Preliminary Talk: DG-categories and Derived Categories

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Introduction

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DG-algebras and their generalisation - DG-categories, is perhaps a more "rudimentary" way to see triangulated categories, and homological properties are expressed more naturally. We will try to include an overview of the definitions and mention some key properties, examples, and a result by Keller which connects DG-categories to triangulated categories as promised.

References

Mainly adapted from Section 3.1 of G. Jasso and F. Muro (With an appendix by B. Keller), "The Derived Auslander-Iyama Correspondence," 2023. https://arxiv.org/abs/2208.14413

Other useful references:

- X. Chen and X.-W. Chen, "An Informal Introduction to DG Categories," 2021. https://arxiv.org/abs/1908.04599
- B. Keller, "Deriving DG categories," *Annales scientifiques de l'École Normale Supérieure*, vol. Ser. 4, 27, Art. no. 1, 1994, doi: https://doi.org/10.24033/asens.1689.
- B. Keller, "On differential graded categories," 2006. https://arxiv.org/abs/math/0601185

Notations

Unless stated otherwise:

- k: A perfect field. All categories are additive, k-linear. All morphisms are k-module (k-vector space) morphisms.
- All gradings are over $\mathbb{Z}$.
- Modules are right modules.
- "Complexes" means cochain complexes of k-modules, and we index cohomologically.

Table of Contents

Gradings

Recall a *graded vector space* is a direct sum decomposition $A = \bigoplus_{i \in \mathbb{Z}} A^i$. An element $a \in A$ is *homogeneous of degree* n if $a \in A^n$ and we write $|a| = n$.

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Given two graded vector space A, B, we say a morphism $f : A \to B$ is *homogeneous with degree* n if $f(A^k) \subseteq B^{k+n}$ for all $k \in \mathbb{Z}$ and write $|f| = n$.

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In this talk, we assume all morphisms of graded vector space is homogeneous, and we generally only consider homogeneous elements in a vector space. So a "degree n" map means homogeneous of degree n.

DG-categories

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That is, this category is specified by:

- Objects: $x, y, ... \in \mathrm{Ob}(\mathcal{A})$.
- Morphisms: The hom-set between objects x, y is a complex $\mathcal{A}(x, y)^{\bullet}$. If $f \in \mathcal{A}(x, y)^n$, we write $|f| = n$.

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- Morphisms: The hom-set between objects x, y is a complex $\mathcal{A}(x, y)^\bullet$. If $f \in \mathcal{A}(x, y)^n$, we write $|f| = n$.
- Composition: For every $x, y, z \in \mathrm{Ob}(\mathcal{A})$, there is a chain map

$$\mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z), \quad g \otimes f \mapsto g \circ f$$

  where $\otimes$ is the tensor product of *chain complexes*.

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Example

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2. The **DG-category of k-modules**, denoted by $\mathsf{C}_{\mathrm{dg}}(k)$:
   - Objects: Complexes over k-modules
   - Morphisms: Given $X, Y \in \mathrm{Ch}(k)$, define:

   $$\mathsf{C}_{\mathrm{dg}}(k)(X,Y)^i := \mathsf{hom}_k^i(X,Y) := \prod_{p \in \mathbb{Z}} \mathrm{Hom}_k(X^p, Y^{p+i})$$

   With differential at degree i as

   $$d^i(\phi^p) = d_Y \circ \phi^p - (-1)^i \phi^{p+1} \circ d_X.$$

More Examples

Example

3. In fact, there is nothing special about k-modules, one can get the **DG-category of** $\mathcal{C}$, $\mathrm{C}_{\mathrm{dg}}(\mathcal{C})$ for any additive category $\mathcal{C}$.

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4. Given a DG-category $\mathcal{A}$, one define its opposite category $\mathcal{A}^{op}$ as usual, but with composition rule as:

$$g \circ^{op} f := (-1)^{|f||g|} f \circ g$$

to make $\mathcal{A}^{op}$ a DG-category.

DG Functors

Definition

A **DG functor** between two DG-categories $\mathcal{A}, \mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that for all $x, y \in \mathcal{A}$, the natural map $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a chain map of complexes.

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Example

The shift functor is a DG functor $\mathsf{C}_{\mathrm{dg}}(k) \to \mathsf{C}_{\mathrm{dg}}(k)$.

Homological Algebra on DG-categories (1)

From now on, we always fix a DG-category $\mathcal{A}$.

Definition

The **underlying category of** $\mathcal{A}$ is the category $\mathsf{Z}^0(\mathcal{A})$ with objects same as those in $\mathcal{A}$, and

$$\mathsf{Z}^0(\mathcal{A})(x,y) := \mathsf{Z}^0(\mathcal{A}(x,y)).$$

That is, the morphisms are the 0-cycles of the complex $\mathcal{A}(x,y)^{\bullet}$.

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Example

1. Underlying category of a DG-algebra A is $\mathsf{Z}^0(A)$.
2. Underlying category of $\mathsf{C}_{\mathrm{dg}}(k)$ is $\mathrm{Ch}(k)$.

Homological Algebra on DG-categories (2)

Definition

The **zeroth cohomology category of** $\mathcal{A}$ is the category $H^0(\mathcal{A})$ with objects same as those in $\mathcal{A}$, and

$$H^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y)).$$

Example

1. Zeroth cohomology category of a DG-algebra A is $H^0(A)$.
2. Zeroth cohomology category of $\mathsf{C}_{\mathrm{dg}}(k)$ is $\mathsf{K}(k)$, the homotopy category of k-module complexes.

Quasi-equivalence

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Definition

For a DG functor $F : \mathcal{A} \to \mathcal{B}$, we say F is a **quasi-equivalence** if the induced functor $H^\bullet(\mathcal{A}) \to H^\bullet(B)$ is a equivalence of graded categories.

Example

Quasi-equivalence is quasi-isomorphism on DG-algebras.

Table of Contents

DG $\mathcal{A}$-modules

Definition

A **(right) DG $\mathcal{A}$-module** is a DG functor $\mathcal{A}^{op} \to \mathsf{C}_{\mathrm{dg}}(k)$. Given two DG $\mathcal{A}$-module M, N, a **morphism of DG $\mathcal{A}$-modules** $M \to N$ is a natural transformation such that there is $n \in \mathbb{Z}$, $M(x) \to N(x)$ is a homogeneous map of degree n for any $x \in \mathcal{A}$,.

We will write $M(x)$ as M_x sometimes. This is a complex.

The Category of DG $\mathcal{A}$-modules

Definition-Proposition

The collection of all DG $\mathcal{A}$-modules and morphism is a DG-category $\mathrm{dgMod_{dg}}(\mathcal{A})$. Given two $\mathcal{A}$-modules M, N, we have $\mathrm{dgMod_{dg}}(\mathcal{A})(M, N)^n$ as the homogeneous morphisms of degree n, with differential:

$$df = d \circ f - (-1)^{|f|} f \circ d.$$

Examples of DG $\mathcal{A}$-modules

Given two $\mathcal{A}$-modules M, N, we denote the hom-set k-module complex $\mathsf{dgMod}_{\mathrm{dg}}(\mathcal{A})(M, N)$ as $\mathsf{hom}_{\mathcal{A}}(M, N)$.

Example

1. For the ordinary k-algebra k (which is a DG-algebra), $\mathsf{dgMod}_{\mathrm{dg}}(\mathcal{A}) = \mathsf{C}_{\mathrm{dg}}(k)$.
2. For any DG-category $\mathcal{A}$ and $x \in \mathcal{A}$, the Yoneda functor $\mathbf{h}_x := \mathcal{A}(-, x)$ is a right DG $\mathcal{A}$-module. This is called the **free DG** $\mathcal{A}$-module.

Homological Algebra on DG Modules (1)

Theorem

The underlying category of $\mathrm{dgMod}_{\mathrm{dg}}(\mathcal{A})$ *is a Frobenius exact, Grothendieck Abelian category.*

We denote the underlying category $\mathsf{Z}^0(\mathrm{dgMod}_{\mathrm{dg}}(\mathcal{A}))$ by $\mathrm{dgMod}(\mathcal{A})$.

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Theorem

The zeroth cohomology category of $\mathrm{dgMod}_{\mathrm{dg}}(\mathcal{A})$ *can be identified as the stable category of* $\mathrm{dgMod}(\mathcal{A})$*, and is canonically triangulated by the shift functor* $[1]$.

We denote the zeroth cohomology category $H^0(\mathrm{dgMod}_{\mathrm{dg}}(\mathcal{A}))$ by $\mathsf{K}(\mathcal{A})$.

Homological Algebra on DG Modules (2)

Given $M, N \in \mathsf{dgMod}(\mathcal{A})$:

Theorem (DG Yoneda Lemma)

There is an isomorphism

$$\mathsf{hom}_{\mathcal{A}}(\mathbf{h}_x, M) \cong M_x$$

natural in M, which is also the case with cohomology:

$$H^i(\mathsf{hom}_{\mathcal{A}}(\mathbf{h}_x, M)) \cong H^i(M_x).$$

This is because we still have the identification:

$$H^i(\mathsf{hom}_{\mathcal{A}}(M, N)) \cong \mathsf{K}(\mathcal{A})(M, N[i]).$$

Homological Algebra on DG Modules (3)

It turns out we want to do homological algebra over $\text{dgMod}(\mathcal{A})$.

Definition

A DG $\mathcal{A}$-module M is **acyclic** if for all $x \in \mathcal{A}$, M_x is an acyclic complex. A morphism of $\mathcal{A}$-module $f : M \to N$ is a **quasi-isomorphism** if $f_x : M_x \to N_x$ is a quasi-isomorphism of complexes for all $x \in \mathcal{A}$.

Definition

A DG $\mathcal{A}$-module M is **DG-projective** if all the epimorphisms $M \to P$ in $\text{dgMod}(\mathcal{A})$ with acyclic kernel are split.

Homological Algebra on DG Modules (4)

Note that the above notation are talking about modules in dgMod$(\mathcal{A})$. There is a good reason we want to consider DG-projective objects:

Example

The free DG $\mathcal{A}$-module $\mathbf{h}_x$ for $x \in \mathcal{A}$ in general is not a projective object in dgMod(A), but is DG-projective!

We can form DG-projective resolutions in dgMod$(\mathcal{A})$.

Motivating Equivalence

Theorem (Keller)

For a triangulated category $\mathcal{T}$, the following are equivalent.

- *$\mathcal{T}$ is algebraic, idempotent complete and compactly generated (say $\mathcal{T} = \mathsf{Thick}(X)$ for some $X \in \mathcal{T}$).*
- *There is a DG-algebra A such that $\mathcal{T}$ is equivalent to $\mathrm{D}^c(A)$.*

Can we generalize this theorem to DG-categories, or produce something similar?

Table of Contents

Various Definition of $\mathsf{D}(\mathcal{A})$

Definition

The **derived category** $\mathsf{D}(\mathcal{A})$ **of a DG-category** $\mathcal{A}$ can be defined in the following ways:

- The triangulated quotient $\mathsf{K}(\mathcal{A})/\mathsf{K}^{ac}(\mathcal{A})$, where $\mathsf{K}^{ac}(\mathcal{A})$ is the full triangulated subcategory consisting of all acyclic DG $\mathcal{A}$-modules.
- The localization respect to quasi-isomorphisms $\mathsf{K}(\mathcal{A})[\mathrm{qiso}^{-1}]$.
- The triangulated subcategory of $\mathsf{K}(\mathcal{A})$ compactly generated by the free DG $\mathcal{A}$-modules.

Perfect Derived Categories

Definition

The **perfect derived category** $\mathsf{D}^c(\mathcal{A})$ is the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ consisting of compact objects (i.e. DG $\mathcal{A}$-modules X such that $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, -)$ commutes with all coproduct.)

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Definition

The **perfect derived DG category** $\mathrm{D}^c_{\mathrm{dg}}(\mathcal{A})$ is the full subcategory of $\mathrm{dgMod}_{\mathrm{dg}}(\mathcal{A})$ consist of DG $\mathcal{A}$-modules that are DG-projective and compact.

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Example

For any $x \in \mathcal{A}$, $\mathbf{h}_{-}$ can be seen as a functor $\mathcal{A} \to \mathsf{D}^c_{\mathrm{dg}}(\mathcal{A})$.

Some “nicer” DG-categories

Note that taking H^0 on $\mathbf{h}_-$ induce a fully faithful functor

$$H^0(\mathbf{h}_-) : H^0(\mathcal{A}) \to \mathsf{D}^c(\mathcal{A}).$$

Definition

If the essential image of $H^0(\mathbf{h}_x)$ is a triangulated subcategory of $\mathsf{D}^c(\mathcal{A})$, then we say $\mathcal{A}$ is **pretriangulated**.

If $H^0(\mathbf{h}_x)$ is an equivalence, then we say $\mathcal{A}$ is **Karoubian pretriangulated.**

Table of Contents

Morita Equivalence

Recall that a quasi-equivalence is a equivalence between $H^\bullet(\mathcal{A}) \to H^\bullet(\mathcal{B})$.

Definition

$F : \mathcal{A} \to \mathcal{B}$ is a **(DG) Morita Equivalence** if the left derived functor

$$\mathbf{L}F : \mathsf{D}^c(\mathcal{A}) \to \mathsf{D}^c(\mathcal{B})$$

is an equivalence.

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Example

All quasi-equivalences are Morita equivalences.

Derived Morita Theorem

Theorem (Rickard)

For rings Λ and Γ, the following are equivalent:

- $\mathbf{D}(\Lambda)$ *and* $\mathbf{D}(\Gamma)$ *are equivalent.*
- $\mathbf{D}^c(\Lambda)$ *and* $\mathbf{D}^c(\Gamma)$ *are equivalent.*
- *There is a* ***tilting object*** *(...)* $T \in \mathbf{D}^c(\Lambda)$.

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- *There is a **tilting object** (...)* $T \in \mathbf{D}^c(\Lambda)$.

Theorem (Keller)

For DG-categories $\mathcal{A}$ and $\mathcal{B}$ such that their homology is concentrated in degree 0, the following are equivalent:

- $\mathbf{D}(\mathcal{A})$ *and* $\mathbf{D}(\mathcal{B})$ *are equivalent.*
- $\mathcal{A}$ *and* $\mathcal{B}$ *are DG Morita equivalent.*
- *There is a **tilting subcategory** (...)* $T \subseteq \mathbf{D}(\mathcal{B})$.

Aside: Model Structure

Definition (Informal)

A **Quillen model category structure**, or just a **model structure**, on a category $\mathcal{M}$, is three collection of morphisms $(\mathrm{Cof}, \mathrm{Fib}, \mathrm{Weq})$, called cofibrations, fibrations, and weak equivalences respectively. They need to satisfy some axioms. Any category $\mathcal{M}$ that has a model structure is called a **model category**.

Definition

In a model category $\mathcal{M}$, the **homotopy category of** $\mathcal{M}$ is the localisation category $\mathrm{ho}(\mathcal{M}) := \mathcal{M}[\mathrm{Weq}^{-1}]$.

Model Structure on dgcat

Let dgcat be the category of all DG-categories with DG functors.

Definition-Theorem

There are two model structures on dgcat:

1. A structure where weak equivalences are quasi-equivalences, called the **Tabuada Model structure**.
2. A structure where weak equivalences are Morita equivalences, called the **Morita Model structure**.

Their homotopy category will be denoted by ho(dgcat) and hmo respectively.

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Fact

The Fibrant objects in the Morita model structure are exactly the Karoubian pretriangulated categories.

Enhancements

Definition

Given a triangulated category $\mathcal{T}$, we say

- $\mathcal{T}$ has **DG enhancement** if there is a pretriangulated DG-category $\mathcal{A}$ such that $\mathcal{T}$ is equivalent to $H^0(\mathcal{A})$ as triangulated categories.
- $\mathcal{T}$ has **unique DG enhancement** if any such $\mathcal{A}$ is isomorphic in ho(dgcat).

Recall that two DG categories $\mathcal{A}, \mathcal{B}$ are isomorphic in ho(dgcat) if there is a zigzag of quasi-equivalence connecting them.

Keller's Theorem

Theorem (Keller)

Let $\mathcal{X}$ be a Karoubian pretriangulated DG category, and $\mathcal{A} \subseteq \mathcal{X}$ a DG subcategory such that $\mathrm{Thick}(H^0(\mathcal{A})) = H^0(\mathcal{X})$. Then the inclusion $\mathcal{A} \to \mathcal{X}$ is a Morita Equivalence.

In particular, we get the canonical triangle equivalence

$$\mathsf{D}^c(\mathcal{A}) \to \mathsf{D}^c(\mathcal{X}) \leftarrow H^0(\mathcal{X}).$$

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Example

For a DG algebra A, $\mathsf{D}^c(A)$ has unique DG enhancement.