CLUSTER CATEGORIES FOR THE LAZY MATHEMATICIAN

DAVID NKANSAH

ABSTRACT. Some very rough unfinished notes...

0. Setup

We fix an algebraically closed field k. In these notes we consider finite dimensional k-algebras kQ that arise from nice quivers (=directed graphs) Q. When we consider modules over kQ, they will always be finite dimensional. We write this category as $\mathsf{mod}(kQ)$.

1. The module category of linear-type A_n

Here we will see the so-called Auslander-Reiten (AR) quiver of the module category of the path algebra of the Dynkin quiver A_n with linear orientation:

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n,$$

which for this section, we fix and label it Q_n. The AR quiver contains essential information of the module category. Namely, the indecomposable modules and the irreducible morphisms. Noticing their names, these things should be thought of as building blocks for all objects and morphisms in $\mathsf{mod}(kQ_n)$.

This specific choice of orientation of A_n will suffice for our lazy needs because the derived category is insensitive to the choice of orientation of arrows in A_n. We restrict ourselves to A_n because this nice case will have a nice connection to the combinatorics of triangulations of a polygon. We will see this in Section 4.

Indecomposable objects in $\mathsf{mod}(kQ_n)$. Recall that the finite dimensional module category of a finite dimensional algebra is a Krull-Schmidt Hom-finite abelian category. This means that all finite dimensional modules are essentially unique finite direct sums of indecomposable modules. This is why the indecomposable modules are like the building blocks for the category.

The indecomposable modules are characterised by intervals $[i, j]$, where $1 \leq i \leq j \leq n$.

Irreducible morphisms in $\mathsf{mod}\, kQ_n$. These should be thought of as the building blocks for the morphisms in the category.

The irreducible morphisms are of the form $[a, b] \to [a, b+1]$ (these are the monomorphisms) and $[c, d] \to [c+1, d]$ (these are the epimorphisms).

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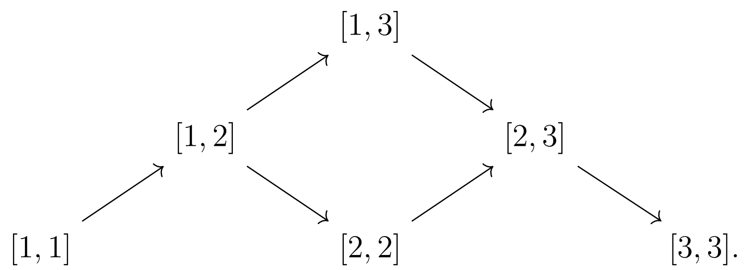
A small example. We present the AR quiver of $\mathsf{mod}\, kQ_3$:



It will be useful to note that the indecomposable projectives are $[1, 1], [1, 2]$ and $[1, 3]$.

Exercise. Draw the Auslander-Reiten quiver for $\mathsf{mod}(kQ_5)$. Make sure it looks like a pyramid.

2. Bounded derived category

The aim of this section is to get comfortable with a certain presentation of the bounded derived category of the path algebra of a simply laced Dynkin quiver. To achieve this, we appeal to a theorem of Happel concerning the so-called *mesh category* which is the the quotient of the path category of an infinite quiver, called the *repetitive quiver*, by the *mesh relations*. This will be our first step towards conquering the cluster category (of a simply laced Dynkin quiver).

Recall that the bounded derived category of a finite dimensional algebra is a Hom-finite triangulated category with the following properties:

- It is Krull-Schmidt (objects are finitely built from indecomposable ones in an essentially unique way).
- When the algebra is hereditary, the indecomposable objects are given by shifts of stalk complexes of indecomposable modules.
- For M and N modules, we have
$$\mathrm{Ext}^i(M, N) \cong \mathrm{Hom}(M, \Sigma^i N),$$
for i an integer and where Σ is the suspension functor.
- When the algebra is of finite global dimension, it has a Serre functor $\tau\Sigma$, where τ is the Auslander-Reiten translate.

Let us record the steps we will take in order to build the mesh category of a quiver:

1. Pick your quiver Q.
2. build $\mathbb{Z} \times Q$ by taking $\mathbb{Z}$ many disjoint copies of Q.
3. Build $\mathbb{Z}Q$ by lacing these disjoint copies of Q together with new arrows. We do this only from a lower copy of Q to the next higher copy.
4. We impose some relations on $\mathbb{Z}Q$.

Path category. Let $Q = (Q_0, Q_1)$ be a quiver. The *path category* $\mathsf{path}\, Q$ is the category with a set of objects Q_0 whose morphism vector spaces are

$$\mathsf{path}\, Q(i, j),$$

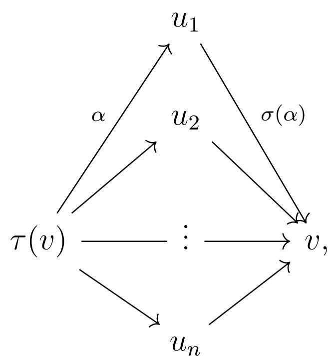
between objects i and j, has a k-basis given by all paths in Q starting at vertex i and ending at vertex j. The composition law in $\mathsf{path}\, Q$ is the unique k-bilinear composition given by concatenation of paths. The identity morphisms are induced by the lazy paths.

Repetitive quiver. Let Q be a simply laced Dynkin quiver. We define the *repetitive quiver* $\mathbb{Z}Q$ of Q as follows:

- The vertices are $\mathbb{Z}Q_0 = \{i_n \mid n \in \mathbb{Z}, i \in Q_0\}$.
- The arrows $\mathbb{Z}Q_1$ consist of two types. Type one: for each arrow $i \xrightarrow{a} j$ in Q, we have an arrow
$$i_n \xrightarrow{a_n} j_n$$
for each integer n. At this point we have built a quiver, one could denote $\mathbb{Z}Q$, that is $\mathbb{Z}$ many disjoint copies of our original quiver Q. Type two: for each arrow in type one $i_n \xrightarrow{a_n} j_n$, we have a connecting arrow
$$j_n \xrightarrow{\sigma(a_n)} i_{n+1}$$
for each integer n. At this point we have connected/laced together the disjoint quiver $\mathbb{Z} \times Q$.

We extend the assignment $a_n \mapsto \sigma(a_n)$ to an assignment on $\mathbb{Z}Q_1$ by defining $\sigma(\sigma(a_n)) = a_{n+1}$ and we endow $\mathbb{Z}Q$ with a quiver automorphism τ given by the action on vertices $\tau(i_n) = i_{n-1}$ and action on arrows $\tau(a_n) = a_{n-1}$.

Mesh relations. We will now impose relations on the path category of $\mathbb{Z}Q$. Let v be a vertex in $\mathbb{Z}Q$. The mesh ending at v is defined to be the full subquiver of $\mathbb{Z}Q$



which consists of vertices $v, \tau(v)$ and all vertices that are the source of an arrow with target v. The mesh endings generate an ideal of the path category of $\mathbb{Z}Q$. Namely, the two-sided ideal generated by the following *mesh relations*:

$$\rho_v = \sum_{\sigma(\alpha): s \to v} \sigma(\alpha) \cdot \alpha,$$

where the sum runs over all arrows with target v.

Definition 2.1. The *mesh category* $\mathsf{mesh}\, Q$ of Q is defined as the quotient of $\mathsf{path}(\mathbb{Z}Q)$ by the two-sided ideal generated by the mesh relations.

Theorem 2.2 (Happel)**.** *Let Q be a simply laced Dynkin quiver and kQ its path algebra.*

1. *There is a bijection*

$$\mathbb{Z}Q_0 \iff \{\textit{Isomorphism classes of indecomposable objects of } \mathsf{D}^{\mathrm{b}}(kQ)\},$$

which takes the vertex i_0 to the indecomposable projective kQ-module P_i considered as a stalk complex in degree 0.

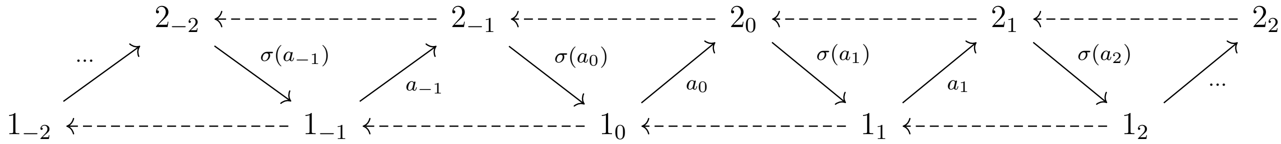
2. *Let* $\operatorname{ind}\mathsf{D}^{\mathrm{b}}(kQ)$ *be the full subcategory of* $\mathsf{D}^{\mathrm{b}}(kQ)$ *consisting of the indecomposable objects. The bijection of* 1. *lifts to an equivalence of categories*

$$\mathsf{mesh}\, Q \simeq \operatorname{ind}\mathsf{D}^{\mathrm{b}}(kQ).$$

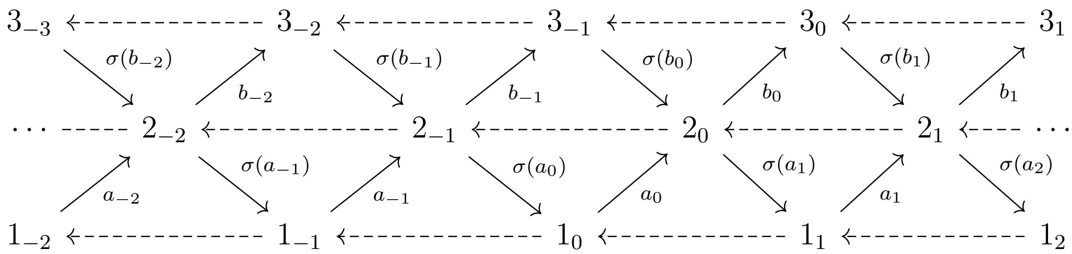
So, as the bounded derived category if Krull-Schmidt, the mesh category is a great approximation of the bounded derived category.

2.1. **Examples.** We present some examples of mesh categories below where we indicate the automorphism τ by leftwards pointing dashed arrows.

Example 2.3. $A_2\colon\ 1 \xrightarrow{a} 2$

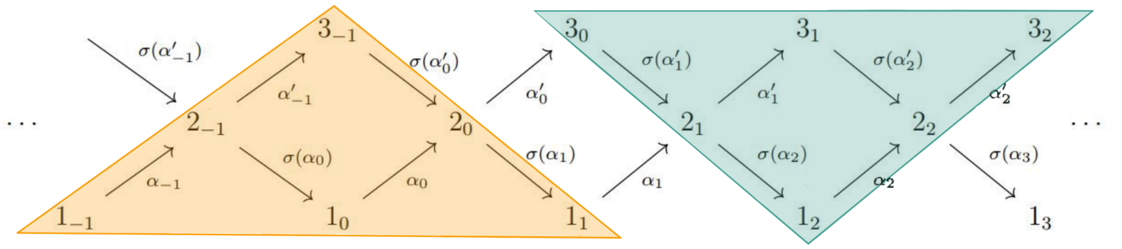


Example 2.4. $A_3\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$



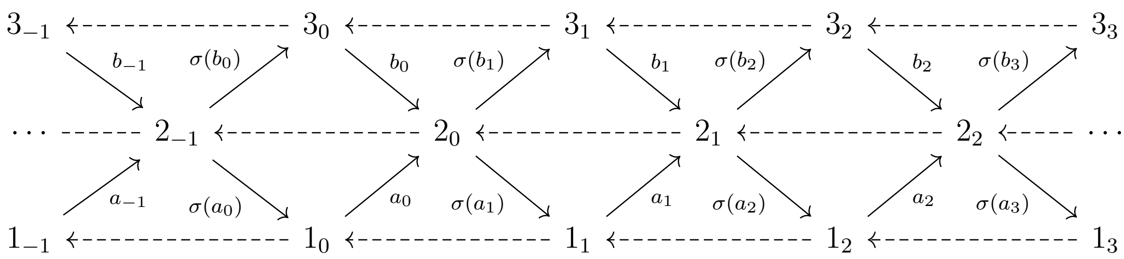
Notice that copies of the AR-quiver of $\mathsf{mod}(kA_3)$ partitions (and almost cover) the repetition quiver (see image below).

Recall at that the start of this section that we noted that extension groups where captured by the Hom spaces in the bounded derived category. This property is witnessed by the non-covered arrows connecting each partition. We depict this in the image below.



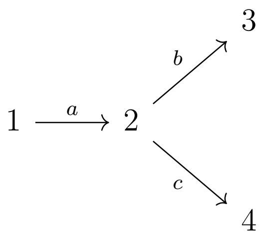
Remark 2.5. This partition plays nicely with the shift in the bounded derived category. So Σ takes one triangle to the next one to the right (as seen in the picture, orange is sent to green). So, the object $\Sigma(1_{-1}) = 3_0$ and $2_2 = \Sigma(2_0)$.

Example 2.6. $\tilde{A}_3$: $1 \xrightarrow{a} 2 \xleftarrow{b} 3$



Notice that the previous two examples convince us that the chosen orientations on the Dynkin quiver A_3 cannot be seen by the derived category. Try this next one out yourself.

Exercise 2.7. D_4:



3. The cluster category

Cluster categories are obtained as *orbit categories* of the bounded derived categories. We will not really define what this means. Instead, we will immediately brush it under the carpet and use what we have just learned instead. We will capture the cluster category via the mesh category.

Definition 3.1 (Buan-Marsh-Reineke-Reiten-Todorov, Caldero-Chapoton-Schiffler)**.** The cluster category $\mathcal{C}_Q$ is the orbit category

$$\mathcal{C}_Q = \mathsf{D}^{\mathrm{b}}(kQ)/\tau^{-1}\Sigma.$$

Whatever the above definition means, the cluster category enjoys the following properties:

- It is Hom-finite.

- It is triangulated (this is due to Keller) with the suspension functor, which we will also denote by Σ, that is isomorphic to τ (remembering that τ shifts objects one step to the left)
- The canonical functor $\mathsf{D}^{\mathrm{b}}(kQ) \to \mathcal{C}_Q$ is a triangulated functor.
- It is Krull-Schmidt.
- It is 2-Calabi Yau, which means it has a Serre functor given by Σ^2.

The category $\mathsf{mesh}\, Q$ has automorphisms Σ and τ. Imposing that objects in the same $\tau^{-1}\Sigma$-orbit are the same gives us a mesh category model for the cluster category:

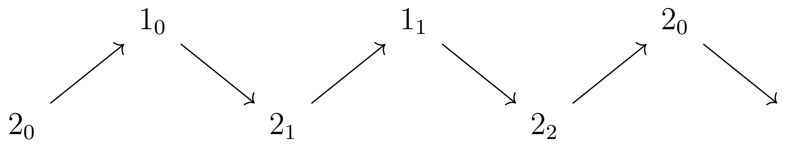
Theorem 3.2. *(Buan-Marsh-Reineke-Reiten-Todorov)*

Let Q be a simply laced Dynkin quiver and kQ its path algebra. There is an equivalence of categories

$$\mathsf{mesh}\, Q/\tau^{-1}\Sigma \simeq \operatorname{ind} \mathcal{C}_Q.$$

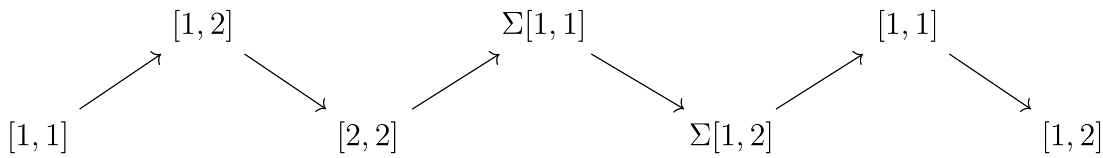
3.1. **Examples.** Recall that in the cluster category, the suspension Σ corresponds to moving one step left in the mesh diagram.

Example 3.3. For $A_2\colon\ 2 \xrightarrow{a} 1$, the cluster category $\mathcal{C}_{A_3}$ is approximated by the following quotient mesh category

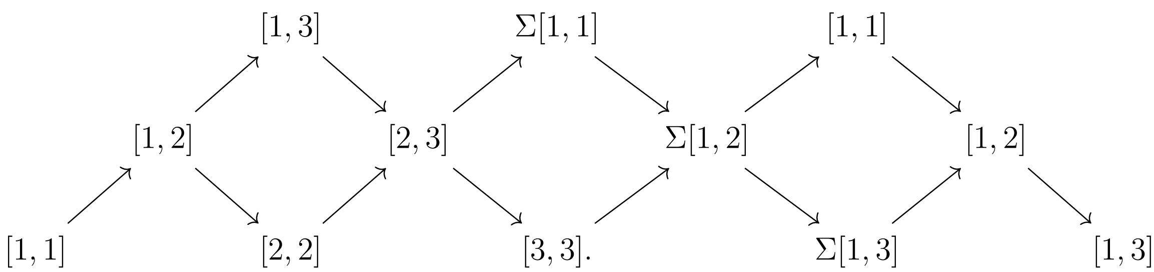


Notice that there is an identification so that this category lies on a Möbius strip.

The cluster category $\mathcal{C}_{A_2}$ can be seen as a splicing of $\mathsf{mod}(kA_2)$ with the shifted projective kA_2-modules. Recalling that modules over kA_2 can be represented by intervals, and in doing so, we have the following AR quiver:



Example 3.4. $A_3\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ Going straight to the idea that $\mathcal{C}_{A_3}$ is connected to $\mathsf{mod}(kA_3)$, we have the following AR quiver:



3.2. **Cluster structure.** In this section, we fix the number of vertices in Q to be n.

Cluster-tilting objects. An object $T = \bigoplus_{i=1}^{n} T_i$ (notice n here) in $\mathcal{C}_Q$ is a cluster tilting object if the following holds:

- Each T_i is indecomposable.
- $T_i \ncong T_j$ for $i \neq j$.
- $\mathrm{Hom}_{\mathcal{C}_Q}(T_i, \Sigma T_j) = 0$ for all $1 \leq i, j \leq n$.

There is always a nice cluster-tilting object which is given by the sum of all the indecomposable projective modules and we will use the combinatorics of triangulations of polygons to find them all!

Exercise. Show that the indecomposable projectives $[1,1] \oplus [1,2] \oplus [1,3]$ in the cluster category $\mathcal{C}_{A_3}$ is a cluster-tilting object.

Mutation. Let T be a cluster-tilting object in $\mathcal{C}_Q$ and write $T = \overline{T} \oplus X$, for X an indecomposable object in $\mathcal{C}_Q$. Then it is known that there exists a unique indecomposable object X^* in $\mathcal{C}_Q$ such that $X^* \ncong X$ and $\overline{T} \oplus X^*$ is a cluster tilting object. We say that $\overline{T} \oplus X^*$ is the mutation of T at X. Notice that if we mutate $\overline{T} \oplus X^*$ at X^*, by uniqueness, we get back T.

Exchange triangles. The above mutation comes equipped with two very nice triangles

$$X^* \xrightarrow{f} B \xrightarrow{g} X \longrightarrow \Sigma(X^*)$$

$$X \xrightarrow{f'} B' \xrightarrow{g'} X^* \longrightarrow \Sigma(X),$$

where the morphisms f, f', g and g' have nice minimal approximation properties.

4. Polygon presentation of cluster-type A_n

This section only works for cluster-type A_n (hence our focus on these examples before). Again, the orientation on arrows does not matter so we choose to work with

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n.$$

To the cluster category $\mathcal{C}_{A_n}$, we consider an $(n+3)$-gon (or a disk with $(n+3)$ marked points). We choose an anticlockwise cyclic ordering of the vertices from 1 to $(n+3)$. We define the following function from the indecomposables of $\mathcal{C}_{A_n}$ to the diagonals of the $(n+3)$-gon. We do this by identifying the indecomposables of $\mathcal{C}_{A_n}$ with the indecomposables of $\mathsf{mod}(kA_n)$ union the shifted projectives $\Sigma[1,k]$. So, for the indecomposables in $\mathsf{mod}(kA_n)$, we have

$$[i,j] \mapsto (i+1, j+3).$$

For the shifted projectives, we have

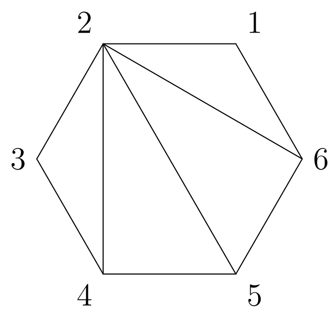
$$\Sigma[1,k] \mapsto (1, k+2).$$

Exercise. Show that the above assignment gives us a well-defined bijection from the indecomposables of $\mathcal{C}_{A_n}$ to the diagonals of the $(n+3)$-gon. This tells us the number of indecomposables of $\mathcal{C}_{A_n}$ is $\frac{1}{2}n(n-3)$.

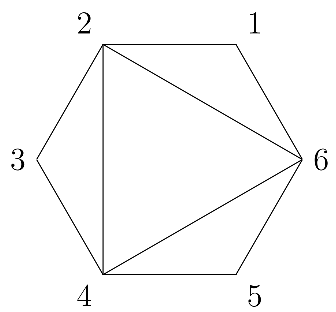
This bijection extends to a bijection between the cluster structure of $\mathcal{C}_{A_n}$ and the combinatorial structure of the $(n+3)$-gon. We summarise the relationships between cluster categories, cluster algebras and n-gons in the following table.

Cluster category $\mathcal{C}_{A_n}$	**Cluster algebra of type A_n**	**$(n+3)$-gon**
Indecomposable objects	Cluster variables	Diagonals
Cluster-tilting objects	Clusters	Triangulations
Mutation	Mutation	Flip
Exchange triangles	Exchange relations	Ptolemy relations

We asserted before that the direct sum of the indecomposable projectives $[1,1]\oplus[1,2]\oplus[1,3]$ in $\mathcal{C}_{A_3}$ gives rise to a cluster-tilting object. Under our bijection, this corresponds to the following triangulation:



Choosing a diagonal in this triangulation, there exists a unique other diagonal resulting in a new valid triangulation. This reflects the mutation of cluster-tilting we talked about before. So replacing the diagonal $(2,5)$ with the diagonal $(4,6)$ gives us



which corresponds to the cluster tilting object given by the direct sum of $[1,1], [1,3]$ and $[3,3]$.

Exercise. Find all cluster-tilting objects via triangulations and flips.

Last exercise. Find all the mistakes in these notes.

References

Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark.
Email address: david.nkansah@math.au.dk