∞-categories: Introduction

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∞-Categories - An Introduction

Literature: Original source

Lurie - Higher Topos Theory
—//— - Higher Algebra
- Spectral Algebraic Geometry

Do not recommend for first read, but amazing resource.

Recommend:

Hebestreit - Higher categories and algebraic K-theory (Book project)

Gnossen - Stable Homotopy Theory and Higher Algebra (Book project)

Motivation / Why use ∞-cats?

- Derived categories are a natural place for homological algebra.

Suppose R is a ring, M right module.

$$\mathrm{Tor}_i^R(M, \bullet) = L^i(M \otimes_R \bullet) \quad \text{... left derived functors}$$

Similarly for Ext-groups. (Co)-Homology also often defined as derived functors.

- Derived ∞-categories allow one to package all derived functors of a given exact functor together.

- Using ∞-categories, we have access to all the tools from ordinary category theory:

  (co)-limits, adjunctions, universal properties, Yoneda lemma, Kan-extensions, etc.

E.g. The ⊗ - Hom adjunction simply holds in ∞-cat land as well:

"All Tor-functors together are left adjoint to all Ext-functors"

- We can treat things like dg-algebras just like ordinary rings (including a working theory of algebra over them).

- This has two advantages
  - Results typically bundle a large amount of "higher" structure at once.
    - Sometimes this would be completely intractable doing by hand.
  - Everything done is automatically "homotopy coherent".
    - This reduces potential for mistakes.

Experimental fact:

- People tend to make mistakes when using simplicial indexing / homotopy groups
- Formal arguments tend to be easier to verify.

What "are" ∞-categories?

Warning: This is like asking what a real number is — One can define a real number as an equivalence class of Cauchy sequences, and prove elementary facts about them using this definition, but in practice one only ever uses basic properties

We will now give a model. The analogy is

real number	Cauchy sequence
∞-category	quasi category

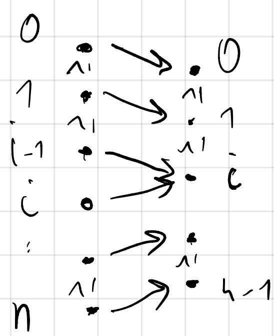
Quasi-groupoids and quasi-categories

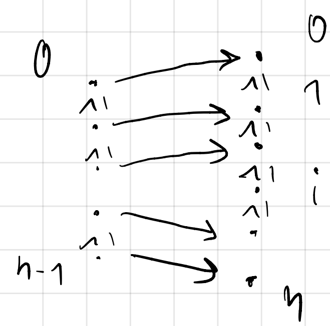
<u>Def:</u> Δ := category of non-empty, totally ordered sets and monotone functions

Write $[n] := \{0 \leq 1 \leq \ldots \leq n-1 \leq n\}$ for totally ordered set with $n+1$ elements.

Have two types of maps

$\sigma_i : [n] \to [n-1]$ $\qquad\qquad$ $\delta_i : [n-1] \to [n]$





collapses $i-1$ and i $\qquad\qquad$ misses i

A simplicial set $X_\bullet$ is a functor $X_\bullet : \Delta^{op} \to Set$ ← contravariant

Write $X_n := X_\bullet([n])$, $s_i := X_\bullet(\sigma_i)$, $d_i := X_\bullet(\delta_i)$

Equivalently, a diagram

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift right=2, "s_0"] \arrow[r, shift left=2, "s_1"'] & X_2 \arrow[l, shift right=5, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=5, "d_2"] & \ldots \end{tikzcd}$$

in Set

with certain identities between s_i and d_j.

Define $\quad sSet := \mathrm{Fun}(\Delta^{op}, Set)$ ... category of simplicial sets.

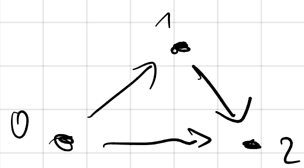
Write $\quad \Delta^n = \hom_\Delta(-, [n]) : \Delta^{op} \to Set$
for the standard n-simplex. By Yoneda lemma,
we have

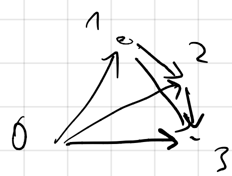
$$\mathrm{Hom}_{sSet}(\Delta^n, X_\bullet) = \mathrm{Nat}(\hom_\Delta(-,[n]), X_\bullet) = X_\bullet([n]) = X_n.$$

Pictures:

$\Delta^0 = \overset{0}{\bullet}$

$\Delta^1 = \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet}$

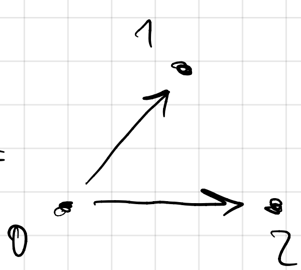
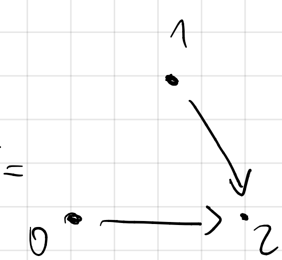
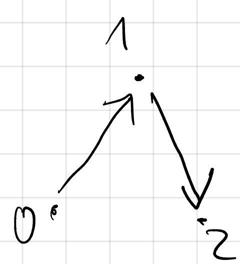
$\Delta^2 =$ 

$\Delta^3 =$ 

Δ^n has 1 n-cell and $n+1$ many $(n-1)$-cells.

Def: Fix $n > 0$. The i-th horn $\Lambda^n_i \subseteq \Delta^n$ for $0 \le i \le n$
is obtained by removing i-th $(n-1)$-cell
(and everything above it)

Ex: $n = 2$

$\Lambda^2_0 =$  $\quad \Lambda^2_1 =$  $\quad \Lambda^2_2 =$ 

Def: (Joyal). A simplicial set $\mathcal{C}_\bullet$ is called quasi-category if $\forall n \geq 0$, $0 < i < n$ and $f: \Lambda^n_i \to \mathcal{C}_\bullet$

strict!!

there exists extension:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C}_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

- A simplicial set $\mathcal{C}_\bullet$ is called quasi-groupoid (also Kan-complex) if same condition $\forall n \geq 0$ $0 \leq i \leq n$ holds.

  includes $i = 0$ and n.

- A morphism of simplicial sets $F: \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ with $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ quasicategories is called a functor

Def: If $\mathcal{C}_\bullet$ is quasi-category:

We call: $X, Y, \ldots \in \mathcal{C}_0$ ... objects of $\mathcal{C}_\bullet$

$f, g, \ldots \in \mathcal{C}_1$ ... morphisms/maps of $\mathcal{C}_\bullet$

$d_0(f) = $ source, $d_1(f) = $ target. Often write

$$X \xrightarrow{f} Y \in \mathcal{C}_\bullet$$

For $X \in \mathcal{C}_0$ have $id_X = s_0(X) \in \mathcal{C}_0$.
Notion of composition:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed, "\exists\, g\circ f"'] & & Z \end{tikzcd} \quad \text{Same as} \quad \begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Two maps $X \xrightarrow{f} Y$, $X \xrightarrow{f'} Y$ are called homotopic, $f \sim f'$, if there exists $\Delta^2 \to \mathcal{C}$ s.t.

$$\begin{tikzcd} & Y \arrow[d, equal, "id_Y"] \\ X \arrow[ur, "f"] \arrow[r, "f'"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & Y \end{tikzcd}$$

$\leadsto$ called homotopy

Call $X \xrightarrow{f} Y \in \mathcal{C}$ an equivalence if $\exists : Y \xrightarrow{g} X$ s.t. $id_X \sim g \circ f$ and $id_Y \sim f \circ g$

Exercise: $\mathcal{C}$ ordinary category.

- Then $N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$ is quasicategory.
- It is quasi groupoid iff $\mathcal{C}$ is a groupoid.
- There is a bijection

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}(N(\mathcal{C})_\bullet, N(\mathcal{D})_\bullet)$$

Subcategories

If $\mathcal{C}$ is ∞-cat, and
- $\mathcal{C}_0' \subseteq \mathcal{C}_0$ collection of objects
- $\mathcal{C}_1' \subseteq \mathcal{C}_1$ collection of morphisms

We can form $\mathcal{C}' \subseteq \mathcal{C}$ smallest sub simplicial set that is a quasicategory, and contains $\mathcal{C}_0'$ and $\mathcal{C}_0'$.
This is the subcategory generated from $\mathcal{C}_0'$, $\mathcal{C}_1'$.

Functor cats and equivalences of ∞-categories

Given two simplicial sets $X_\bullet$, $Y_\bullet$ can define

$$\mathrm{Map}(X_\bullet, Y_\bullet)_n := \mathrm{Map}(X_\bullet \times \Delta^n, Y_\bullet)$$

This is again simplicial set.
If $\mathcal{C}, \mathcal{D}$ quasicat, write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ instead

Prop: $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is again quasicat.
(only needs that $\mathcal{D}$ is quasicat.)

1-cells $\eta: F \Rightarrow G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ called natural transformations
If η equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ then natural equivalence.

Def: $F: \mathcal{C} \to \mathcal{D}$ is equivalence if $\exists G: \mathcal{D} \to \mathcal{C}$ and

$$id \xRightarrow{\sim} G \circ F \qquad id \xRightarrow{\sim} F \circ G$$

nat. equivalence

Comment: In practice, one often doesn't need to give G explicitly. Easier to check that F is essentially surjective + fully faithful.

Central principle of ∞-category theory.

Every statement made about objects of a given ∞-category must be invariant under equivalence.

Every statement made about an ∞-category must be invariant under equivalence.

If this is not the case, you're not studying ∞-cats, you're studying quasicategories.

Dwyer Kan Localization

There is a general method to construct ∞-cats.
Suppose $\mathcal{C}$ is quasicat, $W \subseteq \mathcal{C}_1$ set of morphisms.

Theorem: $\exists$ quasicat. $\mathcal{C}[W^{-1}]$ and a functor $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$
such that

- $\mathcal{C}[W^{-1}]$ has the same set of objects as $\mathcal{C}$.
- γ sends all morphisms in W to equivalences
- Let $\mathcal{D}$ be a quasicategory, then γ induces

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$$

equivalence

subcat. of functors F s.t. F sends morphisms in W to equivalences.

Remark: Up to equivalence, all quasi-cats arise this way from localizations of 1-categories (even posets)!

Examples

- The ∞-category of spaces / ∞-groupoids / anima is

$$\mathrm{Grpd}_\infty := N(\text{Quasigroupoids})[\text{Equivalences}^{-1}]$$

Remark: Often referred to as Spc, $\mathcal{S}$, or An.

- The ∞-category of ∞-categories is

$$\mathrm{Cat}_\infty := N(\text{Quasicategories})[\text{Equivalences}^{-1}]$$

Summarizing: We now have quite a collection of ∞-cats:

- Grpd_∞, Cat_∞, $N(\mathcal{C})$ for $\mathcal{C}$ a 1-category, $\mathrm{Fun}(X, \mathcal{C})$ for X simpl. set and $\mathcal{C}$ ∞-cat. Subcategories of these.

In the following session we will see dg-categories:
If $\mathcal{C}$ is differential graded cat. over some commutative ring k, then

category with 0-cycles

$$D(\mathcal{C}) = N(Z^0(\mathcal{C}))[W^{-1}]$$

quasi-isomorphisms

is derived ∞-category of $\mathcal{C}$.

Remark: There is an ongoing project by

Cisinski - Cnossen - Nguyen - Walde

"Formalization of Higher Categories"

(Link on Cnossen's Webpage).

They list Axiom A - 0 for "synthetic ∞-category theory"

These axioms are meant to be user-friendly and are essentially:

"Category theory + Straightening / Unstraightening"