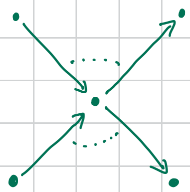
From gentle algebras to surfaces

§1. Representation theory of gentle algebras

Reminder: The path algebra kQ of a quiver ($\hat{=}$ directed graph) Q is the k vector space with paths in Q as basis and composition as multiplication

$$\Longrightarrow \quad \text{mod-}kQ \cong \mathrm{Fun}(Q^{op}, \mathrm{vect}_k) \quad \text{and for } I \subseteq kQ \text{ ideal} \quad \text{mod-}kQ/_I \cong \mathrm{Fun}^I(Q^{op}, \mathrm{vect}_k)$$

Definition: A (graded) algebra A is called gentle algebra if $A \cong kQ/_I$ where
(1) Q is a graded quiver
(2) $I \subseteq kQ$ is generated by paths of length 2 and contains all paths of length $n \gg 0$
(3) Every vertex has at most 2 incoming and at most 2 outgoing arrows
(4) For every arrow $b \in Q$ the following sets are at most singeltons
$\{a \in Q_1 \mid 0 \neq ba \in I\}$ $\{a \in Q_1 \mid ba \notin I\}$ $\{c \in Q_1 \mid 0 \neq ca \in I\}$ $\{c \in Q_1 \mid ca \notin I\}$



Example: (0) $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots \longrightarrow n$
linear $\vec{A}_n$ quiver

(1) Kroneker quiver $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$

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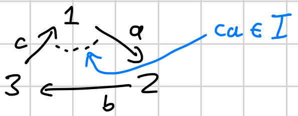

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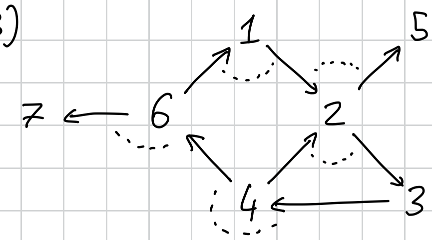

Remark: Gentle algebras are particularly nice glueings of linear A_n $\hat{=}$ maximal paths

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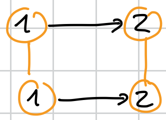

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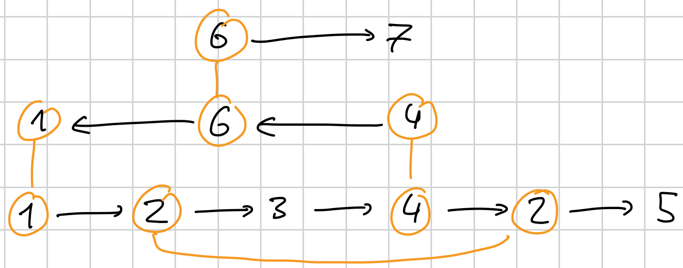
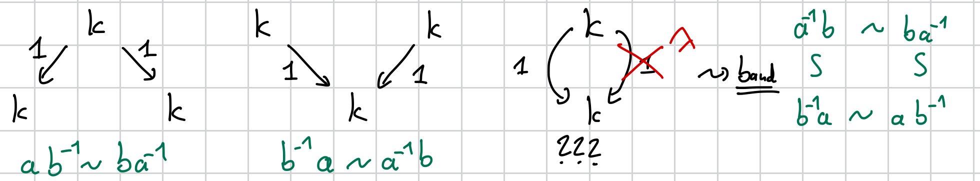


Fact: The indecomposable $\vec{A}_n$ representations are given by intervals $[i,j]$ $1 \le i \le j \le n$

$$\underset{1}{0} \leftarrow \cdots \leftarrow 0 \leftarrow \underset{i}{k} \xleftarrow{1} k \xleftarrow{1} \cdots \xleftarrow{1} \underset{j}{k} \leftarrow 0 \leftarrow \cdots \leftarrow \underset{n}{0}$$

Idea: Glue interval at their start points (respective end points) if there are corresponding "glueing points" to obtain indecomposable representations of a gentle algebra A

Example: $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$ $\qquad a \quad k \xleftarrow{1} k \qquad b \quad k \xleftarrow{1} k$



Definition: Let $Q^{\pm}$ be the graded quiver obtained from Q by adding an inverse arrow a^{-1} for every $a \in Q_1$. We set $|a^{-1}| = -|a|$ and $(a^{-1})^{-1} := a$.

A string for $A \cong kQ/_I$ is a walk $s = s_m \cdots s_1$ in $Q^{\pm}$ s.t. for $1 \le i < n$ $s_{i+1} \neq s_i^{-1}$, $s_{i+1}s_i \notin I$, $s_i^{-1}s_{i+1}^{-1} \notin I$
$\rightsquigarrow$ We consider strings up to inversion $s \sim s^{-1}$

A band is a string $b = b_n \ldots b_1$ s.t. b^2 is a string, $|b| = 0$ and b is not the power of another string
$\rightsquigarrow$ We consider bands up to inversion and rotation

Theorem [Gelfand–Ponomarev 1968, Ringel 1975] For an ungraded gentle algebra A there is a bijection

$$\operatorname{ind}(\operatorname{mod}-A) \longleftrightarrow \text{Strings} \amalg \big(\text{Bands} \times \underbrace{\operatorname{ind}(\operatorname{mod}-k[x^{\pm}])}_{\cong\, k^{\times} \times \mathbb{N}_+ \text{ for } k=\bar{k}}\big)$$

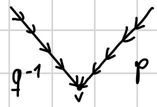
Example: $1 \underset{b}{\overset{a}{\rightrightarrows}} 2$ $\qquad (ab^{-1}, \lambda, n) \longmapsto$ $\begin{tikzcd} k^n \arrow[r, bend left, "{\begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}}"] \arrow[r, bend right, "\mathbb{1}"'] & k^n \end{tikzcd}$

$(ab^{-1}, 0, n) \longmapsto$ $\begin{tikzcd} k^n \arrow[r, bend left, "{\begin{pmatrix} 0 & 1 \\ & \ddots \\ & 0 \end{pmatrix}}"] \arrow[r, bend right, "\mathbb{1}"'] & k^n \end{tikzcd}$ is the string module $b^{-1}(ab^{-1})^{n-1}$

§2. The derived category of a gentle algebra

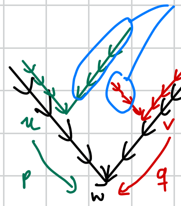
Reminder: $D^b(A) \cong K^{+,b}(\operatorname{proj} A) \rightsquigarrow$ We aim to understand complexes of projectives

Observation: For $v \in Q_0$, $P(v) = q^{-1}p$ for p and q the two possible paths going to v of maximal length (possibly $q = e_v$ or $p = e_v$)

$P(v) \longmapsto$ 

(Yoneda) Lemma $\operatorname{Hom}_A(P(u), P(v)) \cong k\operatorname{Path}^{\mathcal{I}}(u,v)$

$\Rightarrow$ A complex of projectives may be described via a collection of paths and at most two paths may enter a projective without redundancy



$P(u)$ and $P(v)$ may each only obtain a morphism from one projective

$P(u) \oplus P(v) \xrightarrow{(p\ q)} P(w)$

$\rightsquigarrow$ if $u = w$ also $k^n \otimes P(u) \xrightarrow{\mathbb{1}\otimes p - \varphi \otimes q} k^n \otimes P(w)$ possible $\rightsquigarrow$ band!

Definition: A homotopy letter $w = (p, n, \varepsilon)$ consists of a path $\overset{e_v}{\overset{\neq}{p}}$ in Q (which is a string), a shift $n \in \mathbb{Z}$ and a direction $\varepsilon \in \{\pm 1\}$

A <u>homotopy string</u> is a sequence of homotopy letters $w_n \cdots w_1$, $w_i = (p_i, n_i, \varepsilon_i)$, such that

- differential of degree -1 $\quad \bullet\ \forall i \quad n_{i+1} = n_i + \varepsilon_i(|p_i|+1)$
- compatibility $\quad \bullet\ \forall i \quad t(p_i^{\varepsilon_i}) = s(p_{i+1}^{\varepsilon_{i+1}})$ in $Q^{\pm}$
- non-redundancy $\quad \bullet$ If $\varepsilon_{i+1} = -\varepsilon_i$ then $p_i^{\varepsilon_i}$ and $p_{i+1}^{\varepsilon_i}$ do not end with the same arrow
- differential$^2 = 0$ $\quad \bullet$ If $\varepsilon_{i+1} = \varepsilon_{i+1}$ then $(p_{i+1}^{\varepsilon_i}\, p_i^{\varepsilon_i})^{\varepsilon_i} \in \mathcal{I}$

A <u>homotopy band</u> is a homotopy string $w_n \cdots w_1$ which is not the power of another homotopy band such that $w_n \cdots w_1 w_n \cdots w_1$ is a homotopy string and if $\varepsilon_i = \varepsilon_j$ for $1 \le i,j = n$, then one of the p_i is a walk of length at least 2.
See Example 8 in the Appendix
$\rightsquigarrow$ otherwise nullhomotopic band complex

We set $w^{-1} := (p, n + \varepsilon(|p|+1), -\varepsilon)$ for the inverse homotopy letter and consider homotopy up to inversion and homotopy bands up to inversion and rotation.

Moreover, there are special homotopy strings (e_v, n) for $v \in Q_0$ with $n \in \mathbb{Z}$.

Interpretation: $(p_4, n_4, +1)(p_3, n_3, -1)(p_2, n_2, -1)(p_1, n_1, +1)$

We use homological notation as $C[1]_n := C_{n+1}$

$$\begin{tikzcd}[column sep=small] & P(s(p_4))[n_4] \arrow[dl, "p_4"'] \arrow[dr, "p_3"] & & & \\ P(t(s_4))[n_4+|p_4|+1] & & P(s(p_3^{-1}))[n_3] \arrow[dr, "p_2"] & & P(s(p_1))[n_1] \arrow[dl, "p_1"'] \\ & & & P(s(p_2^{-1}))[n_2] & \end{tikzcd}$$

$n_3 = n_2 - |p_2| - 1$

$p_1 \leftarrow$ degree -1 morphism

$p_2 p_3 \in \mathcal{I}$
$\Leftrightarrow d^2 = 0$

$s(p_2^{-1}) = t(p_2)$

$n_2 = n_1 + |p_1| + 1$

$\operatorname{Hom}(P(s(p_1))[n_1], P(t(p_1))[n_2])$
$= \operatorname{Hom}(P(s(p_1)), P(t(p_1))[|p|+1])$

$(e_v, n) \quad \hat{=}\ P(v)[n]$

Theorem [Bekkert – Merklen 2003, Opper – Plamondon – Schroll 2025+]
For a graded gentle algebra there is a bijection

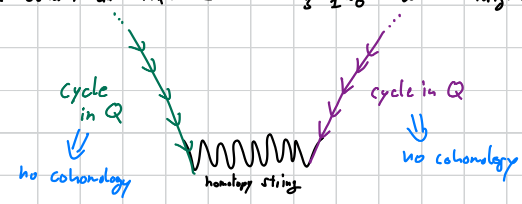
$$\mathrm{ind}\left(K^b(\mathrm{proj} A)\right) \longleftrightarrow \text{ho Strings} \amalg \left(\text{ho Band} \times \mathrm{ind}(k[x^{\pm 1}])\right)$$

Moreover, there is a bijection

$$\mathrm{ind}\left(D^b(A) \setminus K^b(\mathrm{proj} A)\right) \longleftrightarrow \left\{\begin{matrix}\text{primitive right resolvable} \\ \text{homotopy strings}\end{matrix}\right\} \amalg \left\{\begin{matrix}\text{primitive bi resolvable} \\ \text{homotopy strings}\end{matrix}\right\} \Big/ \sim$$

Idea: If Q has an oriented cycle, we obtain an infinite $\dots w_3 w_1 w_0$ which might be pre/post composed to a string
$\Rightarrow$ infinite gldim

$\rightsquigarrow$ biresolvable homotoppy string



$\rightsquigarrow$ The cycles do not contribute any cohomology

§3. The surface of a gentle algebra

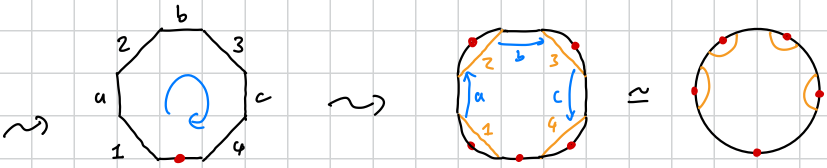
From now on we assume Q has no isolated vertices

Definition: Let p be a maximal path for A a gentle algebra, $p = a_d \cdots a_0$ with $a_i : v_i \to v_{i+1}$.
The marked polygon of p is the $2(d+1)$-gon whose edges are in clockwise order labeled by $v_1, a_1, v_2, a_2, \dots, a_d, v_{d+1}$ with the final edge recieving a marking.

The marked surface S_A of a gentle algebra is optained by glueing the polygons of the maximal paths of A orderrevering along their edges labeled by the same vertex.
Finally, attach to each vertex edge that was not glued a 2-gon with one marked edge and the other labeled by the vertex.

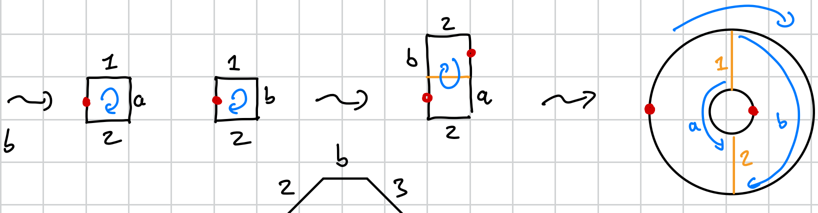
Example: (0) $\vec{A}_4$: $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$

$\rightsquigarrow$ one maximal path bca $\rightsquigarrow$



(1) $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$

$\rightsquigarrow$ maximal paths a, b



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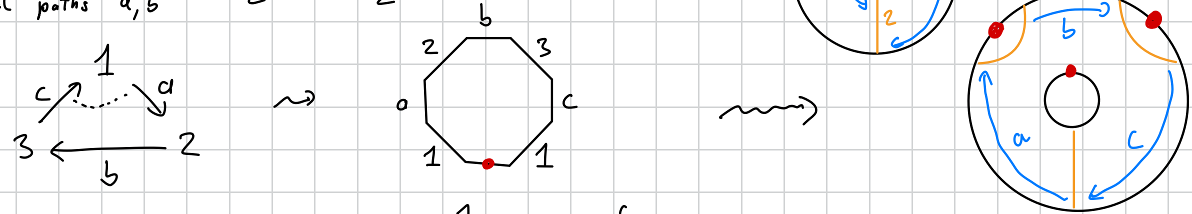

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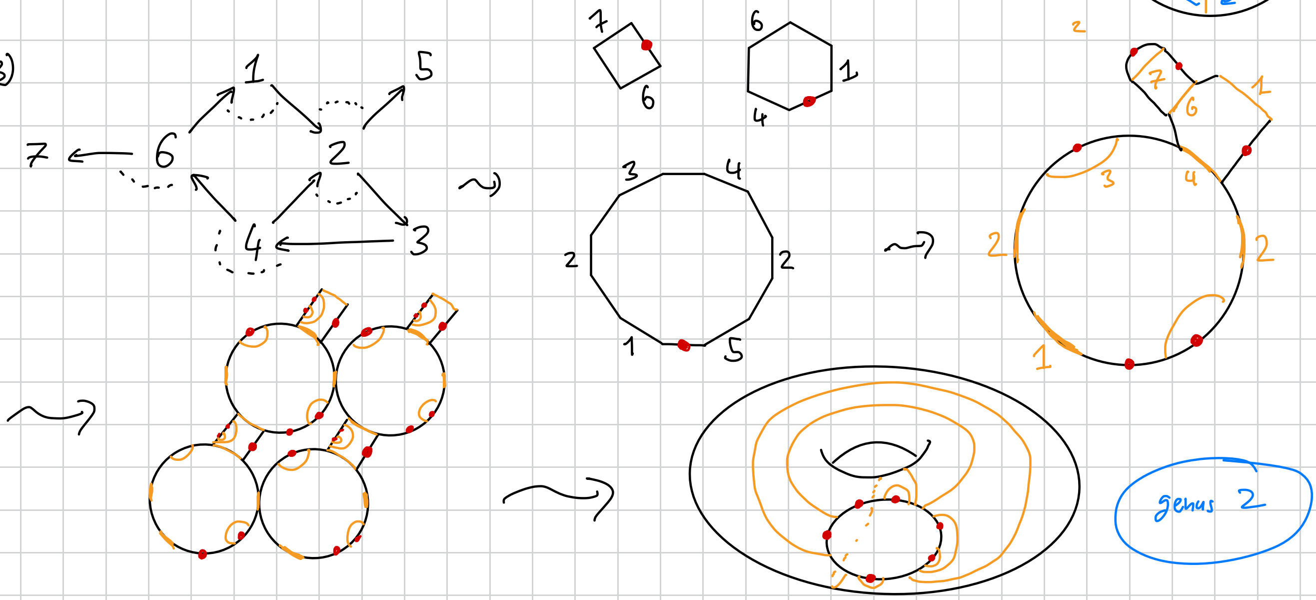
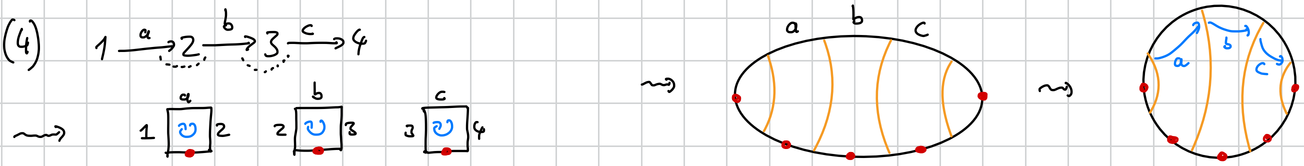


(4) $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$



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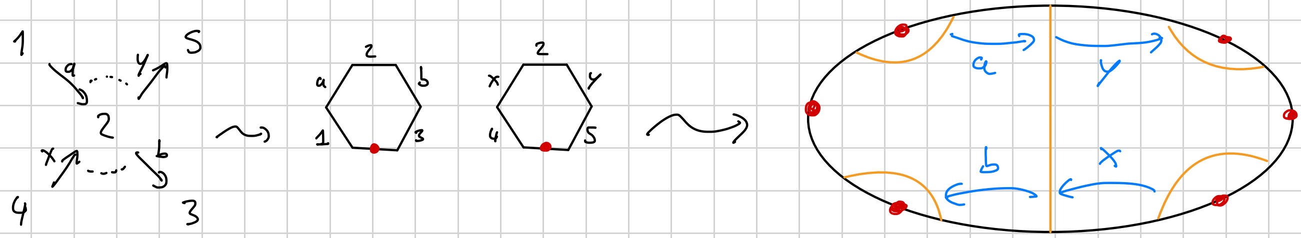


Theorem [OPS 2025+, Haiden–Katzarkov–Kontsevich 2017, ...] There is a bijection

$$\{\text{graded gentle algebras}\}/_{\cong} \longleftrightarrow \left\{\begin{matrix}\text{graded laminations}\\ \text{of marked surfaces}\end{matrix}\right\}/_{\text{homotopy}}$$

Definition: A Lamination of a marked surface $(S,\mathcal{M})$ is a disection of S into polygons such that every polygon contains exactly one point.
A graded lamination is a lamition where every face of the polygon that is part of ∂S but not marked has an associated degree $\in \mathbb{Z}$.

Remark: $ba \in I$ in A $\iff$ a and b are not in order in the same polygon of S_A

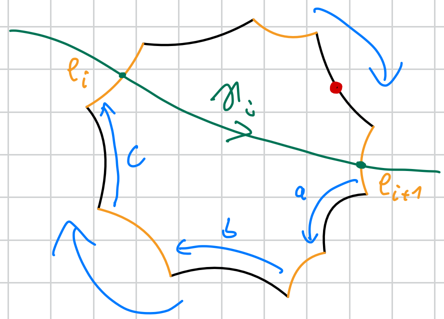
Remark: A is not smooth $\iff$ S_A has an unmarked boundary component $\rightsquigarrow$ unbounded complexes

Observation: Let $\gamma: [0,1] \longrightarrow S_A$ be a smooth curve in S_A for A a gentle algebra
$\implies \gamma$ is completly determined by its retstriction to the Polygons of S_A

Wlog up to homotopy we can assume that γ intersects the Lamination minimally
$\rightsquigarrow \gamma$ exits every polygon at a different edge than it entered

Let $\gamma = \gamma_{n+1} \cdots \gamma_0$ for γ_i the Polygon segments

$0 < i < n+1$



clockwise!

$\implies$ γ_i defines homotopy letter

$$(cba, \ ?, -1)$$

grading

$\rightsquigarrow$ $\epsilon = \pm 1$ depending on the location of the marking relative the direction of γ

Definition: A graded arc is an arc $\gamma: [0,1]: \gamma \longrightarrow S$, $\gamma(\{0,1\}) \in \mathcal{M}$, $\gamma((0,1)) \subseteq \mathring{S}$, $\gamma = \gamma_{n+1} \cdots \gamma_0$ intersecting the lamination at $e_1, \dots, e_n$
together with an assignment $|e_i| \in \mathbb{Z}$ such that $\forall 1 \leq i \leq n$ $|e_{i+1}| = |e_i| + \epsilon_i(|\gamma_i| + 1)$

$\implies \forall 1 < i < n$ $\quad (p_{i+1}, |e_{i+1}|, \epsilon_{i+1})(p_i, |e_i|, \epsilon_i)(p_{i-1}, |e_{i-1}|, \epsilon_{i-1})$ homotopy string

A graded closed curve $\gamma: S^1 \longrightarrow \mathring{S}$ with a grading as above

A graded infinite arc is a ray $\gamma: [0,1) \longrightarrow S$, $\gamma(0) \in \mathcal{M}$, $\gamma((0,1)) \subseteq \mathring{S}$ cirling infinetly often around an unmarked boundary component and a grading as above

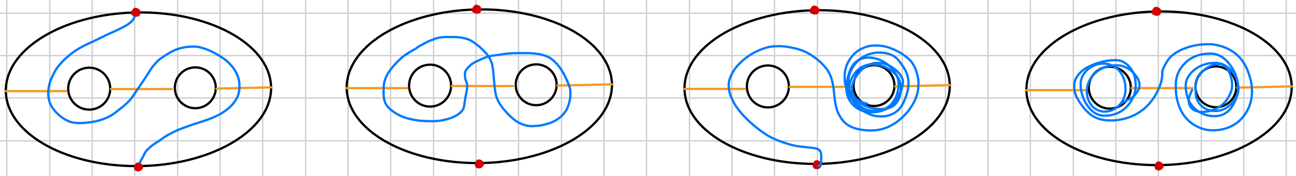
Theorem [OPS 2025+] For a gentle algebra A with surface model S_A there are bijections

homotopy strings	$\longleftrightarrow$	graded arcs in S_A
homotopy bands	$\longleftrightarrow$	primitive graded closed curves in S_A
primitive right resolvable strings	$\longleftrightarrow$	Counter clockwise graded infinite arcs in S_A
primitive biresolvable strings	$\longleftrightarrow$	Counter clockwise graded doubly infinite arcs in S_A

⚠ Not every closed curve can be graded

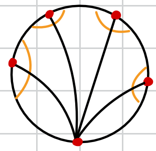
Example:

$1 \rightleftarrows 2 \rightleftarrows 3$

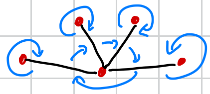


arc   primitive closed curve   infinite arc   doubly infinite arc

Remark: If we connect two marked points with an arc if their polygons glue we get $A \in \mathcal{D}^b(A)$ up to shifts



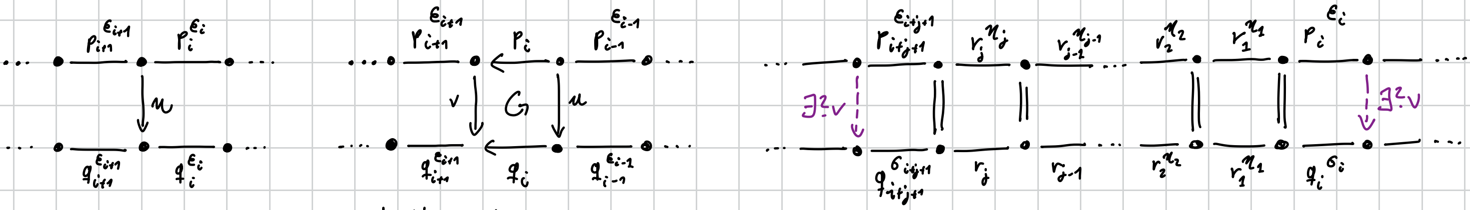
"ribbon graph of A"



§4. Morphisms inside $\mathcal{D}^b(A)$

From now on all homotopy bands are considered with a 1-dim $k[x^{\pm 1}]$-module

There are 4 ways to create morphisms between homotopy bands as above



singleton maps   double singleton maps   (quasi) graph maps

⚠ grading need to match!

Theorem [Arnesen–Laking–Paukstello 2016, OPS 2025+]
The above maps form a Schauder basis of the homomorphism space. We call this the standard basis $\mathcal{B}$.

Slogan: Morphisms in $\mathcal{D}^b(A)$ are described by "meaningful overlap"!

$\rightsquigarrow$ In the surface model "meaningful overlap" occours if the arcs/curves intersect

Theorem [OPS 2025+]
Consider (γ_1, f_1) and (γ_2, f_2) graded arcs or closed curves.
Then there is an injection $(\gamma_1, f_1) \vec{\cap}_{gr} (\gamma_2, f_2) \hookrightarrow \mathcal{B}$.

"graded directed intersections"

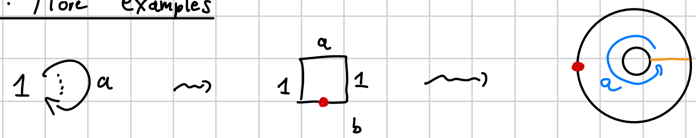
It is bijective unless both are the same closed curve and $f_2 = f_1$ or $f_2 = f_1[1]$.
In these cases the identity respective the AR-sequence are the only missing bases elements.

Remark: Up to homotopy γ_1 and γ_2 are intersecting minimally and transversally.

Remark: Intersections may be in the interior or at marked points

Appendix: More examples

(6)



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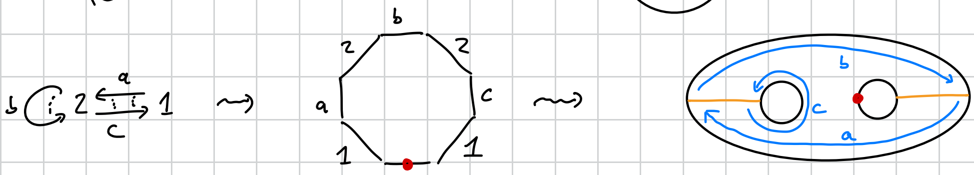

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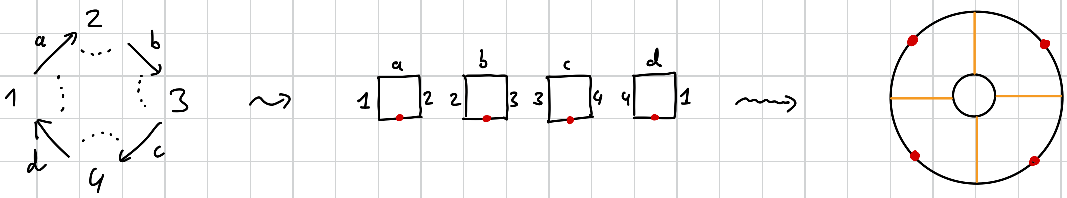

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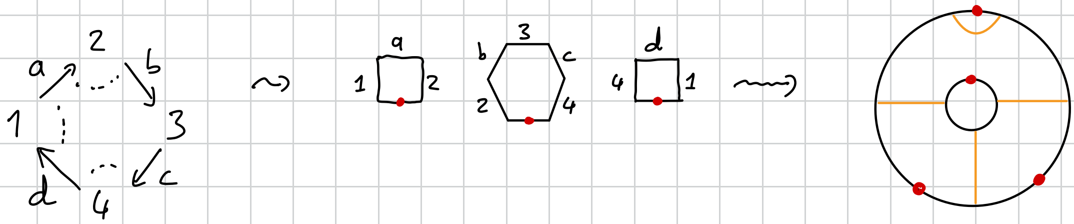

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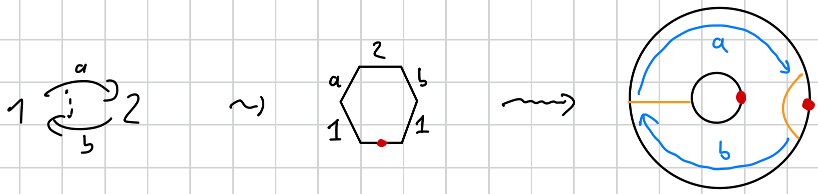

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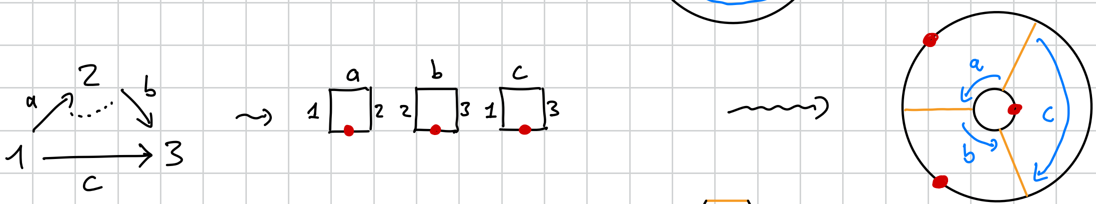

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