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ANDY SENGER: CLASSIFICATION OF HIGHLY-CONNECTED MANIFOLDS

INTRODUCTION TO THE PROBLEM

CONVENTION: ALL MANIFOLDS ARE SMOOTH.

GOAL: UNDERSTAND THE CLASSIFICATION OF HIGHLY-CONN. CLOSED MFDS / DIFFEO

SIMPLEST EXAMPLE: HOMOTOPY SPHERES. $\boxed{n \geq 5}$

THM [KERVAIRE-MILNOR] Θ_n = GROUP OF HOMOTOPY n-SPHERES / DIFFEO UNDER #

$\exists$ EXACT SEQUENCE

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \underbrace{\mathrm{coker}(J)_n}_{\text{COMPLICATED!}} \xrightarrow{\text{kerv}} \mathbb{Z}/2$$

bP_{n+1}: THOSE BOUNDING A PARALLELIZABLE n+1-MFD — CYCLIC OF KNOWN ORDER (0, $\mathbb{Z}/2$ OR $\mathbb{Z}$/BERNOULLI NBR)

WARNING / CONCERN: $\mathrm{coker}(J)_n$ COULD INVOLVE ITSELF IN THE CLASSIFICATION OF HIGHLY-CONN MFDS IN A WAY THAT WE CAN'T DESCRIBE.

Def: A compact n-mfd N is **almost closed** if $\partial N \cong S^{n-1}$. "AC"

Eg: M closed $\rightsquigarrow M - \mathring{D}^n$ is almost closed.

Also: Milnor's $D^4 \to E \to S^4$ whose boundaries are Milnor's examples of exotic S^7's.

Plan: (Following CTC Wall)

1) Classify almost closed manifolds
2) Study $\{\text{closed mfds}\} \to \{\text{almost closed mfds}\}$
$$M \longmapsto M \setminus \mathring{D}^n.$$

§1 Almost closed mfds

Thm: (Corollary to the h-cobordism thm)
$n \geq 6$. Suppose N is a contractible AC manifold $\Rightarrow N \cong D^n$.

Pf: $N \setminus \mathring{D}^n$ is an h-cobordism between S^{n-1} and $\partial N \xRightarrow{\text{h-cob thm}}$ trivial cob.

Next simplest case: AC $(n-1)$-conn. $(2n)$-mfd N

$\Leftrightarrow \pi_1(N) = 0 + H_*(N) = 0$ UNLESS $* = n$

$+ H_n(N)$ FIN. DIM. FREE $\mathbb{Z}$-MOD.

THM (WALL) ASSUME $n \geq 9$.

$\{$AC $(n-1)$-CONN $2n$-MFDS$\}$ / DIFFEO

$\Updownarrow$

$\left\{\begin{array}{l} H \quad \text{f. di FREE ABELIAN GP } (= H_n(N)) \\ \mathbb{Z} \to (H \otimes H)_{\Sigma_2} \quad \text{UNIMODULAR } (-1)^n\text{-QUADRATIC FORM} \end{array}\right.$

$(-1)^n$-SWAP

(WRITTEN DUALLY...)

$\Leftrightarrow$ QUADRATIC FORM

HERMITIAN FORM WITH QUADRATIC REFINEMENT

$H \to \pi_n BO$

(CLASSIFYING THE STABLE NORMAL BDL) $\}$ / ISO

RMK: WALL PROVED SOMETHING FOR $n \geq 3$ BUT THE MORE GENERAL STATEMENT IS MORE COMPLICATED (THE LAST TWO THINGS INTERACT)

THM (BURKLUND-S) ASSUME $k \geq \frac{n}{3} + \underbrace{O(\log n)}_{\text{CAN BE MADE EFFECTIVE}}$

THEN $\{$AC. k-CONN n-MFDS$\}$ / DIFFEO

$\Updownarrow$

$\{$ k-CONN. FINITE SPECTRA N, $S^n \to (N \otimes N)_{h\Sigma_2}$ UNIMODULAR QUAD. FORM, $N \to KO$ $\}$ / TANH(?)

WHEN $k = \frac{n}{2} - 1$ WITH n EVEN, RECOVER WALL'S RESULT

$$N \simeq \oplus S^{n/2}$$

NOTE: NOT SO EASY TO DESCRIBE THE CORRESPONDENCE GIVEN BY THE THM.

NOTE: $S^n \longrightarrow (N \otimes N)_{h\mathbb{Z}_2}$ UNIMODULAR

$\Longrightarrow \Sigma^n \mathbb{D}(N) \simeq N \Longrightarrow$ CELLS OF N IN dim $[k, n-k]$

SPANIER-WHITEHEAD DUAL

$\rightsquigarrow$ GIVES SOME DIMENSION TO N.

§2 ALMOST CLOSED $\longrightarrow$ CLOSED

$$\{\text{CLOSED } n\text{-MANIFOLD}\} \longrightarrow \{\text{AC } n\text{-MFD}\}$$

$\Theta_n : M \mapsto M \# \Sigma \quad$ $M \longmapsto M - \mathring{B}^n$

QUESTION A) WHEN IS N IN THE IMAGE?

B) IF IT IS, HOW MANY WAYS DOES IT LIFT?

LEMMA A: N IS IN THE IMAGE
$\Longleftrightarrow \partial N \underset{\text{DIFFEO}}{\cong} S^n$ STANDARD SPHERE.

LEMMA B: FIBERS OF THIS MAP ARE Θ_n-ORBITS

PF: SUPPOSE M, M' s.t. $M - \mathring{D}^n \cong M' - \mathring{D}^n$

THIS RESTRICTS TO $\delta^{n-1} \cong S^{n-1}$

A') GIVEN AC N, WHEN IS $\partial N \cong \delta^{n-1}$

B') GIVEN AC N WITH $M - \mathring{D}^n \cong N$, WHAT IS THE INERTIA GROUP $I(M) = \{\Sigma \in \Theta_n \mid M \# \Sigma \cong M\}$?

MAIN INPUT:

THM (BURKLUND-HAHN-S) ASSUME $k \geq \frac{13}{30} + \overbrace{\mathcal{O}(\log n)}^{\text{EFFECTIVE}}$

OR $k \geq \frac{n}{3} + \underbrace{o(n)}_{\text{VERY INEFFECTIVE}}$ [BURKLUND]

SUPPOSE M IS AC k-CONN. n-MFD. THEN $\partial N \in bP_{n+1}$, i.e. $[\partial N] \in \operatorname{coker}(J)_n$
$\|$
0

COR (COMBINING WITH A FORMULA OF S. STOLZ)
UNDER SAME ASSUMPTIONS,

$N \cong M - \mathring{D}^n$ IFF
- ALWAYS IF $n = $ ODD. ($bP_{n} = 0$ THEN)
- $\operatorname{Kerv}(N) = 0$ IF $n = 2\ (4)$
- $\frac{1}{8} \operatorname{sig}(N) \equiv s_n(N)$ mod $|bP_n|$ IF $n = 0\ (4)$

SIGNATURE

LEMMA: IF $\Sigma \in I(M)$, M k-CONN n-MFD

$\Rightarrow \Sigma \cong \partial T$ FOR T k-CONN. $(n+1)$-MFD

COR: UNDER THE ASSUMPT OF THEOREM,

$I(M) \subseteq bP_{n+1}$ $\Rightarrow I(M) = 0$ n EVEN

REM: IF n ODD, NOT KNOWN WHAT $I(M)$ IS. FOR 2-CONN 7-MFDS, WORKED OUT BY CROWLEY.

ANDY SENGER : TALK 2

BAR SPECTRAL SEQUENCE

THM (last time — version we will prove today)
BURKLUND–HAHN–S

For $k \geq \frac{13}{30} n + O(\log n)$

Given N AC (almost closed) k-conn. n-MFD

then $0 = [\partial N] \in \mathrm{coker}(J)_{n-1}$.

If Σ is the boundary of k-conn. N, it's the boundary of a MFD with $BO\langle k\rangle$-structure

$$\begin{tikzcd} & BO\langle k\rangle = \tau_{\geq k} BO \arrow[d] \\ N \arrow[r] \arrow[ur, dashed] & BO \end{tikzcd}$$

I.e. Σ is null-bordant as a MFD with such structure.

If we choose a framing of Σ,

$$\pi_{n-1} \mathbb{S} \longrightarrow \pi_{n-1} MO\langle k\rangle$$
$$[\Sigma, fr] \longmapsto 0 \ ?$$

by Pontryagin–Thom

QUESTION: What is $\ker(\pi_{n-1}(\mathbb{S}) \to \pi_{n-1} MO\langle k\rangle)$

Rem: 1) If $n \geq \ell$, then $\exists!$ $BO\langle \ell \rangle$-struct on S^{n-1}, so

$$
\begin{tikzcd}
\Omega^{fr}_{n-1} = \pi_{n-1} \mathbb{S} \arrow[rr] \arrow[dr] & & \pi_{n-1} MO\langle \ell \rangle = \Omega^{BO\langle \ell \rangle}_{n-1} \\
& \operatorname{coker}(J)_{n-1} \arrow[ur] &
\end{tikzcd}
$$

measures choice of framing

THM This map is injective.

2) If $n \geq 2k$, then surgery below the middle dimension

$\Rightarrow$ [If $\Sigma = \partial N$, N $BO\langle \ell \rangle$-struct
$\rightarrow \Sigma = \partial \tilde{N}$, $\tilde{N}$ k-conn]

So we need to prove the following:

THM: If $k \geq \frac{13}{30} n + O(\log(n))$, then

$$\ker(\pi_{n-1} \mathbb{S} \to \pi_{n-1} MO\langle \ell \rangle) = \operatorname{Im}(J)_{n-1}$$

No more geometry! ??

New goal: understand $\pi_0 \mathbb{S} \to \pi_0 MO\langle \ell \rangle$.

STEP 1 : REWRITE THE DEFINITION OF A THOM SPEC.

WANT TO ASSOCIATE A THOM SPECTRUM TO A MAP

$$\underset{\text{SPACE}}{\xi : X} \longrightarrow BGL_1(\mathbb{S}) = BG$$

$$\rightsquigarrow$$

$$\Omega X \longrightarrow \Omega BGL_1(\mathbb{S}) = GL_1(\mathbb{S})$$

$$\rightsquigarrow \quad \text{GROUP ALG } \cancel{\text{.....}} \text{ UNITS}$$

$$\bar{\xi} : \Sigma^\infty_+ \Omega X \longrightarrow \mathbb{S} \qquad E_1\text{-MAP}.$$

THEN

$$\varepsilon : \Sigma^\infty_+ \Omega X \to \Sigma^\infty_+(*) \simeq \mathbb{S}$$

$$Th(\xi) := \mathbb{S} \underset{\Sigma^\infty_+ \Omega X}{\otimes} \mathbb{S}$$

(ε and $\bar{\xi}$ the two module structures)

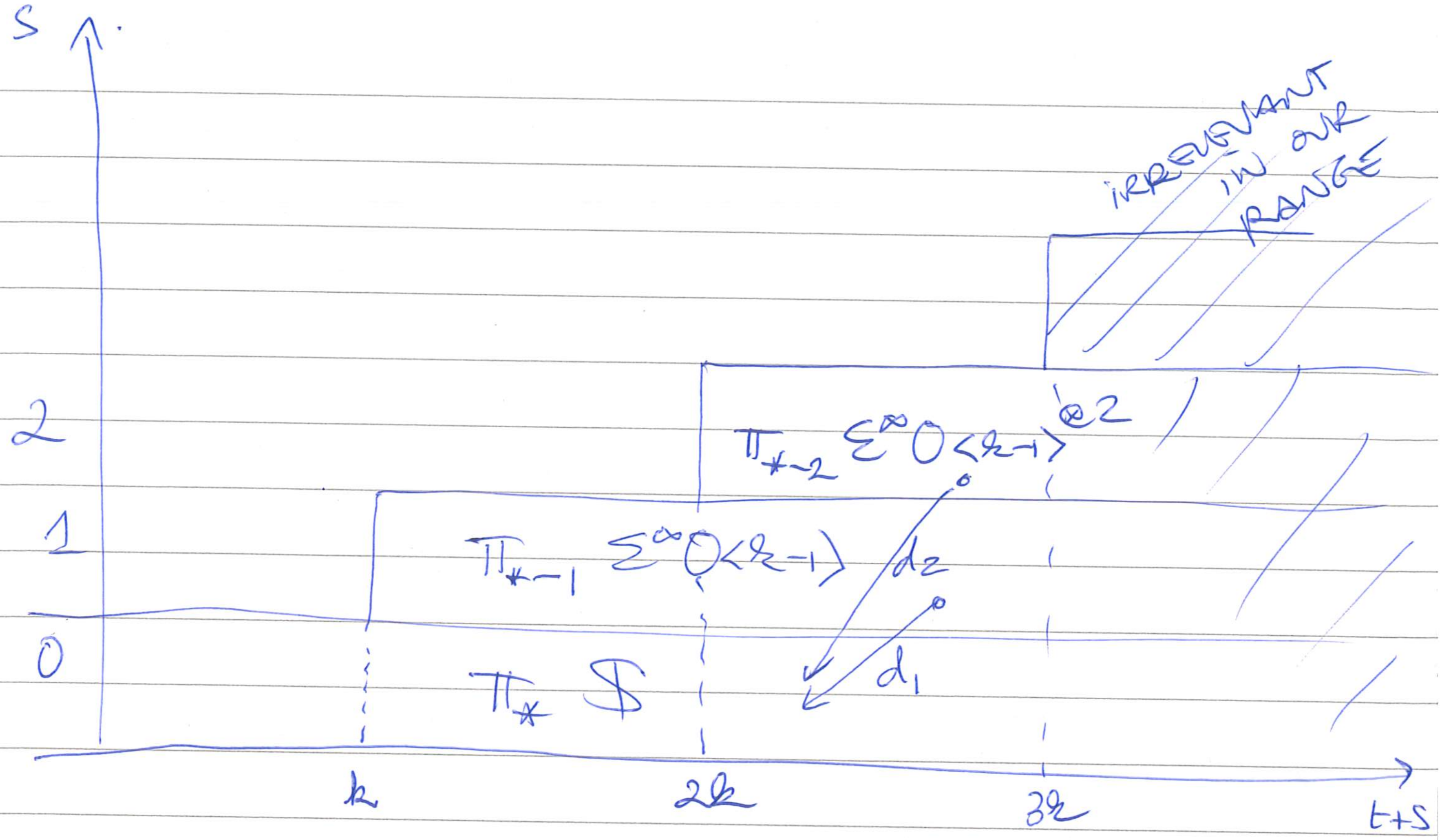
$$\simeq \left| \mathbb{S} \underset{\bar{\xi}}{\overset{\varepsilon}{\leftleftarrows}} \Sigma^\infty_+ \Omega X \underset{1\otimes\bar{\xi}}{\overset{\varepsilon\otimes 1}{\overset{m}{\leftleftarrows}}} \Sigma^\infty_+ \Omega X^{\otimes 2} \cdots \right|$$

$$\Rightarrow MO\langle k\rangle \simeq \left| \mathbb{S} \underset{J}{\overset{\varepsilon}{\leftleftarrows}} \Sigma^\infty_+ O\langle k-1\rangle \underset{1\otimes J}{\overset{\varepsilon\otimes 1}{\overset{m}{\leftleftarrows}}} \Sigma^\infty_+ O\langle k-1\rangle^{\otimes 2} \cdots \right|$$

$$J : BO\langle k\rangle \longrightarrow BO \xrightarrow{J} BGL_1(\mathbb{S})$$

GET A SPECTRAL SEQUENCE

$$\pi_t \Sigma^\infty O\langle k-1\rangle^{\otimes s} \Rightarrow \pi_{t+s} MO\langle k\rangle .$$



$\pi_* \mathbb{S} \longrightarrow \pi_* MO\langle k\rangle$ is an edge homomorphism.

Differentials: d_1: down 1, left 1
d_2: down 2, left 1

Most interested in those, landing on the 0-line.

<u>CONCLUSION</u>: $\ker(\pi_* \mathbb{S} \to \pi_* MO\langle k\rangle)$ is generated by

1) $\operatorname{im}(d_1 : \pi_* \Sigma^\infty O\langle k-1\rangle \to \pi_* \mathbb{S})$

2) $\operatorname{im}\left(\begin{array}{ccc} \pi_* \Sigma^\infty O\langle k-1\rangle^{\otimes 2} & & \pi_{*+1} \mathbb{S} \\ \cup & & \downarrow \\ \ker(d_1) & \xrightarrow{d_2} & \operatorname{coker}(d_1) \end{array}\right)$.

<u>NEED</u>: $\mathrm{im}(d_1) \subseteq \mathrm{im}(J)$ $\qquad \mathrm{im}(J)_n \subseteq \pi_n \mathbb{S}$

$\mathrm{im}(d_2) \subseteq \mathrm{im}(J)$ $\qquad = \mathrm{im}(J : \pi_n O \to \pi_n \mathbb{S})$

<u>NOTE</u>: ALSO HAD "J": $\Sigma^\infty O\langle \ell-1\rangle \longrightarrow \mathbb{S}$

BUT $\quad \pi_* \Sigma^\infty O\langle \ell-1\rangle \neq \pi_* O\langle \ell-1\rangle$

SO NOT A RELEVANT VERSION OF J ...

∞-CATEGORICAL DOLD-KAN:

SIMPLICIAL OBJECTS

$\updownarrow$

COHERENT CHAIN CPXS

WE OBTAIN

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^\infty O\langle \ell-1\rangle^{\otimes 2} \arrow[r, "m - \mathrm{id}\otimes J"'] \arrow[rr, bend left, "0"] & \Sigma^\infty O\langle \ell-1\rangle \arrow[r, "J"] & \mathbb{S} \\
& 2 & 1 & 0
\end{tikzcd}$$

$\Updownarrow \mathrm{can}$

($\varepsilon = 0$ BY SOME NORMALISATION)

$$\begin{tikzcd}
\Sigma^\infty O\langle \ell-1\rangle^{\otimes 2} \arrow[r, "1\otimes J"] \arrow[d, "m"'] \arrow[dr, phantom, "\swarrow"] & \Sigma^\infty O\langle \ell-1\rangle \arrow[d, "J"] \\
\Sigma^\infty O\langle \ell-1\rangle \arrow[r, "J"] & \mathbb{S}
\end{tikzcd}$$

THIS HOMOTOPY IS PART OF

1) SYMM. MON STRUCT ON Sp

2) $J : \Sigma^\infty O\langle \ell-1\rangle \to \mathbb{S}$ IS E_1-RING.

THIS IMPLIES:

1) $d_1 : \pi_* \Sigma^\infty O\langle k-1\rangle \longrightarrow \pi_* \mathbb{S}$

IS THE MAP INDUCED BY J

2) d_2 IS GIVEN BY THE TODA BRACKET

$$\begin{tikzcd}
\mathbb{S}^a \arrow[r, "x"] \arrow[rr, bend right, "0"'] & \Sigma^\infty O\langle k-1\rangle^{\otimes 2} \arrow[r, "m-1\otimes J"] \arrow[rr, bend left, "0"] & \Sigma^\infty O\langle k-1\rangle \arrow[r, "J"] & \mathbb{S}
\end{tikzcd}$$

WITNESSES THAT $d_1 x = 0$

COMBINES TWO NULL-HOMOTOPIES

$$\mathbb{S}^{a+1} \simeq \Sigma \mathbb{S}^a \xrightarrow{d_2(x)} \mathbb{S}^0 .$$

$$\Sigma^\infty O\langle k-1\rangle \simeq \Sigma^\infty \Omega^\infty o\langle k-1\rangle \longrightarrow o\langle k-1\rangle$$

COUNIT OF $\Sigma^\infty \Omega^\infty$ ADJ.

$(\Omega^\infty o\langle k-1\rangle \simeq O\langle k-1\rangle)$

<u>Def</u>: $D_2(X) := (X^{\otimes 2})_{h\Sigma_2}$

$\mathbb{E}_\infty$-STRUCT MAP FOR $\Sigma^\infty O\langle k-1\rangle$

<u>Prop</u>: $D_2(\Sigma^\infty O\langle k-1\rangle) \xrightarrow{\hat{m}} \Sigma^\infty O\langle k-1\rangle \longrightarrow o\langle k-1\rangle$

1) IS A FIBER SEQUENCE AFTER $\tau_{\leq 3k-4}$ IN $Sp_{\leq 3k-4}$

2) INDUCES A SPLIT SES ON HOMOTOPY GPS.

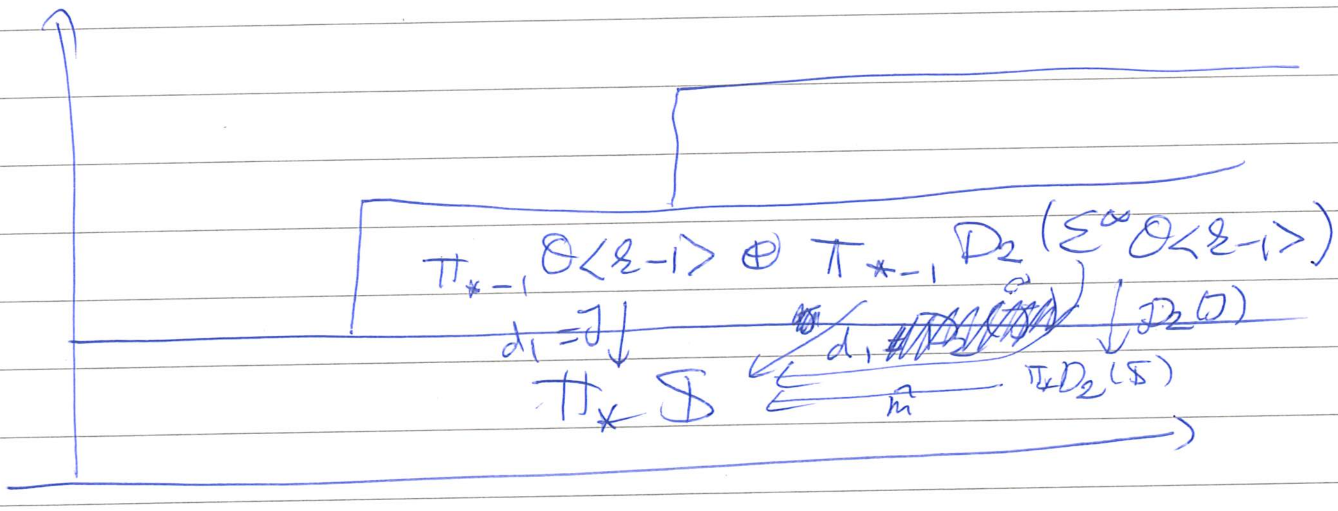
PROOF: 1) Goodwillie calculus

2) is because $\Omega^\infty \Sigma^\infty \Omega^\infty_{\theta}\langle \ell-1\rangle \longrightarrow \Omega^\infty_{\Theta}\langle \ell-1\rangle$

admits a splitting for adjoint reasons.

CONCLUSION: $\pi_* \Sigma^\infty \Theta\langle \ell-1\rangle \cong \pi_* \Theta\langle \ell-1\rangle \oplus \pi_* D_2(\Sigma^\infty \Theta\langle \ell-1\rangle)$

$\rightsquigarrow$ in SS



$\to$ NEED 1) $\mathrm{im}(D_2(J)) \subseteq \mathrm{im}\, J$

2) $\mathrm{im}(d_2) \subseteq \mathrm{im}\, J$.

NOT TRUE IN GENERAL! LOW DIM COUNTER-EX:

Eg IF $x \in \mathrm{im}(J)$, $x^2 \in \mathrm{im}(D_2(J))$

$\nu, \sigma \in \mathrm{im}(J)$, $\nu^2, \sigma^2 \notin \mathrm{im}(J)$

$\eta\sigma \in \mathrm{im}(J)$ $\quad \eta\zeta \in \mathrm{Im}(D_2(\eta J)) \notin \mathrm{Im}(J)$.

ANDY SENGER: LECTURE III

$\mathbb{F}_2$-ADAMS SS AND SYNTHETIC SPECTRA

~~[illegible]~~ RECALL: WANTED TO PROVE THAT

$\mathrm{Im}(J)_* \subseteq \pi_*(\mathbb{S})$ IS CLOSED

UNDER 1) $D_2(J)$

2) TODA BRACKET DEFINING d_2.

WANT: WAY OF THINKING ABOUT $\mathrm{Im}(J)$ THAT INTERFACES NICELY WITH D_2, ETC.

TODAY: VIEWING $\mathrm{Im}(J)$ AS "HIGH ADAMS FILTRATION" IS SUCH A WAY.

CONSTRUCTION: $X, Y \in Sp$ X FINITE.

$\mathbb{F}_p$-ADAMS SS FOR $\mathrm{Hom}(X,Y)$ IS:

$$\mathrm{Ext}^{s,t}_{A_*}\left(H_*(X;\mathbb{F}_p),\ H_*(Y,\mathbb{F}_p)\right) \Rightarrow \pi_{t-s}\mathrm{Hom}(X, Y_p^\wedge)$$

DUAL STEENROD ALG (GRADED HOPF ALGEBRA)

$\# H_*$-GRADING

INTERNAL GRADING FROM THE GRADING OF THE ORIGINAL OBJECTS

THE ASSOCIATED FILTRATION IS:

$F^s\pi_*\mathrm{Hom}(X,Y)$, $F^{s+1} \subseteq F^s$

is THE $\mathbb{F}_p$-ADAMS FILTRATION ON $\pi_* \mathrm{Hom}(X,Y)$.

<u>Def/ NOTATION</u> IF $f : \Sigma^n X \longrightarrow Y$ LIES IN

$F^s \pi_n \mathrm{Hom}(X,Y)$, WE WRITE

$$AF(f) \geq s$$

ADAMS FILTRATION

Eg $AF(f) \geq 1 \iff H_*(f; \mathbb{F}_p) = 0$.

AF HIGHER & HIGHER IF IT TAKE MORE WORK FOR $H_*(\, ; \mathbb{F}_p)$ TO DETECT YOU...

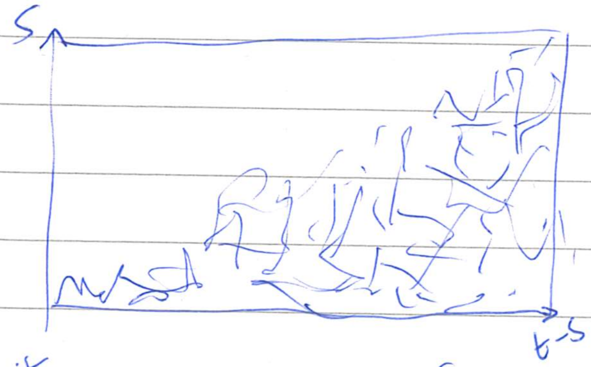
<u>KEY INSIGHT</u> (STOLZ): $\mathrm{im}(J) \subseteq \pi_*(\mathbb{S})$

$\Updownarrow$

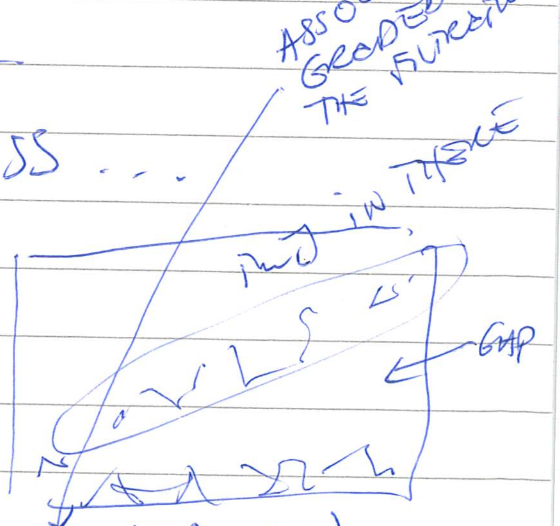
ELEMENTS OF HIGH $\mathbb{F}_p$-AF in $\pi_*(\mathbb{S})$ $\forall p$

ALMOST TRUE
$\exists$ ANOTHER FAMILY OF HIGH AF : μ-FAMILY
BUT CAN BE IGNORED HERE...

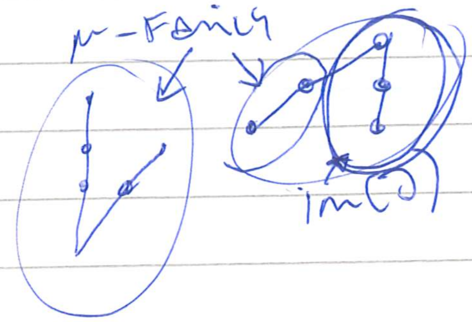
$\mathbb{F}_2$-ADAMS SS ...
$X = Y = \mathbb{S}$



$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathbb{F}_2) = E_2$ PAGE $\rightsquigarrow$ E_∞ PAGE $= gr_*(F^\bullet \pi_* S_2^\wedge)$



ZOOMING ON THIS DIAGONAL, HAVE A REPEATED PATTERN OF



ODD PRIMES ARE SIMILAR, BUT A LITTLE EASIER.

TAKE AWAYS: 1) im(J) LIVES ON A SLOPE $\frac{1}{2}$-LINE ($-\log$(STEM))

2) THERE IS A HUGE GAP BETWEEN THESE AND NEXT ELEMENTS.

IDEA OF PROOF GIVEN THE ABOVE:

- START WITH im(J) - IT'S IN HIGH AF.
- DO SOME CONSTRUCTION (eg D_2, TODA BRACKET, ...)
- SHOW THAT THIS CONSTRUCTION ONLY DROPS AF A LITTLE
- USING THE HUGE GAP, CONCLUDE IT STILL LIES IN im(J).

LEMMA: LET $M_\ell \longrightarrow \Sigma^\infty O\langle \ell-1\rangle$ ~~BE~~ AN ℓ-SKELETON OF $\Sigma^\infty O\langle \ell-1\rangle$. THEN

$$M_\ell \longrightarrow \Sigma^\infty O\langle \ell-1\rangle \xrightarrow{J} \mathbb{S} \text{ HAS}$$

$AF \geq h(\ell-1) - \lfloor \log_2(\ell+1) \rfloor + 1$, $\geq \frac{\ell}{2} - \lfloor \log_2(\ell+1)\rfloor$ WHERE

$h(n)$ = # OF INTEGERS $0 < s \leq n$ s.t. $s \equiv 0,1,2,4\ (8)$

PROOF SKETCH: $\Sigma^\infty \Theta\langle k-1\rangle$

$$\begin{array}{c}\Sigma^\infty \Theta\langle k-1\rangle \\ \downarrow \\ \Sigma^\infty \Theta\langle k-2\rangle \\ \downarrow \\ \vdots \\ \downarrow \\ \Sigma^\infty \Theta \xrightarrow{J} \mathbb{S}\end{array}$$

Claim: IF $\Sigma^\infty \Theta\langle n\rangle \longrightarrow \Sigma^\infty \Theta\langle n-1\rangle$ IS NOT AN ISO ($\Leftrightarrow \pi_n \Theta \neq 0$), THEN IT IS ZERO ON $H(-;\mathbb{F}_2)$ IN A RANGE EXPONENTIALLY INCREASING IN n.

THM (DAVIS – MAHOWALD)

SUPPOSE $x \in \pi_n(\mathbb{S})^\wedge_2$ SATISFIES

$$AF(x) \geq \frac{3}{10} n + 4 + \nu_2(n+2) + \nu_2(n+1)$$

ν_2 2-ADIC VALUATION

THEN $x \in \mathrm{im}(J) \oplus \mu$-FAMILY.

THM (BURKLUND) $x \in \pi_n(\mathbb{S})^\wedge_2$, THEN $\exists$ SUBLINEAR FUNCTION $\varepsilon(n)$ s.t. $AF(x) \geq \frac{n}{6} + \varepsilon(n)$ IMPLIES $x \in \mathrm{im}(J) \oplus \mu$-FAMILY.

CONJECTURE: TRUE FOR $AF(x) \geq \frac{n}{6} + O(\log n)$

(HOLDS AT ODD PRIMES BUT NOT KNOWN FOR $p=2$)

SYNTHETIC SPECTRA (in 15 min!)

SLOGAN: SYNTHETIC SPECTRA CATEGORIFY THE ADAMS SS.

THM: $\exists$ SYM. MONOIDAL ~~STABLE~~ PRESENTABLE ∞-CAT OF SYNTHETIC SPECTRA $\mathrm{Syn}_{\mathbb{F}_p}$ S.T.

$$X \xrightarrow{\nu} \mathrm{ASS}(X)$$

ADAMS SS OF X

$$\begin{tikzcd}
Sp_p \arrow[d, "\nu"] \arrow[rr, "H_*(-;\mathbb{F}_p)"] \arrow[dd, bend right, "id"'] & & \\
\mathrm{Syn}_{\mathbb{F}_p} \arrow[rr] \arrow[d] & & \mathrm{STAB}_{A_*} \\
Sp_p & &
\end{tikzcd}$$

$(\simeq D(A_*))$

= CATEGORY OF FORMAL SS THAT ARE NOT ASSUMED TO CONVERGE

PROPERTIES: – ν FULLY FAITHFUL, SYM. MONOIDAL, FILTERED COLIMIT PRESERVING

– ν PRESERVES COFIB. SEQU. $X \to Y \to Z$ WHICH INDUCE A S.E.S. ON $H_*(-;\mathbb{F}_p)$.

– $\mathbb{S}^{a,b} = \Sigma^{-b}\nu(\mathbb{S}^{a+b})$ "BIGRADED SPHERE" ARE COMPACT GENERATORS FOR $\mathrm{Syn}_{\mathbb{F}_p}$

$\Longrightarrow$ $\pi_{*,*}$ BIGRADED HTPY GROUPS FOR SYNTH. SPEC.

- $\tau: \mathbb{S}^{0,-1} \longrightarrow \mathbb{S}^{0,0}$ $\quad \tau \in \pi_{0,-1}\mathbb{S}^{0,0}$

is ~~ASSEMBLY MAP~~

$$\Sigma \nu(\mathbb{S}^{-1}) \longrightarrow \nu(\Sigma \mathbb{S}^{-1}) \simeq \nu(\mathbb{S}^{0})$$

- $\mathrm{Syn}_{\mathbb{F}_p}[\tau^{-1}] \simeq \mathrm{Sp}$.

- $\mathbb{S}/\tau$ is E_∞-RING & $\mathrm{Mod}_{\mathbb{S}/\tau}(\mathrm{Syn}_{\mathbb{F}_p}) \simeq \mathrm{Stable}_{A_*}$

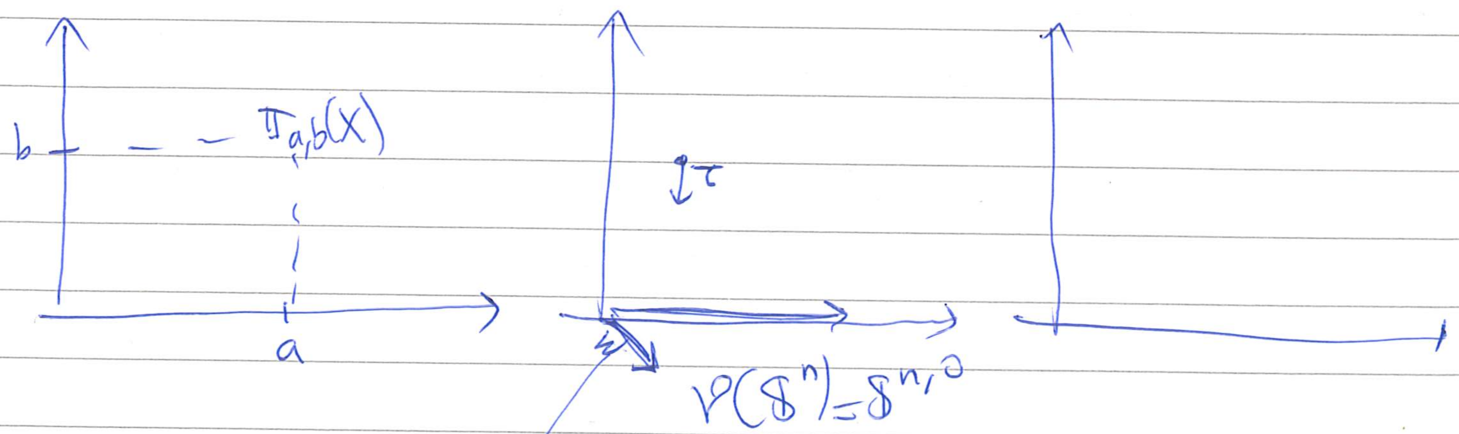
$$\pi_{t-s,s}(\nu(X)/\tau) \cong \mathrm{Ext}^{s,t}(\mathbb{F}_p, H_*(X;\mathbb{F}_p))$$

- τ-BSS FOR $\nu(X)$ $\Longleftrightarrow$ $\mathbb{F}_p$-ASS FOR X

"TAU-BOCKSTEIN" SS $\qquad$ ADAMS SS

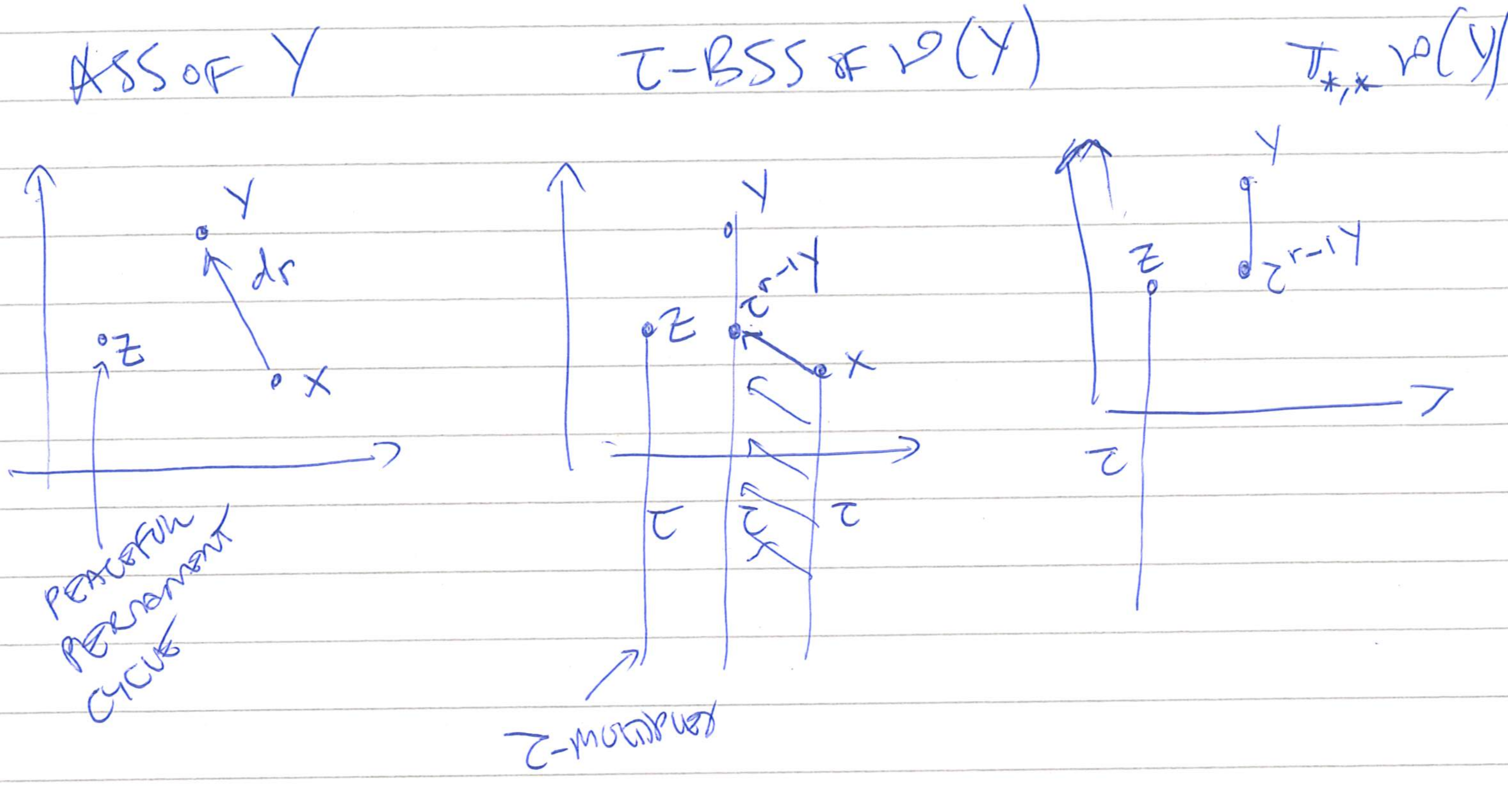
$d_r(x) = \tau^{r-1} y \qquad\qquad d_r(x) = y$

- $F^s \pi_n(X_p^\wedge) = \mathrm{im}(\pi_{n,s}\nu(X_p^\wedge) \to \pi_n(X_p^\wedge))$.



$X \in \mathrm{Syn}_{\mathbb{F}_p}$, $\quad \Sigma = - \otimes \mathbb{S}^{1,-1}$

$Y \in Sp$.



$$AF(f \circ g) \geq AF(f) + AF(g).$$

ANDY LECTURE 4: PUTTING IT TOGETHER

SENGER

RECALL: **THM** $\ker(\pi_n \mathbb{S} \to \pi_n MO\langle k\rangle) \subseteq \mathrm{Im}(J)$

GOAL:

IF $k \geq \frac{13}{30} n + O(\log n)$

HAVE SHOWN: THE KERNEL IS GEN. BY

1) $\mathrm{im}(J)$
2) $\mathrm{im}\left(D_2(J) : D_2(\Sigma^\infty O\langle k-1\rangle) \to \mathbb{S}\right)$
3) TODA BRACKETS

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[rr, bend left, "0"] & \Sigma^\infty O\langle k-1\rangle^{\otimes 2} \arrow[r, "\mu - 1 \otimes J"'] \arrow[rr, bend right, "0"'] & \Sigma^\infty O\langle k-1\rangle \arrow[r, "J"] & \mathbb{S} \end{tikzcd}$$

CAN

PLAN: DO THIS BY SHOWING THEY HAVE LARGE AF.

THM (DAVIS-MAHOWALD) $x \in \pi_n(\mathbb{S})^\wedge_2$ SATISFIES

$AF(x) \geq \frac{3}{10} n + O(\log n) \Rightarrow x \in \mathrm{im}(J_n) \oplus \{\mu\text{-family}\}$

μ-FAMILY:
$$\begin{tikzcd} (\mu\text{-fam})_n \arrow[r, phantom, "\subseteq"] \arrow[dr, "\cong"'] & \pi_n(\mathbb{S}) \arrow[r] & \pi_n(ko) \\ & \pi_n(ko)_{tors} \arrow[ur] & \end{tikzcd}$$

CAN IGNORE THE μ-FAMILY BECAUSE OF THE SPIN

ORIENTATION OF ko:

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & MSpin & \longrightarrow & ko \\ \downarrow & & \| & & \\ MO\langle 2\rangle & \longrightarrow & MO\langle 4\rangle & & \end{array}$$

IMPLYING THAT THE μ-FAMILY CANNOT BE PART OF THE KERNEL OF $\mathbb{S} \longrightarrow MO\langle 2\rangle$.

$\longrightarrow$ WE ONLY NEED TO SHOW HIGH AF.

RECALL: $Sp \xrightarrow{\nu} Syn_{\mathbb{F}_2} \xrightarrow{\tau^{-1}} Sp$

$X \longmapsto ASS(X)$

$SS \longmapsto$ WHERE IT'S SUPPOSE TO CONVERGE

$f: X \longrightarrow Y$ in Sp

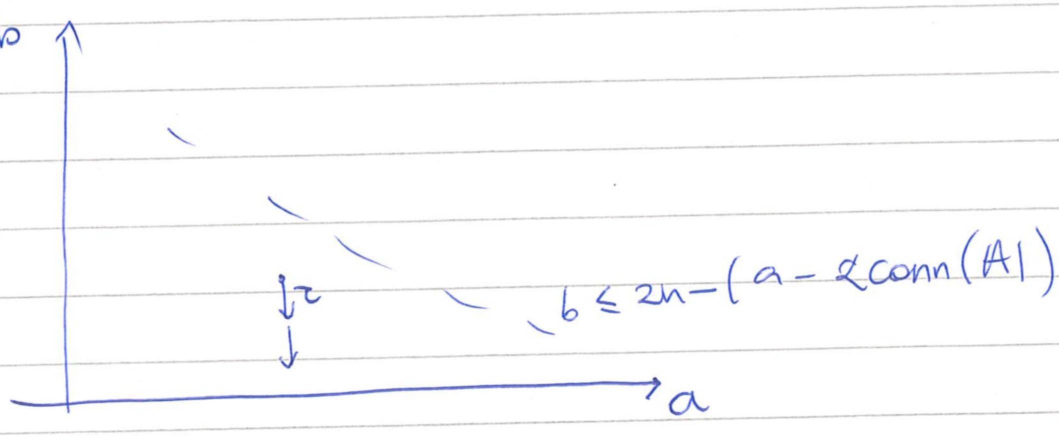
$AF(f) \geq n \iff \exists \tilde{f}: \Sigma^{0,n}\nu(X) \longrightarrow \nu(Y)$ s.t. $\tau^{-1}(\tilde{f}) \simeq f$

$\left(\tau^n \tilde{f} \simeq \nu(f) \right)$

LEMMA: $A \in Sp$: $\pi_{a,b}\left[D_2\left(\Sigma^{0,n}\nu(A)\right)\right] \xrightarrow[\tau^{-1}]{\cong} \pi_a(D_2(A))$

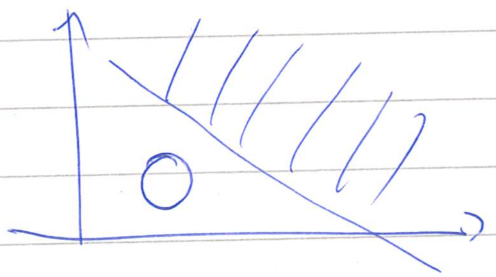
IF $b \leq 2n - (a - 2\,\mathrm{conn}(A))$

PROOF: $\pi_{a,b} D_2(\Sigma^{0,n}\nu(A))$



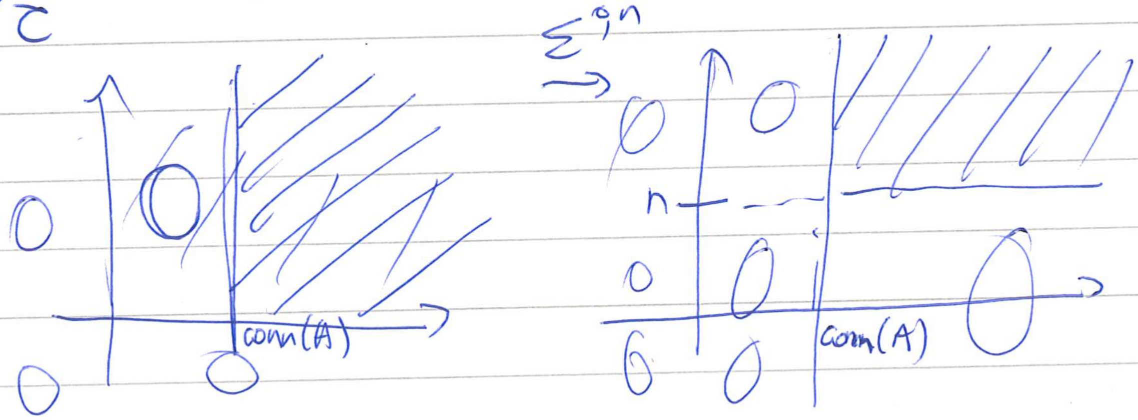
WANT $\downarrow\tau$ ISO BELOW ...

$\Leftrightarrow$

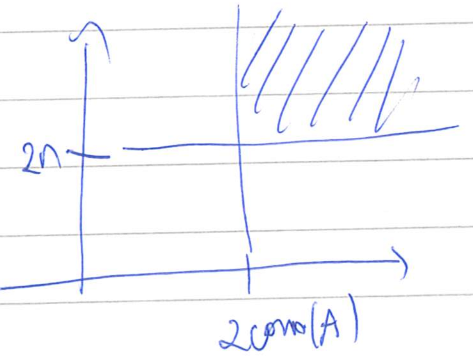


$\pi_{a,b}(D_2(\Sigma^{0,n}\nu(A))/\tau)$

$\nu(A)/\tau$ = E_2-PAGE OF $\mathbb{F}_2$-ASS FOR A



$(\Sigma^{0,n}\nu A)^{\otimes 2}/\tau$

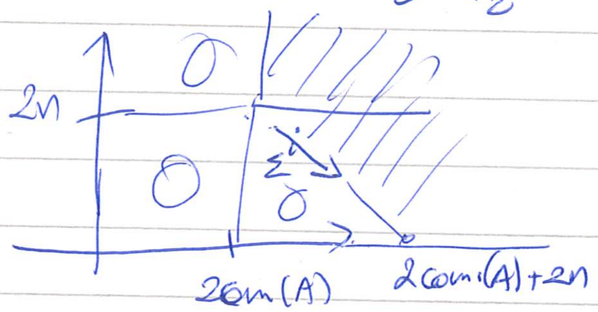


$D_2(\Sigma^{0,n}\nu A)/\tau := ((\Sigma^{0,n}\nu A)^{\otimes 2})_{h\Sigma_2}/\tau$

CLAIM: $D_2(X)$ HAS A FILTRATION WITH ASS. GRADED $\Sigma^i X^{\otimes 2}$ $i \geq 0$ (IN ANY SYM. MON. CATEGORY)

$\operatorname{colim}_{B\Sigma_2}(-)$, $B\Sigma_2 = \mathbb{R}P^\infty$

RECALL: In $\mathrm{Syn}_{\mathbb{F}_2}$, $\Sigma \simeq \Sigma^{1,-1}$



CONDITION IN STATEMENT $\Leftrightarrow$ BEING BELOW THAT LINE.

COR: $x \in \pi_n(D_2 \Sigma^\infty O\langle k-1\rangle)$ ~~[illegible]~~ AND $n \leq 3k$, THEN $D_2(J)(x)$ ~~[illegible]~~ HAS AF $\geq 3k - n - O(\log k)$ ~~[illegible]~~.

PROOF: $M_{3k} \longrightarrow \Sigma^\infty O\langle k-1\rangle$ $3k$-SKELETON

$\exists\ \tilde{J} : \Sigma^{0,\frac{k}{2} - O(\log k)} \nu(M_{3k}) \longrightarrow \mathbb{S}^{0,0}$

BY LEMMA, x LIFTS TO (TAKING BORDMAL)

$\tilde{x} \in \pi_{n, k-(n-2k)-O(\log(k))} D_2(\Sigma^{0,\frac{k}{2}-O(\log k)} \nu M_{3k})$

$D_2(\tilde{J})(\tilde{x}) \in \pi_{n, k-(n-2k)-O(\log k)} \mathbb{S}^{0,0}$ LIFTS $D_2(J)(x)$.

COR: $\mathcal{D}_2(J)(x) \in \mathrm{im}(J)$ IF $k > \frac{13}{30}n + O(\log k)$

PROOF: NEED $3k - n - O(\log k) > \frac{3}{10}n + O(\log k)$

$\left[O(\log k) \sim O(\log n), \; k \le n \le 3k\right]$

$\iff 3k > \frac{13}{10}n + O(\log k)$

$\iff k > \frac{13}{30}n + O(\log k)$.

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend left, "0"] & X \arrow[r, "g"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

$\Rightarrow \langle f, g, h \rangle : \Sigma W \longrightarrow Z$.

SUPPOSE $AF(f) \geq a$, $AF(g) \geq b$, $AF(h) \geq c$.

LIFTS TO $\mathrm{Syn}_{\mathbb{F}_2}$:

$$\begin{tikzcd} \Sigma^{0,a+b+c}\nu W \arrow[r, "\tilde{f}"] \arrow[rr, bend left] & \Sigma^{0,b+c}\nu X \arrow[r, "\tilde{g}"] \arrow[rr, bend right] & \Sigma^{0,c}\nu Y \arrow[r, "\tilde{h}"] & \nu Z \end{tikzcd}$$

SUPPOSE CAN CHOOSE $\tilde{f}, \tilde{g}, \tilde{h}$ s.t. $\tilde{g} \circ \tilde{f}$ AND $\tilde{h} \circ \tilde{g}$ ARE NULL, LIFTING THE ORIGINAL NULL-HOMOTOPIES, THEN WOULD GET

$\langle \tilde{f}, \tilde{g}, \tilde{h} \rangle : \Sigma(\Sigma^{0,a+b+c}\nu W) \longrightarrow \nu Z$

IS

$\Sigma^{1,a+b+c-1}\nu W$

$\rightsquigarrow AF(\langle f, g, h \rangle) \geq a+b+c-1$.

OUR SITUATION:

FROM $x \in \ker(d')$ $\Leftrightarrow$ $m \circ x \simeq (1 \otimes J) \circ x$

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "x"] \arrow[rr, bend left, ""{name=U}] & \Sigma^\infty O\langle \mathcal{E}-1\rangle^{\otimes 2} \arrow[r, "m-1\otimes J"] \arrow[rr, bend right, "0"', ""{name=D}] \arrow[from=U, phantom, "\Downarrow"] & \Sigma^\infty O\langle \mathcal{E}-1\rangle \arrow[r, "J"] \arrow[from=D, phantom, "\Uparrow \text{can}"] & \mathbb{S}
\end{tikzcd}$$

FROM E_1-STRUCT

PROBLEM: THE CANONICAL 0-HTPY USED THAT J IS AN E_1-RING MAP, BUT $\tilde{J}$ WILL NOT BE A RING MAP.

ASSUMPTION (1): $m \circ x \simeq 0$, $(1 \otimes J) \circ x \simeq 0$.

$$\begin{tikzcd}
\mathbb{S}^n \arrow[dr, "x"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& \Sigma^\infty O\langle \mathcal{E}-1\rangle^{\otimes 2} \arrow[r, "1\otimes J"] \arrow[d, "m"] & \Sigma^\infty O\langle \mathcal{E}-1\rangle \arrow[d, "J"] \arrow[dl, phantom, "\Leftarrow \text{can}"] \\
& \Sigma^\infty O\langle \mathcal{E}-1\rangle \arrow[r, "J"'] & \mathbb{S}
\end{tikzcd}$$

NOTE: $\Sigma^\infty O\langle \mathcal{E}-1\rangle^{\otimes 2} \xrightarrow{c} D_2(\Sigma^\infty O\langle \mathcal{E}-1\rangle) \xrightarrow{\hat{m}} \Sigma^\infty O\langle \mathcal{E}-1\rangle$, with composite m

SPLIT INJECTIVE IN OUR RANGE

ASSUMPTION (2) (INSTEAD OF (1)): $c \circ x \simeq$, $(1 \otimes J) \circ x \simeq 0$.

$$
\begin{tikzcd}
S^n \arrow[rr, bend left, "0"] \arrow[dr, "x"] \arrow[ddr, bend right, "0"'] & & \Sigma^\infty O\langle k-1\rangle \arrow[d, "J\otimes 1"] & \\
& \Sigma^\infty O\langle k-1\rangle^{\otimes 2} \arrow[ur, "1\otimes J"] \arrow[r, "J\otimes J"] \arrow[d, "c"] \arrow[dr, phantom, "\text{NAT TRANSF}"] & S^{\otimes 2} \arrow[d, "c"] \arrow[ddr, bend left, "m"] \arrow[dr, phantom, "S\ E_\infty"] & \\
& D_2(\Sigma^\infty O\langle k-1\rangle) \arrow[r, "D_2(J)"] \arrow[d, "\tilde m"] & D_2(S) \arrow[dr, "\tilde m"] & \\
& \Sigma^\infty O\langle k-1\rangle \arrow[rr, "J", "E_\infty\text{-RING MAP}"'] & & S
\end{tikzcd}
$$

CAN BE DELETED TO DEFINE THE TODA BRACKET

↳ NO NEED FOR THE E_1-STRUCT OF J

⇝ SYNTHETIC LIFT:

$M_{3k} \longrightarrow \Sigma^\infty O\langle k-1\rangle$ WRITE $\underline{m} = \frac{k}{2} - O(\log k)$

3-SKELETON

$$
\begin{tikzcd}
S^{n,2\underline{m}} \arrow[rr, bend left, "0"] \arrow[dr, "\nu x"] \arrow[ddr, bend right, "0"'] & & \Sigma^{0,\underline{m}}\nu M_{3k} \arrow[d, "\tilde J\otimes 1"] & \\
& (\Sigma^{0,\underline{m}}\nu M_{3k})^{\otimes 2} \arrow[ur, "1\otimes\tilde J"] \arrow[r, "\tilde J\otimes\tilde J"] \arrow[d, "c"] & (S^{0,0})^{\otimes 2} \arrow[d, "c"] \arrow[ddr, bend left, "m"] & \\
& D_2(\Sigma^{0,\underline{m}}\nu M_{3k}) \arrow[r, "D_2(\tilde J)"] & D_2(S^{0,0}) \arrow[dr, "\tilde m"] & \\
& & & S^{0,0}
\end{tikzcd}
$$

$\Updownarrow$?

$\Updownarrow$?

NEED TO MULTIPLY BY τ A FEW TIMES FOR THIS TO HAPPEN

PROP: SUPPOSE $x \in \ker(d_1) \subseteq \pi_n \Sigma^\infty \mathcal{O}\langle \ell - 1\rangle^{\otimes 2}$

THEN 1) $x \circ x \simeq 0$

2) $(1 \otimes J) \circ x \simeq 0$

3) $\cancel{x \circ x} \; c \circ \; \tau^{n-2\ell+2} \rho x$

4) $(1 \otimes \tilde{J}) \circ \tau^2 \rho x \simeq 0$.

((4) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3) + PROVE (4)).

$\Longrightarrow d_2(x)$ HAS AF $\geq 2\underline{m} - n + 2\ell - 3$

$\geq 3\ell - n - O(\log \ell)$

SAME AS FOR d_1 $\Longrightarrow$ CONCLUDE!