begin{array}{ccc} \mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k \mathcal{B} & \longrightarrow & \mathrm{Fun}_k(\mathcal{A}, \mathcal{B}) \\ & \searrow \quad \swarrow & \\ & \mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B} & \end{array}$$

is fully faithful.

Proof. This is an exercise. □

Corollary 6.9. *The class of nuclear objects of $\mathrm{Cat}_k^{\mathrm{perf}}$ is closed under:*
(a) filtered colimits;
(b) semi-orthogonal decompositions;
(c) taking full subcategories; indeed, if $\mathcal{B}' \subseteq \mathcal{B}$, where $\mathcal{B}$ is nuclear, then

$$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ \mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k \mathcal{B}' & \hookrightarrow & \mathrm{Fun}_k(\mathcal{A}, \mathcal{B}') \\ \downarrow & & \downarrow \\ \mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k \mathcal{B} & \xrightarrow{\simeq} & \mathrm{Fun}_k(\mathcal{A}, \mathcal{B}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_k(\mathcal{A}, \mathrm{Perf}(k)) \otimes_k (\mathcal{B}/\mathcal{B}') & \hookrightarrow & \mathrm{Fun}_k(\mathcal{A}, \mathcal{B}/\mathcal{B}') \\ \downarrow & & \\ 0 & & \end{array}$$

and this formally implies that $\mathcal{B}'$ is nuclear.

Suppose $\mathcal{C} = \mathrm{Perf}(R)$. Define a k-enriched category $\mathcal{B}$ with $\mathrm{ob}(\mathcal{B}) = \mathbb{N}$ and

$$\mathrm{Hom}_{\mathcal{B}}(n, m) = \begin{cases} R, & \text{if } n < m, \\ k, & \text{if } n = m, \\ 0, & \text{if } n > m. \end{cases}$$

[2]This is due to BGT.

Then we have an exact sequence

$$0 \to \mathrm{Ker} \to \mathrm{Fun}_k(\mathcal{B}^{\mathrm{op}}, \mathrm{Mod}(k))^{\omega} \to \mathrm{Perf}(R) \to 0,$$

where Ker is generated by $\mathrm{Cone}(h_{x_n} \to h_{x_{n+1}})$, and the right map is induced by $h_{x_n} \mapsto R$. (This should be "familiar" to symplectic geometers.)

By the above, the category $\mathrm{Fun}_k(\mathcal{B}^{\mathrm{op}}, \mathrm{Mod}(k))^{\omega}$ is nuclear (it has a countable semi-orthogonal decomposition into $\mathrm{Perf}(k)$). Hence, also Ker is nuclear.

Let $\mathcal{C}$ be a rigid category. We want a good notion of nuclearity with the required properties (in particular, full subcategories of nuclear categories should be nuclear).

Fact 6.10. The category $\mathrm{Cat}_{\mathcal{C}}^{\mathrm{perf}} \simeq \mathrm{Cat}_{\mathcal{C}}^{\mathrm{cg}}$ is compactly assembled, and the functor

$$\widehat{Y}\colon \mathrm{Cat}_{\mathcal{C}}^{\mathrm{perf}} \to \mathrm{Ind}(\mathrm{Cat}_{\mathcal{C}}^{\mathrm{perf}})$$

is symmetric monoidal.

Definition 6.11. A category $\mathcal{B}$ is called *nuclear* if for any $\mathcal{A} \in (\mathrm{Cat}_{\mathcal{C}}^{\mathrm{perf}})^{\omega_1}$ such that $\widehat{Y}(\mathcal{A}) = \varinjlim(\mathcal{A}_1 \to \mathcal{A}_2 \to \cdots)$, the natura lmap

$$\text{“}\varprojlim_n\text{”}\, \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{A}_n, \mathcal{C}) \otimes \mathcal{B} \xrightarrow{\sim} \text{“}\varprojlim_n\text{”}\, \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{A}_n, \mathcal{B})$$

is an isomorphim.

The most difficult part is proving that, if $\mathcal{B}$ is nuclear, then any subcategory $\mathcal{B}' \subseteq \mathcal{B}$ is nuclear (where "subcategory" means "generated by relatively (to $\mathcal{C}$) compact objects").

We need to show the following: for any small $\mathcal{C}$-enriched category $\mathcal{B}$ and any compact map $R \to S$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$, consider the commutative square

$$\begin{array}{ccc}
\mathrm{Rep}(R, \mathcal{C}^{\omega}) \otimes \mathcal{B} & \longrightarrow & \mathrm{Rep}(R, \mathcal{B}) \\
\uparrow & \nwarrow & \uparrow \\
\mathrm{Rep}(S, \mathcal{C}^{\omega}) \otimes \mathcal{B} & \xrightarrow[G]{} & \mathrm{Im}(G) \subseteq \mathrm{Rep}(S, \mathcal{B}).
\end{array}$$

We want to construct a functor $\mathrm{Im}(G) \to \mathrm{Rep}(R, \mathcal{C}^{\omega}) \otimes \mathcal{B}$ such that all triangles commute.

This reduces to proving that there exists a map of $\mathbb{E}_1$-coalgebras in $\mathrm{Ind}(\mathrm{BiMod}(S, S))$:

$(*)$ $Y(S \otimes_R S) \to \widehat{Y}(S)$

such that after applying $\varinjlim$, we get the canonical map $S \otimes_R S \to S$ as $\mathbb{E}_1$-coalgebras in $\mathrm{BiMod}(S, S)$.

The map $(*)$ reduces to a version of an argument of Toën–Vaquié for "$\varinjlim_i$"$(S \oplus M_i)$.

Theorem 6.12. *Let R be a connective $\mathbb{E}_1$-ring. Then*

$$\mathrm{Hom}_{\mathrm{Mot}^{\mathrm{loc}}}\big(\widetilde{\mathcal{U}}_{\mathrm{loc}}(\mathbb{S}[x]), \mathcal{U}_{\mathrm{loc}}(R)\big) = \mathrm{TR}(R) = \Omega \varinjlim_n \widetilde{K}(R[x^{-1}]/x^{-n}).$$

Idea. Prove

$$\widetilde{\mathcal{U}}_{\mathrm{loc}}(\mathbb{S}[x]) = \Sigma \widetilde{\mathcal{U}}_{\mathrm{loc}}(\mathrm{Perf}_{\{\infty\}}(\mathbb{P}^1_{\mathbb{S}}))$$

and

$$\mathrm{Perf}_{\{\infty\}}(\mathbb{P}^1_{\mathbb{S}}) = \varinjlim_n \mathcal{A}_n, \qquad \text{where } \mathcal{A}_n = \mathrm{Perf}(\mathrm{Cobar}(\mathbb{S}[x^{-1}]/x^{-n})^*).$$

It then follows that

$$\mathrm{Hom}(\Sigma\widetilde{\mathcal{U}}_{\mathrm{loc}}(\mathrm{Perf}_{\{\infty\}}(\mathbb{P}^1_{\mathbb{S}}),\mathcal{U}_{\mathrm{loc}}(R))) = \Omega \varprojlim_n \widetilde{K}(\mathrm{Fun}(\mathcal{A}_n,\mathcal{U}_{\mathrm{loc}}(R[x^{-1}]/x^{-n}))).$$

Next, we use the following

Lemma 6.13. *Suppose $\mathcal{B} = \varinjlim_n \mathcal{B}_n$ where the transition maps $\mathcal{B}_n \to \mathcal{B}_{n+1}$ are trace class. Then*

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mot}^{\mathrm{loc}}}(\mathcal{U}_{\mathrm{loc}}(\mathcal{B}),\mathcal{U}_{\mathrm{loc}}(\mathcal{C})) &= \varprojlim_n K(\mathrm{Fun}(\mathcal{B}_n,\mathcal{C})) \\ &= \varprojlim_n K(\mathrm{Fun}(\mathcal{B}_n,\mathrm{Sp}^{\omega})\otimes\mathcal{C}).\end{aligned}$$

□

Theorem 6.14. *Consider a smooth scheme X over k, and suppose that there exists a smooth compactification $X \subset \overline{X}$. Then*

$$KK^{\mathcal{K}}(\mathrm{Perf}(X),\mathrm{Perf}(k)) = \mathrm{fib}\big(K(\overline{X}) \to K^{\mathrm{cont}}(\overline{X}_{\widehat{(\overline{X}\setminus X)}})\big) \simeq \mathrm{fib}\big(K(X) \to K^{\mathrm{cont}}(\widehat{X}_{\infty})\big).$$

Theorem 6.15. *Suppose that k is a regular noetherian ring and X is a proper scheme over k. Then*

$$KK^{\mathcal{K}}(\mathrm{Perf}(X),\mathrm{Perf}(k)) = G(X) = K(\mathrm{Coh}\,X).$$

Consider the following functors:

$$\mathrm{Mot}^{\mathrm{loc,cyc}} \xrightarrow{j_!} \mathrm{Mot}^{\mathrm{loc}} \xrightarrow{i^*} \mathrm{Mot}^{\mathrm{loc},\mathbb{A}^1} = \mathrm{Mod}(|\mathcal{U}_{\mathrm{loc}}(\Delta)|).$$

Theorem 6.16. *For a connective $\mathbb{E}_1$-ring R we have*

$$\mathrm{Map}(j_!\mathbf{1},\mathcal{U}_{\mathrm{loc}}(R)) = \mathrm{TC}(R).$$

Suppose k is a $\mathbb{Q}$-algebra. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mot}^{\mathrm{loc}}_k & \xrightarrow{\mathrm{HC}^-} & \mathrm{Mod}_{\widehat{u}}(k[[u]]) \\ & \searrow^{\mathrm{HC}^{-,\mathrm{ref}}} & \uparrow \\ & & \mathrm{Nuc}(k[[u]]), \end{array}$$

where $\deg u = 2$.

Exercise 6.17. For $k = \mathbb{Q}[x]$, we have

$$\mathrm{HC}^{-,\mathrm{ref}}(\mathbb{Q}[x,x^{-1}]/\mathbb{Q}[x]) = \mathcal{O}\Big(\bigcap_{n>0}\{|u| \le |x|^n \neq 0\}\Big),$$

where the right hand side is not generated by compact objects of $\mathrm{Nuc}(\mathbb{Q}[x][[u]])$.

Corollary 6.18. $\mathrm{Mot}^{\mathrm{loc}}_{\mathbb{Q}[x]}$ *is not compactly generated.*