

From gentle algebras to surfaces

§1. Representation theory of gentle algebras

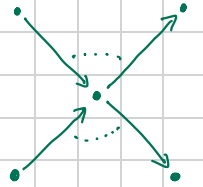
Reminder: The path algebra kQ of a quiver $(\hat{Q}, \text{directed graph})$ is the k vector space with paths in Q as basis and composition as multiplication.

$$\Rightarrow \text{mod-}kQ \cong \text{Fun}(Q^{\text{op}}, \text{vect}_k) \quad \text{and for } I \subseteq kQ \text{ ideal} \quad \text{mod-}kQ_I \cong \text{Fun}^I(Q^{\text{op}}, \text{vect}_k)$$

Definition: A (graded) algebra A is called gentle algebra if $A \cong kQ/I$ where

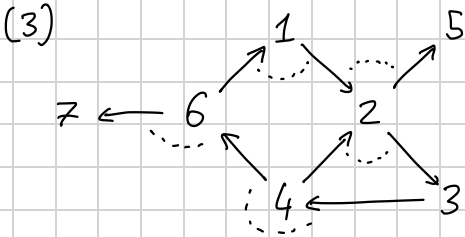
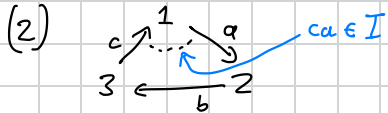
- (1) Q is a graded quiver
- (2) $I \subseteq kQ$ is generated by paths of length 2 and contains all paths of length $n, n \geq 0$
- (3) Every vertex has at most 2 incoming and at most 2 outgoing arrows
- (4) For every arrow $b \in Q$ the following sets are at most singletons

$$\{a \in Q_1 \mid 0 \neq ba \in I\} \quad \{a \in Q_1 \mid ba \notin I\} \quad \{c \in Q_1 \mid 0 \neq ca \in I\} \quad \{c \in Q_1 \mid ca \notin I\}$$

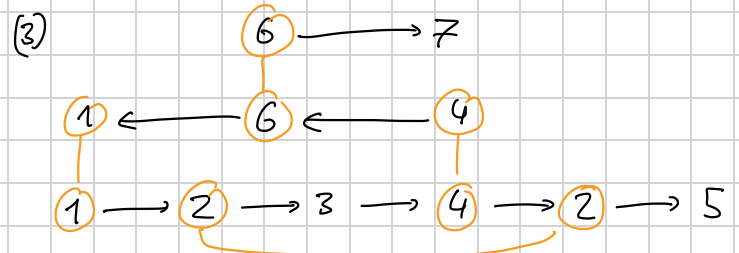
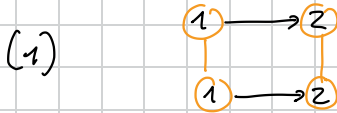


Example: (0) $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n$
linear A_n quiver

(1) Kronecker quiver $1 \begin{matrix} \xrightarrow{a} \\ \xleftarrow{b} \end{matrix} 2$



Remark: Gentle algebras are particularly nice glueings of linear $A_n \hat{=}$ maximal paths



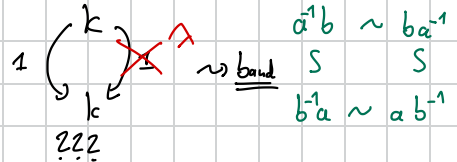
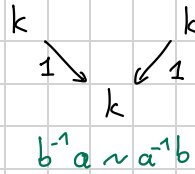
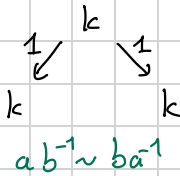
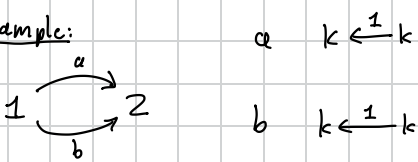
Fact: The indecomposable $\vec{\Lambda}_n$ representations are given by intervals $[i, j]$ $1 \leq i \leq j \leq n$

$$0 \leftarrow \dots \leftarrow 0 \leftarrow k \xleftarrow{1} k \xleftarrow{1} \dots \xleftarrow{1} k \leftarrow 0 \leftarrow \dots \leftarrow 0$$

i j n

Idea: Glue interval at their start points (respective end points) if there are corresponding "glueing points" to obtain indecomposable representations of a gentle algebra A

Example:



Definition: Let Q^\pm be the graded quiver obtained from Q by adding an inverse arrow a^{-1} for every $a \in Q_1$. We set $|a^{-1}| = -|a|$ and $(a^{-1})^{-1} := a$.

A string for $A \cong kQ/I$ is a walk $S = s_m \dots s_1$ in Q^\pm s.t. for $1 \leq i \leq n$ $s_{i+1} \neq s_i^{-1}$, $s_{i+1} s_i \notin I$, $s_i^{-1} s_{i+1}^{-1} \notin I$
 \rightsquigarrow We consider strings up to inversion $S \sim S^{-1}$

A band is a string $b = b_n \dots b_1$ s.t. b^2 is a string, $|b| = 0$ and b is not the power of another string
 ~~~ We consider bands up to inversion and rotation

Theorem [Gelfand-Ponomarev 1968, Ringel 1975] For an ungraded gentle algebra  $A$  there is a bijection

$$\text{ind}(\text{mod-}A) \longleftrightarrow \text{Strings} \amalg \left( \text{Bands} \times \underbrace{\text{ind}(\text{mod-}k[x^{\pm 1}])}_{\cong k^* \times \mathbb{N}_+ \text{ for } k=\bar{k}} \right)$$

Example:

$$1 \xrightarrow[q]{q} 2 \quad (ab^{-1}, \lambda, n) \mapsto k^n \xrightarrow[\amalg]{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} k^n \quad \triangle! \quad (ab^{-1}, 0, n) \mapsto k^n \xrightarrow[\amalg]{\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}} k^n$$

is the string module  $b^{-1}(ab^{-1})^{n-1}$

## §2. The derived category of a gentle algebra

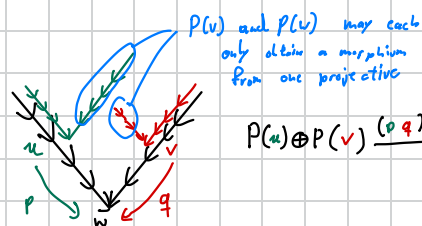
Reminder:  $D^b(A) \cong k^{+,b}(\text{proj } A) \rightsquigarrow$  We aim to understand complexes of projectives

Observation: For  $v \in Q_0$ ,  $P(v) = q^{-1}p$  for  $p$  and  $q$  the two possible paths going to  $v$  of minimal length (possibly  $q=e, w=p=e_v$ )

$$P(v) \mapsto \begin{array}{c} \swarrow q^{-1} \\ \searrow p \end{array}$$

(Yoneda) Lemma  $\text{Hom}_A(P(u), P(v)) \cong k \text{Path}^I(u, v)$

$\Rightarrow$  A complex of projectives may be described via a collection of paths and at most two paths may enter a projective without redundancy



$$P(u) \oplus P(v) \xrightarrow{(p, q)} P(w)$$

$\leadsto$  if  $u=w$  also  $k \oplus P(u) \xrightarrow{\text{non-trivial}} k \oplus P(u)$  possible  $\leadsto$  band!

Definition: A homotopy letter  $w = (p, n, \epsilon)$  consists of a path  $p$  in  $Q$  (which is a string), a shift  $n \in \mathbb{Z}$  and a direction  $\epsilon \in \{\pm 1\}$

A homotopy string is a sequence of homotopy letters  $w_n \dots w_1$ ,  $w_i = (p_i, n_i, \epsilon_i)$ , such that

- $\bullet \forall i \quad n_{i+1} = n_i + \epsilon_i (|p_i| + 1)$  (differential of degree -1)
- $\bullet \forall i \quad t(p_i^{\epsilon_i}) = s(p_{i+1}^{\epsilon_{i+1}})$  in  $Q^{\pm}$  (compatibility)
- $\bullet$  If  $\epsilon_{i+1} = -\epsilon_i$  then  $p_i^{\epsilon_i}$  and  $p_{i+1}^{\epsilon_{i+1}}$  do not end with the same arrow (non-redundancy)
- $\bullet$  If  $\epsilon_{i+1} = \epsilon_i$  then  $(p_{i+1}^{\epsilon_{i+1}} p_i^{\epsilon_i})^{\epsilon_i} \in I$  (differential  $^2 = 0$ )

A homotopy band is a homotopy string  $w_n \dots w_1$  which is not the power of another homotopy band such that  $w_n \dots w_1 w_n \dots w_1$  is a homotopy string and if  $\epsilon_i = \epsilon_j$  for  $1 \leq i, j \leq n$ , then one of the  $p_i$  is a walk of length at least 2.

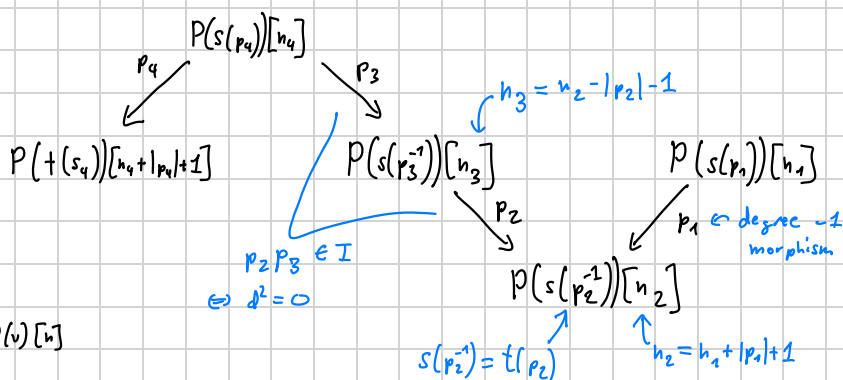
See Example 8 in the Appendix  $\leadsto$  otherwise nullhomotopic band complex

We set  $w^{-1} := (p, n + \epsilon(|p| + 1), -\epsilon)$  for the inverse homotopy letter and consider homotopy up to inversion and homotopy bands up to inversion and rotation.

Moreover, there are special homotopy strings  $(e_v, n)$  for  $v \in Q_0$  with  $n \in \mathbb{Z}$ .

Interpretation:  $(p_4, n_4, +1) (p_3, n_3, -1) (p_2, n_2, -1) (p_1, n_1, +1)$

We use homological notation as  $C[1]_n := C_{n+1}$



$$\text{Hom}(P(s(p_2))[n_2], P(t(s_1))[n_2 + 1]) = \text{Hom}(P(s(p_2)), P(t(s_1)[|p_1| + 1]))$$

$$(e_v, n) \cong P(v)[n]$$

# Theorem [Bekker-Merkle 2003, Oppen-Plamondon-Schroll 2025+]

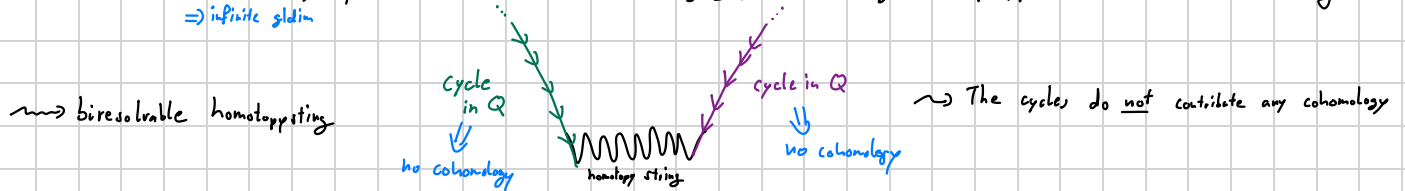
For a graded gentle algebra there is a bijection

$$\text{ind}(\text{lc}^b(\text{proj } A)) \longleftrightarrow \text{ho Strings} \amalg (\text{ho Band} \times \text{ind}(k[x^{\pm 1}]))$$

moreover, there is a bijection

$$\text{ind}(\text{D}^b(A) \setminus \text{K}^b(\text{proj } A)) \longleftrightarrow \left\{ \begin{array}{l} \text{primitive right resolvable} \\ \text{homotopy strings} \end{array} \right\} \amalg \left\{ \begin{array}{l} \text{primitive bi resolvable?} \\ \text{homotopy strings} \end{array} \right\} / \sim$$

Idea: If  $Q$  has an oriented cycle, we obtain an infinite  $\dots w_3 w_2 w_1$  which might be pre/post composed to a string  
 $\Rightarrow$  infinite glidin



## §3. The surface of a gentle algebra

From now on we assume  $Q$  has no isolated vertices

Definition: Let  $p$  be a maximal path for  $A$  a gentle algebra,  $p = a_1 \dots a_n$  with  $a_i: v_i \rightarrow v_{i+1}$

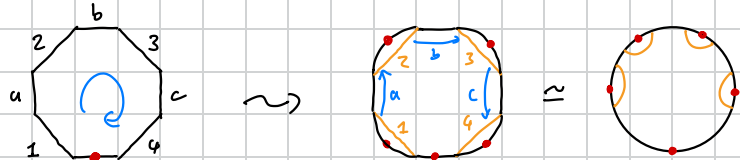
The marked polygon of  $p$  is the  $2(d+1)$ -gon whose edges are in clockwise order labeled by  $v_1, a_1, v_2, a_2, \dots, a_n, v_{n+1}$  with the final edge receiving a marking.

The marked surface  $S_A$  of a gentle algebra is obtained by gluing the polygons of the maximal paths of  $A$  orderreversing along their edges labeled by the same vertex.

Finally, attach to each vertex edge that was not glued a 2-gon with one marked edge and the other labeled by the vertex.

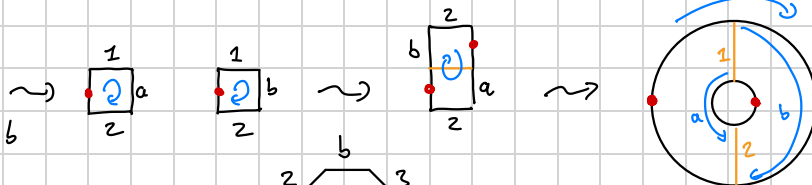
Example: (b)  $\vec{A}_4: 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$

$\leadsto$  one maximal path  $bca$

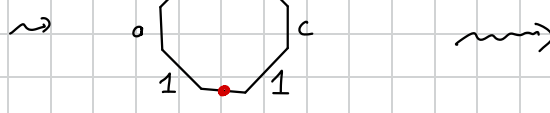


(1)  $1 \xrightarrow{a} 2 \xrightarrow{b} 1$

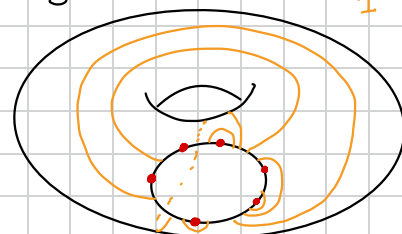
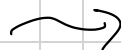
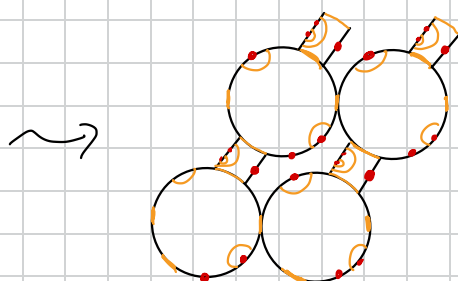
$\leadsto$  maximal paths  $a, b$



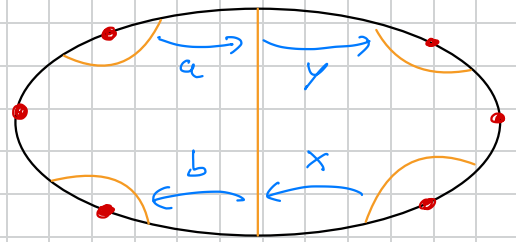
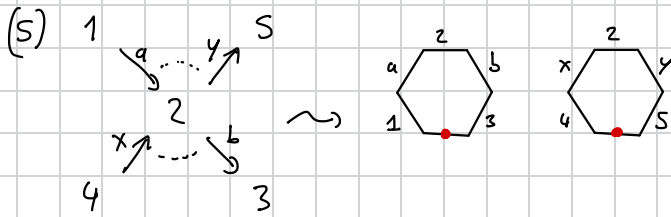
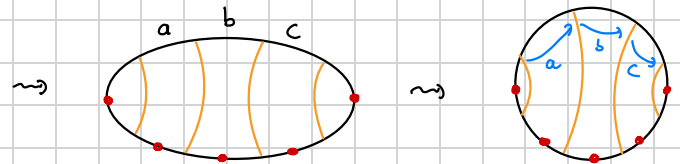
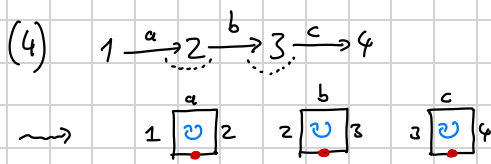
(2)  $1 \xrightarrow{c} 2 \xrightarrow{a} 3 \xrightarrow{b} 2 \xrightarrow{c} 1$



(3)  $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5 \xrightarrow{e} 6 \xrightarrow{f} 7 \xrightarrow{g} 8 \xrightarrow{h} 9 \xrightarrow{i} 10 \xrightarrow{j} 11 \xrightarrow{k} 12 \xrightarrow{l} 13 \xrightarrow{m} 14 \xrightarrow{n} 15 \xrightarrow{o} 16 \xrightarrow{p} 17 \xrightarrow{q} 18 \xrightarrow{r} 19 \xrightarrow{s} 20 \xrightarrow{t} 21 \xrightarrow{u} 22 \xrightarrow{v} 23 \xrightarrow{w} 24 \xrightarrow{x} 25 \xrightarrow{y} 26 \xrightarrow{z} 27 \xrightarrow{aa} 28 \xrightarrow{ab} 29 \xrightarrow{ac} 30 \xrightarrow{ad} 31 \xrightarrow{ae} 32 \xrightarrow{af} 33 \xrightarrow{ag} 34 \xrightarrow{ah} 35 \xrightarrow{ai} 36 \xrightarrow{aj} 37 \xrightarrow{ak} 38 \xrightarrow{al} 39 \xrightarrow{am} 40 \xrightarrow{an} 41 \xrightarrow{ao} 42 \xrightarrow{ap} 43 \xrightarrow{aq} 44 \xrightarrow{ar} 45 \xrightarrow{as} 46 \xrightarrow{at} 47 \xrightarrow{au} 48 \xrightarrow{av} 49 \xrightarrow{aw} 50 \xrightarrow{ax} 51 \xrightarrow{ay} 52 \xrightarrow{az} 53 \xrightarrow{ba} 54 \xrightarrow{bb} 55 \xrightarrow{bc} 56 \xrightarrow{bd} 57 \xrightarrow{be} 58 \xrightarrow{bf} 59 \xrightarrow{bg} 60 \xrightarrow{bh} 61 \xrightarrow{bi} 62 \xrightarrow{bj} 63 \xrightarrow{bk} 64 \xrightarrow{bl} 65 \xrightarrow{bm} 66 \xrightarrow{bn} 67 \xrightarrow{bo} 68 \xrightarrow{bp} 69 \xrightarrow{bq} 70 \xrightarrow{br} 71 \xrightarrow{bs} 72 \xrightarrow{bt} 73 \xrightarrow{bu} 74 \xrightarrow{bv} 75 \xrightarrow{bw} 76 \xrightarrow{bx} 77 \xrightarrow{by} 78 \xrightarrow{bz} 79 \xrightarrow{ca} 80 \xrightarrow{cb} 81 \xrightarrow{cc} 82 \xrightarrow{cd} 83 \xrightarrow{ce} 84 \xrightarrow{cf} 85 \xrightarrow{cg} 86 \xrightarrow{ch} 87 \xrightarrow{ci} 88 \xrightarrow{cj} 89 \xrightarrow{ck} 90 \xrightarrow{cl} 91 \xrightarrow{cm} 92 \xrightarrow{cn} 93 \xrightarrow{co} 94 \xrightarrow{cp} 95 \xrightarrow{cq} 96 \xrightarrow{cr} 97 \xrightarrow{cs} 98 \xrightarrow{ct} 99 \xrightarrow{cu} 100 \xrightarrow{cv} 101 \xrightarrow{cw} 102 \xrightarrow{cx} 103 \xrightarrow{cy} 104 \xrightarrow{cz} 105 \xrightarrow{da} 106 \xrightarrow{db} 107 \xrightarrow{dc} 108 \xrightarrow{dd} 109 \xrightarrow{de} 110 \xrightarrow{df} 111 \xrightarrow{dg} 112 \xrightarrow{dh} 113 \xrightarrow{di} 114 \xrightarrow{dj} 115 \xrightarrow{dk} 116 \xrightarrow{dl} 117 \xrightarrow{dm} 118 \xrightarrow{dn} 119 \xrightarrow{do} 120 \xrightarrow{dp} 121 \xrightarrow{dq} 122 \xrightarrow{dr} 123 \xrightarrow{ds} 124 \xrightarrow{dt} 125 \xrightarrow{du} 126 \xrightarrow{dv} 127 \xrightarrow{dw} 128 \xrightarrow{dx} 129 \xrightarrow{dy} 130 \xrightarrow{dz} 131 \xrightarrow{ea} 132 \xrightarrow{eb} 133 \xrightarrow{ec} 134 \xrightarrow{ed} 135 \xrightarrow{ee} 136 \xrightarrow{ef} 137 \xrightarrow{eg} 138 \xrightarrow{eh} 139 \xrightarrow{ei} 140 \xrightarrow{ej} 141 \xrightarrow{ek} 142 \xrightarrow{el} 143 \xrightarrow{em} 144 \xrightarrow{en} 145 \xrightarrow{eo} 146 \xrightarrow{ep} 147 \xrightarrow{eq} 148 \xrightarrow{er} 149 \xrightarrow{es} 150 \xrightarrow{et} 151 \xrightarrow{eu} 152 \xrightarrow{ev} 153 \xrightarrow{ew} 154 \xrightarrow{ex} 155 \xrightarrow{ey} 156 \xrightarrow{ez} 157 \xrightarrow{fa} 158 \xrightarrow{fb} 159 \xrightarrow{fc} 160 \xrightarrow{fd} 161 \xrightarrow{fe} 162 \xrightarrow{ff} 163 \xrightarrow{fg} 164 \xrightarrow{fh} 165 \xrightarrow{fi} 166 \xrightarrow{fj} 167 \xrightarrow{fk} 168 \xrightarrow{fl} 169 \xrightarrow{fm} 170 \xrightarrow{fn} 171 \xrightarrow{fo} 172 \xrightarrow{fp} 173 \xrightarrow{fq} 174 \xrightarrow{fr} 175 \xrightarrow{fs} 176 \xrightarrow{ft} 177 \xrightarrow{fu} 178 \xrightarrow{fv} 179 \xrightarrow{fw} 180 \xrightarrow{fx} 181 \xrightarrow{fy} 182 \xrightarrow{fz} 183 \xrightarrow{ga} 184 \xrightarrow{gb} 185 \xrightarrow{gc} 186 \xrightarrow{gd} 187 \xrightarrow{ge} 188 \xrightarrow{gf} 189 \xrightarrow{gg} 190 \xrightarrow{gh} 191 \xrightarrow{gi} 192 \xrightarrow{gj} 193 \xrightarrow{gk} 194 \xrightarrow{gl} 195 \xrightarrow{gm} 196 \xrightarrow{gn} 197 \xrightarrow{go} 198 \xrightarrow{gp} 199 \xrightarrow{gq} 200 \xrightarrow{gr} 201 \xrightarrow{gs} 202 \xrightarrow{gt} 203 \xrightarrow{gu} 204 \xrightarrow{gv} 205 \xrightarrow{gw} 206 \xrightarrow{gx} 207 \xrightarrow{gy} 208 \xrightarrow{gz} 209 \xrightarrow{ha} 210 \xrightarrow{hb} 211 \xrightarrow{hc} 212 \xrightarrow{hd} 213 \xrightarrow{he} 214 \xrightarrow{hf} 215 \xrightarrow{hg} 216 \xrightarrow{hh} 217 \xrightarrow{hh} 218 \xrightarrow{hi} 219 \xrightarrow{hj} 220 \xrightarrow{hk} 221 \xrightarrow{hl} 222 \xrightarrow{hm} 223 \xrightarrow{hn} 224 \xrightarrow{ho} 225 \xrightarrow{hp} 226 \xrightarrow{hq} 227 \xrightarrow{hr} 228 \xrightarrow{hs} 229 \xrightarrow{ht} 230 \xrightarrow{hu} 231 \xrightarrow{hv} 232 \xrightarrow{hw} 233 \xrightarrow{hx} 234 \xrightarrow{hy} 235 \xrightarrow{hz} 236 \xrightarrow{ia} 237 \xrightarrow{ib} 238 \xrightarrow{ic} 239 \xrightarrow{id} 240 \xrightarrow{ie} 241 \xrightarrow{if} 242 \xrightarrow{ig} 243 \xrightarrow{ih} 244 \xrightarrow{ii} 245 \xrightarrow{ii} 246 \xrightarrow{ij} 247 \xrightarrow{ik} 248 \xrightarrow{il} 249 \xrightarrow{im} 250 \xrightarrow{in} 251 \xrightarrow{io} 252 \xrightarrow{ip} 253 \xrightarrow{iq} 254 \xrightarrow{ir} 255 \xrightarrow{is} 256 \xrightarrow{it} 257 \xrightarrow{iu} 258 \xrightarrow{iv} 259 \xrightarrow{iw} 260 \xrightarrow{ix} 261 \xrightarrow{iy} 262 \xrightarrow{iz} 263 \xrightarrow{ja} 264 \xrightarrow{jb} 265 \xrightarrow{jc} 266 \xrightarrow{jd} 267 \xrightarrow{je} 268 \xrightarrow{jf} 269 \xrightarrow{jg} 270 \xrightarrow{jh} 271 \xrightarrow{ji} 272 \xrightarrow{ji} 273 \xrightarrow{jj} 274 \xrightarrow{jk} 275 \xrightarrow{jl} 276 \xrightarrow{jm} 277 \xrightarrow{jn} 278 \xrightarrow{jo} 279 \xrightarrow{jp} 280 \xrightarrow{jq} 281 \xrightarrow{jr} 282 \xrightarrow{js} 283 \xrightarrow{jt} 284 \xrightarrow{ju} 285 \xrightarrow{jv} 286 \xrightarrow{jw} 287 \xrightarrow{jx} 288 \xrightarrow{jy} 289 \xrightarrow{jz} 290 \xrightarrow{ka} 291 \xrightarrow{kb} 292 \xrightarrow{kc} 293 \xrightarrow{kd} 294 \xrightarrow{ke} 295 \xrightarrow{kf} 296 \xrightarrow{kg} 297 \xrightarrow{kh} 298 \xrightarrow{ki} 299 \xrightarrow{ki} 300 \xrightarrow{kl} 301 \xrightarrow{km} 302 \xrightarrow{kn} 303 \xrightarrow{ko} 304 \xrightarrow{kp} 305 \xrightarrow{kq} 306 \xrightarrow{kr} 307 \xrightarrow{ks} 308 \xrightarrow{kt} 309 \xrightarrow{ku} 310 \xrightarrow{kv} 311 \xrightarrow{kw} 312 \xrightarrow{kx} 313 \xrightarrow{ky} 314 \xrightarrow{kz} 315 \xrightarrow{la} 316 \xrightarrow{lb} 317 \xrightarrow{lc} 318 \xrightarrow{ld} 319 \xrightarrow{le} 320 \xrightarrow{lf} 321 \xrightarrow{lg} 322 \xrightarrow{lh} 323 \xrightarrow{li} 324 \xrightarrow{li} 325 \xrightarrow{lj} 326 \xrightarrow{lk} 327 \xrightarrow{ll} 328 \xrightarrow{lm} 329 \xrightarrow{ln} 330 \xrightarrow{lo} 331 \xrightarrow{lp} 332 \xrightarrow{lq} 333 \xrightarrow{lr} 334 \xrightarrow{ls} 335 \xrightarrow{lt} 336 \xrightarrow{lu} 337 \xrightarrow{lv} 338 \xrightarrow{lw} 339 \xrightarrow{lx} 340 \xrightarrow{ly} 341 \xrightarrow{lz} 342 \xrightarrow{ma} 343 \xrightarrow{mb} 344 \xrightarrow{mc} 345 \xrightarrow{md} 346 \xrightarrow{me} 347 \xrightarrow{mf} 348 \xrightarrow{mg} 349 \xrightarrow{mh} 350 \xrightarrow{mi} 351 \xrightarrow{mi} 352 \xrightarrow{mj} 353 \xrightarrow{mk} 354 \xrightarrow{ml} 355 \xrightarrow{mn} 356 \xrightarrow{mo} 357 \xrightarrow{mp} 358 \xrightarrow{mq} 359 \xrightarrow{mr} 360 \xrightarrow{ms} 361 \xrightarrow{mt} 362 \xrightarrow{mu} 363 \xrightarrow{mv} 364 \xrightarrow{mw} 365 \xrightarrow{mx} 366 \xrightarrow{my} 367 \xrightarrow{mz} 368 \xrightarrow{na} 369 \xrightarrow{nb} 370 \xrightarrow{nc} 371 \xrightarrow{nd} 372 \xrightarrow{ne} 373 \xrightarrow{nf} 374 \xrightarrow{ng} 375 \xrightarrow{nh} 376 \xrightarrow{ni} 377 \xrightarrow{ni} 378 \xrightarrow{nj} 379 \xrightarrow{nk} 380 \xrightarrow{nl} 381 \xrightarrow{nm} 382 \xrightarrow{nn} 383 \xrightarrow{no} 384 \xrightarrow{np} 385 \xrightarrow{nq} 386 \xrightarrow{nr} 387 \xrightarrow{ns} 388 \xrightarrow{nt} 389 \xrightarrow{nu} 390 \xrightarrow{nv} 391 \xrightarrow{nw} 392 \xrightarrow{nx} 393 \xrightarrow{ny} 394 \xrightarrow{nz} 395 \xrightarrow{oa} 396 \xrightarrow{ob} 397 \xrightarrow{oc} 398 \xrightarrow{od} 399 \xrightarrow{oe} 400 \xrightarrow{of} 401 \xrightarrow{og} 402 \xrightarrow{oh} 403 \xrightarrow{oi} 404 \xrightarrow{oi} 405 \xrightarrow{oj} 406 \xrightarrow{ok} 407 \xrightarrow{ol} 408 \xrightarrow{om} 409 \xrightarrow{on} 410 \xrightarrow{oo} 411 \xrightarrow{op} 412 \xrightarrow{op} 413 \xrightarrow{oq} 414 \xrightarrow{or} 415 \xrightarrow{os} 416 \xrightarrow{ot} 417 \xrightarrow{ou} 418 \xrightarrow{ov} 419 \xrightarrow{ow} 420 \xrightarrow{ox} 421 \xrightarrow{oy} 422 \xrightarrow{oz} 423 \xrightarrow{pa} 424 \xrightarrow{pb} 425 \xrightarrow{pc} 426 \xrightarrow{pd} 427 \xrightarrow{pe} 428 \xrightarrow{pf} 429 \xrightarrow{pg} 430 \xrightarrow{ph} 431 \xrightarrow{pi} 432 \xrightarrow{pi} 433 \xrightarrow{pj} 434 \xrightarrow{pk} 435 \xrightarrow{pl} 436 \xrightarrow{pm} 437 \xrightarrow{pn} 438 \xrightarrow{po} 439 \xrightarrow{pp} 440 \xrightarrow{pq} 441 \xrightarrow{pr} 442 \xrightarrow{ps} 443 \xrightarrow{pt} 444 \xrightarrow{pu} 445 \xrightarrow{pv} 446 \xrightarrow{pw} 447 \xrightarrow{px} 448 \xrightarrow{py} 449 \xrightarrow{pz} 450 \xrightarrow{qa} 451 \xrightarrow{qb} 452 \xrightarrow{qc} 453 \xrightarrow{qd} 454 \xrightarrow{qe} 455 \xrightarrow{qf} 456 \xrightarrow{qg} 457 \xrightarrow{qh} 458 \xrightarrow{qi} 459 \xrightarrow{qi} 460 \xrightarrow{qj} 461 \xrightarrow{qk} 462 \xrightarrow{ql} 463 \xrightarrow{qm} 464 \xrightarrow{qn} 465 \xrightarrow{qo} 466 \xrightarrow{qp} 467 \xrightarrow{qq} 468 \xrightarrow{qr} 469 \xrightarrow{qs} 470 \xrightarrow{qt} 471 \xrightarrow{qu} 472 \xrightarrow{qv} 473 \xrightarrow{qw} 474 \xrightarrow{qx} 475 \xrightarrow{qy} 476 \xrightarrow{qz} 477 \xrightarrow{ra} 478 \xrightarrow{rb} 479 \xrightarrow{rc} 480 \xrightarrow{rd} 481 \xrightarrow{re} 482 \xrightarrow{rf} 483 \xrightarrow{rg} 484 \xrightarrow{rh} 485 \xrightarrow{ri} 486 \xrightarrow{ri} 487 \xrightarrow{rj} 488 \xrightarrow{rk} 489 \xrightarrow{rl} 490 \xrightarrow{rm} 491 \xrightarrow{rn} 492 \xrightarrow{ro} 493 \xrightarrow{rp} 494 \xrightarrow{rq} 495 \xrightarrow{rr} 496 \xrightarrow{rs} 497 \xrightarrow{rt} 498 \xrightarrow{ru} 499 \xrightarrow{rv} 500 \xrightarrow{rw} 501 \xrightarrow{rx} 502 \xrightarrow{ry} 503 \xrightarrow{rz} 504 \xrightarrow{sa} 505 \xrightarrow{sb} 506 \xrightarrow{sc} 507 \xrightarrow{sd} 508 \xrightarrow{se} 509 \xrightarrow{sf} 510 \xrightarrow{sg} 511 \xrightarrow{sh} 512 \xrightarrow{si} 513 \xrightarrow{si} 514 \xrightarrow{sj} 515 \xrightarrow{sk} 516 \xrightarrow{sl} 517 \xrightarrow{sm} 518 \xrightarrow{sn} 519 \xrightarrow{so} 520 \xrightarrow{sp} 521 \xrightarrow{sq} 522 \xrightarrow{sr} 523 \xrightarrow{ss} 524 \xrightarrow{st} 525 \xrightarrow{su} 526 \xrightarrow{sv} 527 \xrightarrow{sw} 528 \xrightarrow{sx} 529 \xrightarrow{sy} 530 \xrightarrow{sz} 531 \xrightarrow{ta} 532 \xrightarrow{tb} 533 \xrightarrow{tc} 534 \xrightarrow{td} 535 \xrightarrow{te} 536 \xrightarrow{tf} 537 \xrightarrow{tg} 538 \xrightarrow{th} 539 \xrightarrow{ti} 540 \xrightarrow{ti} 541 \xrightarrow{tj} 542 \xrightarrow{tk} 543 \xrightarrow{tl} 544 \xrightarrow{tm} 545 \xrightarrow{tn} 546 \xrightarrow{to} 547 \xrightarrow{tp} 548 \xrightarrow{tq} 549 \xrightarrow{tr} 550 \xrightarrow{ts} 551 \xrightarrow{tt} 552 \xrightarrow{tu} 553 \xrightarrow{tv} 554 \xrightarrow{tw} 555 \xrightarrow{tx} 556 \xrightarrow{ty} 557 \xrightarrow{tz} 558 \xrightarrow{ua} 559 \xrightarrow{ub} 560 \xrightarrow{uc} 561 \xrightarrow{ud} 562 \xrightarrow{ue} 563 \xrightarrow{uf} 564 \xrightarrow{ug} 565 \xrightarrow{uh} 566 \xrightarrow{ui} 567 \xrightarrow{ui} 568 \xrightarrow{uj} 569 \xrightarrow{uk} 570 \xrightarrow{ul} 571 \xrightarrow{um} 572 \xrightarrow{un} 573 \xrightarrow{uo} 574 \xrightarrow{up} 575 \xrightarrow{uq} 576 \xrightarrow{ur} 577 \xrightarrow{us} 578 \xrightarrow{ut} 579 \xrightarrow{uu} 580 \xrightarrow{uv} 581 \xrightarrow{uw} 582 \xrightarrow{ux} 583 \xrightarrow{uy} 584 \xrightarrow{uz} 585 \xrightarrow{va} 586 \xrightarrow{vb} 587 \xrightarrow{vc} 588 \xrightarrow{vd} 589 \xrightarrow{ve} 590 \xrightarrow{vf} 591 \xrightarrow{vg} 592 \xrightarrow{vh} 593 \xrightarrow{vi} 594 \xrightarrow{vi} 595 \xrightarrow{vj} 596 \xrightarrow{vk} 597 \xrightarrow{vl} 598 \xrightarrow{vm} 599 \xrightarrow{vn} 600 \xrightarrow{vo} 601 \xrightarrow{vp} 602 \xrightarrow{vq} 603 \xrightarrow{vr} 604 \xrightarrow{vs} 605 \xrightarrow{vt} 606 \xrightarrow{vu} 607 \xrightarrow{vv} 608 \xrightarrow{vw} 609 \xrightarrow{vx} 610 \xrightarrow{vy} 611 \xrightarrow{vz} 612 \xrightarrow{wa} 613 \xrightarrow{wb} 614 \xrightarrow{wc} 615 \xrightarrow{wd} 616 \xrightarrow{we} 617 \xrightarrow{wf} 618 \xrightarrow{wg} 619 \xrightarrow{wh} 620 \xrightarrow{wi} 621 \xrightarrow{wi} 622 \xrightarrow{wj} 623 \xrightarrow{wk} 624 \xrightarrow{wl} 625 \xrightarrow{wm} 626 \xrightarrow{wn} 627 \xrightarrow{wo} 628 \xrightarrow{wp} 629 \xrightarrow{wq} 630 \xrightarrow{wr} 631 \xrightarrow{ws} 632 \xrightarrow{wt} 633 \xrightarrow{wu} 634 \xrightarrow{wv} 635 \xrightarrow{ww} 636 \xrightarrow{wx} 637 \xrightarrow{wy} 638 \xrightarrow{wz} 639 \xrightarrow{xa} 640 \xrightarrow{xb} 641 \xrightarrow{xc} 642 \xrightarrow{xd} 643 \xrightarrow{xe} 644 \xrightarrow{xf} 645 \xrightarrow{xg} 646 \xrightarrow{xh} 647 \xrightarrow{xi} 648 \xrightarrow{xi} 649 \xrightarrow{xj} 650 \xrightarrow{xk} 651 \xrightarrow{xl} 652 \xrightarrow{xm} 653 \xrightarrow{xn} 654 \xrightarrow{xo} 655 \xrightarrow{xp} 656 \xrightarrow{xq} 657 \xrightarrow{xr} 658 \xrightarrow{xs} 659 \xrightarrow{xt} 660 \xrightarrow{xu} 661 \xrightarrow{xv} 662 \xrightarrow{xw} 663 \xrightarrow{xx} 664 \xrightarrow{xy} 665 \xrightarrow{xz} 666 \xrightarrow{ya} 667 \xrightarrow{yb} 668 \xrightarrow{yc} 669 \xrightarrow{yd} 670 \xrightarrow{ye} 671 \xrightarrow{yf} 672 \xrightarrow{yg} 673 \xrightarrow{yh} 674 \xrightarrow{yi} 675 \xrightarrow{yi} 676 \xrightarrow{yj} 677 \xrightarrow{yk} 678 \xrightarrow{yl} 679 \xrightarrow{ym} 680 \xrightarrow{yn} 681 \xrightarrow{yo} 682 \xrightarrow{yp} 683 \xrightarrow{yq} 684 \xrightarrow{yr} 685 \xrightarrow{ys} 686 \xrightarrow{yt} 687 \xrightarrow{yu} 688 \xrightarrow{yv} 689 \xrightarrow{yw} 690 \xrightarrow{yx} 691 \xrightarrow{yy} 692 \xrightarrow{yz} 693 \xrightarrow{za} 694 \xrightarrow{zb} 695 \xrightarrow{zc} 696 \xrightarrow{zd} 697 \xrightarrow{ze} 698 \xrightarrow{zf} 699 \xrightarrow{zg} 700 \xrightarrow{zh} 701 \xrightarrow{zi} 702 \xrightarrow{zi} 703 \xrightarrow{zj} 704 \xrightarrow{zk} 705 \xrightarrow{zl} 706 \xrightarrow{zm} 707 \xrightarrow{zn} 708 \xrightarrow{zo} 709 \xrightarrow{zp} 710 \xrightarrow{zq} 711 \xrightarrow{zr} 712 \xrightarrow{zs} 713 \xrightarrow{zt} 714 \xrightarrow{zu} 715 \xrightarrow{zv} 716 \xrightarrow{zw} 717 \xrightarrow{zx} 718 \xrightarrow{zy} 719 \xrightarrow{zz} 720$



genus 2



Theorem [OPS 2025+, Haiden-Kontsevich-Kontsevich 2017, ...] There is a bijection

$$\{\text{graded gentle algebras}\} / \cong \longleftrightarrow \{\text{graded laminations of marked surfaces}\} / \text{homotopy}$$

Definition: A lamination of a marked surface  $(S, \mathcal{M})$  is a dissection of  $S$  into polygons such that every polygon contains exactly one point.  
A graded lamination is a lamination where every face of the polygon that is part of  $\partial S$  but not marked has an associated degree  $\in \mathbb{Z}$ .

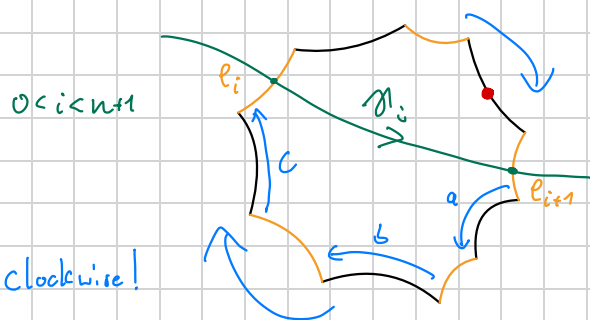
Remark:  $ba \in I$  in  $A \iff a$  and  $b$  are not in order in the same polygon of  $S_A$

Remark:  $A$  is not smooth  $\iff S_A$  has an unmarked boundary component  $\leadsto$  unbounded complexes

Observation: Let  $\gamma: [0, 1] \rightarrow S_A$  be a smooth curve in  $S_A$  for  $A$  a gentle algebra  $\Rightarrow \gamma$  is completely determined by its restriction to the polygons of  $S_A$

Wlog up to homotopy we can assume that  $\gamma$  intersects the lamination minimally  $\leadsto \gamma$  exits every polygon at a different edge than it entered

Let  $\gamma = \gamma_{n+1} \dots \gamma_0$  for  $\gamma_i$  the polygon segments



$\gamma_i$  defines homotopy letter

$$(cba, \overset{\text{grading}}{?}, -1)$$

$\leadsto \epsilon = \pm 1$  depending on the location of the marking relative the direction of  $\gamma$

Definition: A graded arc is an arc  $\gamma: [0, 1] \rightarrow S$ ,  $\gamma([0, 1]) \in \mathcal{M}$ ,  $\gamma([0, 1]) \subseteq \bar{S}$ , together with an assignment  $|p_i| \in \mathbb{Z}$  such that  $\forall i \in \{0, \dots, n\}$   $|p_{i+1}| = |p_i| + \epsilon_i (|p_i| + 1)$ ,  $\gamma = \gamma_{n+1} \dots \gamma_0$  intersecting the lamination at  $e_1, \dots, e_n$

$$\Rightarrow \forall i \in \{0, \dots, n\} \quad (p_{i+1}, |p_{i+1}|, \epsilon_{i+1}) (p_i, |p_i|, \epsilon_i) (p_{i-1}, |p_{i-1}|, \epsilon_{i-1}) \text{ homotopy string}$$

A graded closed curve  $\gamma: S^1 \rightarrow \bar{S}$  with a grading as above

A graded infinite arc is a ray  $\gamma: [0, 1) \rightarrow S$ ,  $\gamma(0) \in \mathcal{M}$ ,  $\gamma([0, 1)) \in \bar{S}$  circling infinitely often around an unmarked boundary component and a grading as above

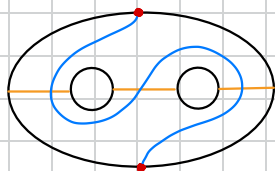
Theorem [OPS 2025+] For a gentle algebra  $A$  with surface model  $S_A$  there are bijections

homotopy strings  $\longleftrightarrow$  graded arcs in  $S_A$   
 homotopy bands  $\longleftrightarrow$  primitive graded closed curves in  $S_A$   
 primitive right resolvable strings  $\longleftrightarrow$  counter clockwise graded infinite arcs in  $S_A$   
 primitive biresolvable strings  $\longleftrightarrow$  counter clockwise graded doubly infinite arcs in  $S_A$

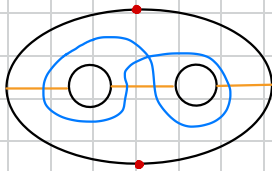
⚠ Not every closed curve can be graded

Example:

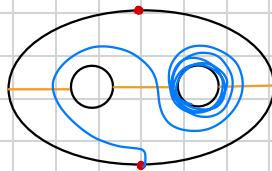
1  $\rightleftharpoons$  2  $\rightleftharpoons$  3



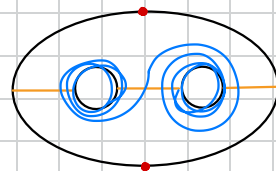
arc



primitive closed curve

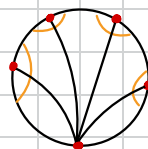


infinite arc



doubly infinite arc

Remark: If we connect two marked points with an arc if their polygons glue we get  $A \text{Ext}^1(A)$  up to shifts



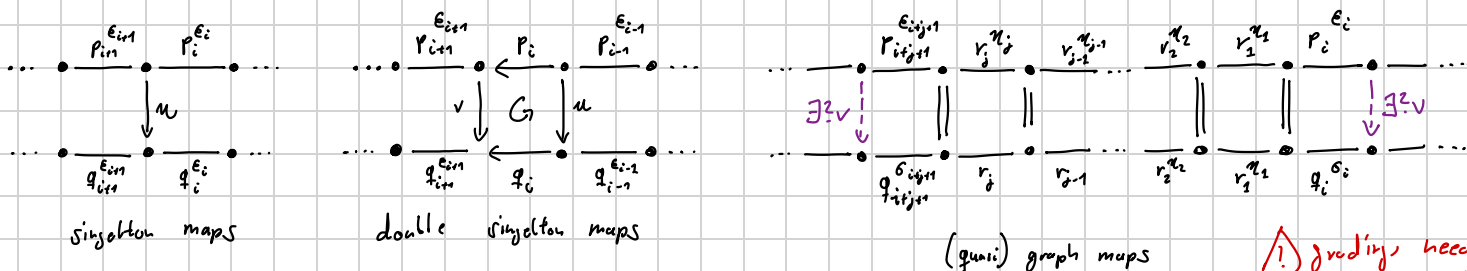
"ribbon graph of  $A$ "



## §4. Morphisms inside $\mathcal{D}^b(A)$

From now on all homotopy bands are considered with a 1-dim  $k[x^{\pm 1}]$ -module

There are 4 ways to create morphisms between homotopy bands as above



⚠ grading, need to match!

Theorem [Arnesen-Laking-Pauksztello 2016, OPS 2025+]

The above maps form a Schauder basis of the homomorphism space. We call this the standard basis  $\mathcal{B}$ .

Slogan: Morphisms in  $\mathcal{D}^b(A)$  are described by "meaningful overlap"!

$\rightsquigarrow$  In the surface model "meaningful overlap" occurs if the arcs/curves intersect

Theorem [OPS 2025+]

Consider  $(\pi_1, f_1)$  and  $(\pi_2, f_2)$  graded arcs or closed curves.

Then there is an injection  $(\pi_1, f_1) \xrightarrow{\pi_2} (\pi_2, f_2) \hookrightarrow \mathcal{B}$ .

"graded directed intersections"

It is bijective unless both are the same closed curve and  $f_2 = f_1$  or  $f_2 = f_1[1]$ .

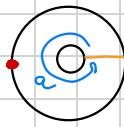
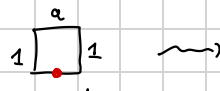
In these cases the identity respective the AR-sequence are the only missing basis elements.

Remark: Up to homotopy  $\pi_1$  and  $\pi_2$  are intersecting minimally and transversally.

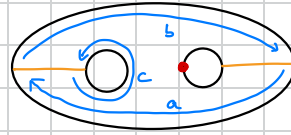
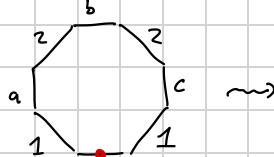
Remark: Intersections may be in the interior or at marked points

# Appendix: More examples

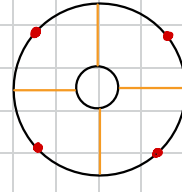
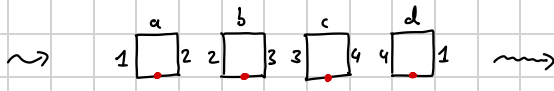
(6)  $1 \xrightarrow{a} 1 \rightsquigarrow$



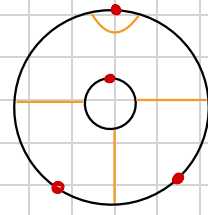
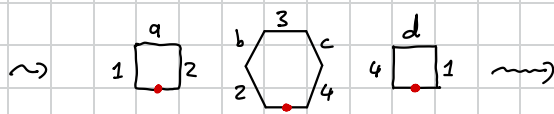
(7)  $1 \xrightarrow{a} 2 \xrightarrow{c} 1 \rightsquigarrow$



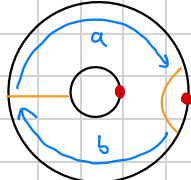
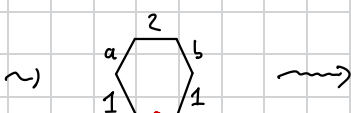
(8)  $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 1 \rightsquigarrow$



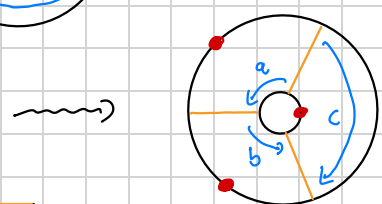
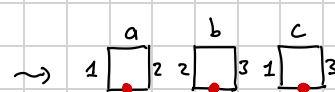
(9)  $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 1 \rightsquigarrow$



(10)  $1 \xrightarrow{a} 2 \xrightarrow{b} 1 \rightsquigarrow$



(11)  $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 1 \rightsquigarrow$



(12)  $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \rightsquigarrow$

