Preliminary-Preliminary Slides: DG-Algebras

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Introduction

In the talk, I introduce DG-algebras as DG-categories with one object. Here is a more detailed write-up and basic facts about them.

References

- X. Chen and X.-W. Chen, "An Informal Introduction to DG Categories," 2021. https://arxiv.org/abs/1908.04599
- H. Krause, *Homological Theory of Representations*. Cambridge: Cambridge University Press, 2021.

Notations

Unless stated otherwise:

- k: A perfect field. All categories are additive, k-linear. All morphisms are k-module (k-vector space) morphisms.
- All gradings are over $\mathbb{Z}$, $\otimes$ means $\otimes_k$.
- Modules are right modules.
- "Complexes" means cochain complexes of k-modules, and we index cohomologically.

Gradings

Recall a *graded vector space* is a direct sum decomposition $A = \bigoplus_{i \in \mathbb{Z}} A^i$. An element $a \in A$ is *homogeneous of degree* n if $a \in A^n$ and we write $|a| = n$.

Given two graded vector spaces A, B, we say a morphism $f : A \to B$ is *homogeneous with degree* n if $f(A^k) \subseteq B^{k+n}$ for all $k \in \mathbb{Z}$ and write $|f| = n$.

In this talk, we assume all morphisms of graded vector spaces are homogeneous, and we generally only consider homogeneous elements in a vector space. So a "degree n" map means homogeneous of degree n.

DG-algebra (1)

DG-algebras stand for "**D**ifferentially **G**raded" algebras.

Definition

A **DG-(k-)algebra** is a k-algebra A with grading $A = \bigoplus_{i \in \mathbb{Z}} A^i$ and "differential" $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$, $A^i A^j \subseteq A^{i+j}$, and the following (called the *Leibniz Rule*) is satisfied:

$$d^{i+j}(ab) = d^i(a)b + (-1)^i a d^j(b) \quad \text{for any } a \in A^i, b \in A^j.$$

Or, put it more compactly, $d : A \to A$ is degree 1 with:

$$d(ab) = d(a)b + (-1)^{|a|} a d(b) \quad \text{for any } a, b \in A.$$

DG-algebra (2)

Definition

A **morphism** between two DG-algebras $(A, d_A), (B, d_B)$ is a degree 0 algebra morphism $f : A \to B$ with $d_B \circ f = f \circ d_A$.

Example

Any ordinary k-algebra is a DG-algebra, with the grading concentrated at degree 0, and the map d being identically zero.

Examples of DG-algebra

Fix a DG-algebra A.

Example

The opposite algebra of a DG-algebra A is A^{op}, with multiplication

$$a \cdot_{op} b := (-1)^{|a||b|} b \cdot a.$$

We can think of DG-algebras as an algebra and a complex at the same time.

Example

The shift $[1] : A \to A$ is a morphism of DG-algebras.

The Endomorphism DG Algebra

Example

Consider a complex X of k-modules, define the **endomorphism DG Algebra of** X $\mathcal{E}nd_k(X)$. For the grading, at degree n, we define:

$$\mathcal{E}nd_k(X)^n := \prod_{p \in \mathbb{Z}} \mathrm{Hom}_k(X^p, X^{p+n})$$

with differential

$$d^n(\phi^p) = d_X \circ \phi^p - (-1)^n \phi^{p+1} \circ d_X.$$

Where the multiplicative algebra structure is the composition of maps.

Note that this motivates our definition of the DG-category $\mathsf{C}_{dg}(k)$.

Modules over DG-Algebras

Definition

A **DG** A**-module** is an A-module X with grading $X = \bigoplus_{i \in \mathbb{Z}} X^i$ with a homogeneous degree 1 map d such that $X^i A^j \subseteq X^{i+j}$ for all $i, j \in \mathbb{Z}$, and

$$d(xa) = d(x)a + (-1)^{|x|} x d(a) \quad \text{for any } x \in X, a \in A.$$

A **morphism** between DG A-modules (X, d_X) and (Y, d_Y) is a degree 0 A-module morphism $f : X \to Y$ such that $d_Y \circ f = f \circ d_X$.

Modules over DG-Algebras (2)

Example

Similar to $\mathcal{E}nd_k(X)$, for k-module complexes X, Y, define $\mathcal{H}om_k(X,Y)^n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_k(X^p, Y^{p+i})$ and differential at degree i as

$$d^i(\phi^p) = d_Y \circ \phi^p - (-1)^i \phi^{p+1} \circ d_X.$$

Then $\mathcal{H}om_k(X,Y)$ is a DG $\mathcal{E}nd_k(X)$-module.

Definition

Denote $\mathrm{dgMod}_{\mathrm{dg}}(A)$ as the category of DG A-modules. This is an abelian category with all limits and colimits.

Homological Algebra in DG-algebras and Modules

We have some familiar functors from homological algebra:

Example

1. The shift functor: $[1] : \mathsf{dgMod}_{\mathrm{dg}}(A) \to \mathsf{dgMod}_{\mathrm{dg}}(A)$.
2. The forgetful functor $U : \mathsf{dgMod}_{\mathrm{dg}}(A) \to \mathrm{Ch}(k)$, which sees a DG A-module (X, d_X) as a complex.

We can do homological algebra on A-modules as they are complexes. Given an A-module X, we can define $H^n(X)$, quasi-isomorphisms, whether X is acyclic, projective, etc. We can also form resolutions and the derived category $\mathsf{D}(A)$.