

GROTHENDIECK-WITT THEORY

Def: $GW^{eq/es}(R) = (\text{Unimod}^{eq/es}(R), \oplus)^{grp} \in (Grp(\text{Spec}))$

" $\coprod_{[P, b]} \text{Bisom}(P, b)$

A) SET-UP CATEGORICAL LANGUAGES

- $q: \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}_f = \mathcal{S}_f^{\text{EUTRA}}$

$\phi(\phi) = 0$

$$\rho(x \oplus y) = \rho(x) \oplus \rho(y) \oplus B_\phi(x, y)$$

$$B_p(X, Y): \mathcal{C}^{op} \times \mathcal{C}^{op} \rightarrow \mathcal{S}p$$

2-EXCITE $\Rightarrow$ BIFURC.

$G \rightarrow A \rightarrow P$

$$\begin{array}{ccccc}
 B_q(X, X) & \xrightarrow{h_2} & \varphi(X) & \longrightarrow & 1_\varphi(X) \\
 & \searrow \text{norm} & \downarrow & & \downarrow h_2 \\
 & & B_q(X, X) & \xrightarrow{h_2} & B_q(X, X)
 \end{array}$$

$$\begin{array}{c}
 \uparrow 1_\varphi \\
 \downarrow h_2
 \end{array}$$

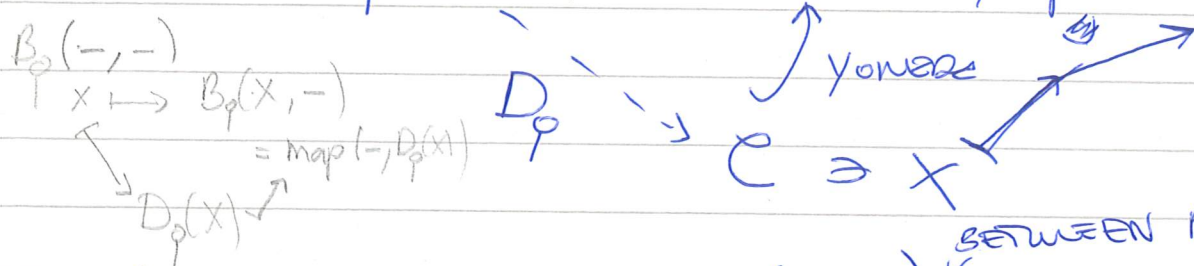
1_φ EXACT
 $(=)$ 2-EXCISIVE

A HERMIDIAN FUNCTOR: $(\mathcal{C}, \varphi) \rightarrow (\mathcal{C}', \varphi')$ consists of (f, η) :

- $f: \mathcal{C} \rightarrow \mathcal{C}'$
- $\eta: \varphi \rightarrow f^* \varphi'$

$\leadsto \text{Cat}^h = \text{category of Hermitian Cat}$

DEF: $(\mathcal{C}, \varphi)$ is POINCARÉ if $\exists D_\varphi: \mathcal{C}^{op} \xrightarrow{\sim} \mathcal{C}$ s.t. $B_\varphi: \mathcal{C}^{op} \rightarrow \text{Fun}(\mathcal{C}^{op}, S_p) \ni \text{map}(-, X)$



A HERMIDIAN FUNCTOR (f, η) is POINCARÉ if AN ASSOCIATED MAP $\forall D_\varphi \xrightarrow{\sim} D_{\varphi'} f$

(EX: This map (ev on X) is induced by $\text{Id}_{D_\varphi X}$)

$\leadsto \text{Cat}^P \rightarrow \text{Cat}^h$

EX: R conn. RING. $\text{Perf}(R) = [{}^b \text{Proj}(R)] [q, \text{iso}^{-1}]$

$$\begin{aligned} \mathcal{O}_R^{\text{Eq}}(X) &= \text{map}_R(X \otimes_R X, {}_{\mathcal{E}} R) {}^{h\mathcal{C}_2} \\ \mathcal{O}_R^{\text{ES}}(X) &= \text{map}_R(X \otimes_R X, {}_{\mathcal{E}} R) {}^{h\mathcal{C}_2} \end{aligned} \quad \downarrow \text{norm}$$

EXERCISE: $B_{\varphi^{\text{Eq}}} \xrightarrow{\sim} B_{\varphi^{\text{ES}}} = ((X, Y) \mapsto \text{map}_R(X \otimes_R Y, R))$

$\Rightarrow \mathcal{O}_R^{\text{Eq/ES}}$

POINCARÉ s.t. on $\text{Perf}(R)$ with DUALITY

$$D_\varphi(X) = \text{map}_R(X, R).$$

PROP: $\text{Fun}^{\text{red}, 2\text{-exc}}(\text{Perf}(R)^{\text{op}}, \mathcal{S}_P) \xrightarrow{\cong} \text{Fun}^{\text{red}, 2\text{-add}}(\text{Proj}(R)^{\text{op}}, \mathcal{S}_P)$ ②
 THE RESTRICTION
 IS AN EQUIV. OF CATEGORIES.

→ TO UNDERSTAND REINCKE'S ANSWER ON $\text{Perf}(R)$,
 CAN LOOK AT $\text{Proj}(R)$ INSTEAD.

$\text{Fun}^{\text{red}, 2\text{-add}}(\text{Proj}(R)^{\text{op}}, \mathcal{A}_b) \subseteq \text{Fun}^{\text{red}, 2\text{-add}}(\text{Proj}(R)^{\text{op}}, \mathcal{S}_P)$
 IS [SUCHTING]
 "TURN PARAMETERS"
 IN THE CLASSICAL SENSE

IN PARTICULAR, $P \mapsto \begin{cases} \text{Hom}_R(P \otimes_R P, R) \\ \text{Hom}_R(P \otimes_R P, R) \end{cases} \begin{matrix} C_2 \\ C_2 \end{matrix}$

GIVE RISE TO $\begin{matrix} \text{egg/egs} \\ \uparrow \\ \text{"GENUINE"} \end{matrix}$

IF $R \rightarrow S$ IS A MAP OF COMM. RINGS
CLAIM: $- \otimes_R S : \text{Perf}(R) \rightarrow \text{Perf}(S)$ REFINES TO
 A REINCKE FUNCTOR FOR $\begin{matrix} \text{egg/egs/egsYES} \\ \uparrow \\ R \end{matrix}$ + SAME FOR S .

$\mathcal{S}_P^u = \text{FINITE SPECTRA } (\text{Perf}(\mathbb{Z})) \quad D = \text{map}_{\mathbb{Z}}(-, \mathbb{Z})$
 "universal"

$\mathcal{P}^u(X) \rightarrow D(X)$ — TAKE DIAGONAL

$(DX \otimes DX)^{hC_2} \rightarrow (D\mathbb{Z} \otimes DX)^{tC_2}$

THE FACT: $\text{Fun}^{\text{red}, 2\text{-exc}}(\mathcal{S}_P^u, \mathcal{S}_P) \xrightarrow{\text{ev}_{\mathbb{Z}}} \mathcal{S}_P$
 CORRESPONDENCE BY $\mathcal{P}^u$.

Def: A Hermitian object in $(\mathcal{C}, \mathcal{P})$ consists of
A pair (X, q)

- $X \in \mathcal{C}$

- $q \in \Sigma_1^\infty \mathcal{P}(X)$ Form on X .

IF $(\mathcal{C}, \mathcal{P})$ Poincaré, we say (X, q) is Poincaré

if $q_\# : X \xrightarrow{\sim} D_q X$

$$\begin{array}{c} \text{map}(X, D_q X) \cong B(X, X) \xleftarrow{1G_2} B(X, X) \xleftarrow{1G_2} \mathcal{P}(X) \\ \downarrow q_\# \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow q \end{array}$$

Exercise: The sum of Poincaré objects is again Poincaré.

Observation: $\text{Cat}^{\text{herm}} \xrightarrow{\quad} \text{Cat} \rightarrow \text{Spec}$

(REMEMBER THE FORM)

$(\mathcal{C}, \mathcal{P}) \longmapsto \{ \text{Hermitian objects in } \mathcal{C} \}$

represented by $(\Sigma_p^w, \mathcal{P}^u)$

Similarly $\text{Cat}^{\text{Poinc.}} \xrightarrow{\quad} \text{Spec}$

$(\mathcal{C}, \mathcal{P}) \longmapsto \{ (X, q) \text{ Poincaré obj in } (\mathcal{C}, \mathcal{P}) \}$

corep. by $(\Sigma_p^w, \mathcal{P}^u)$

Def: A Cobordism between Poincaré obj. (X', q') and (X, q) consists of

$$- X' \xleftarrow{f'} W \xrightarrow{f} X \quad \text{in } \mathcal{C}$$

$$- \alpha \in \underset{\mathcal{Q}(W)}{\mathcal{Q}(X')} \times \mathcal{Q}(X) \quad (f^*(q) \xrightarrow{\text{PATH}} f'^*(q'))$$

S.t. THE SQUARE

$$\begin{array}{ccc} W & \xrightarrow{f} & X \simeq DX \\ f' \downarrow & \circlearrowleft & \downarrow \\ X' \simeq DX' & \longrightarrow & DW \end{array} \quad (*)$$

IS A PULL-BACK.

($\leadsto$ EQUIV BETWEEN RELATIVE HOMOLOGY (LHS) AND COHOMOLOGY (RHS))

EX: $P \in \text{Proj}(R) \quad q: P \otimes_R \rightarrow P$
 λ -FORM
 $\xrightarrow{\text{TYPE OF FORM}} (P, q) \in \text{Forns}(\text{Perf}(R), \underset{R}{\mathcal{Q}}^{\text{eg}} \lambda)$

q unimodular $\leadsto (P, q) \in \text{Poinc}(\text{---})$

• $L \hookrightarrow P$ IS A NULL COBORDISM (LAGRANGIAN) IF

$q|_L = 0 \quad \text{AND} \quad 0 \rightarrow L \rightarrow P \equiv DP \rightarrow DL \rightarrow 0$
 EXACT SEQUENCE

(OR A POINCARÉ CYC)

• M CLOSED OR. MFD, $\dim M = n$
 $q_M: (C^*(M) \otimes_{\mathbb{Z}} C^*(M))_{\mathbb{Z}} \xrightarrow{\text{MULT}} C^*(M)$
 $\downarrow \text{ev}[M]$
 $\Leftrightarrow q_M \in \Omega^n \text{map}_{\mathbb{Z}}(C^*(M) \otimes_{\mathbb{Z}} C^*(M), \mathbb{Z})$
 $\leadsto \text{Poinc}(\text{Perf}(\mathbb{Z}), \Omega^n \mathcal{Q}_{\mathbb{R}}^s)$

$$M' \xrightarrow{i'} W \xleftarrow{i} M$$

$$\rightsquigarrow C^*(M') \leftarrow C^*(W) \rightarrow C^*(M)$$

From $i'_* [M'] = i_* [M] \rightsquigarrow q_M|_W \cong q_{M'}|_W$

$\rightsquigarrow$ GET $C^*(W, M') \cong \Omega^{n+1} C_*(W, M)$ ~~LEFT~~ ~~RIGHT~~
 $(\Leftrightarrow (*) \text{ is a Poincaré duality})$

Def: $(\mathcal{C}, \varphi)$ Poincaré. $\mathcal{Q}_1(\mathcal{C}, \varphi) = \text{Poincaré cat}$
 Aim $\nearrow$

$$D_{\varphi^{\text{cob}}} (X' \leftarrow W \rightarrow X)$$

$$= (D_{X'} \xleftarrow{D_{X'} \times D_X} D_X)$$

$$(\text{Fun}(\Lambda^2, \mathcal{C}), \varphi^{\text{cob}})$$

$$\{X' \leftarrow W \rightarrow X\}$$

$$\varphi^{\text{cob}} (X' \leftarrow W \rightarrow X) = \varphi(X') \times_{\varphi(W)} \varphi(X)$$

$\rightsquigarrow$ Poincaré objects $\text{Poinc}(\mathcal{Q}_1(\mathcal{C}, \varphi)) = \{ \text{cobordisms in } (\mathcal{C}, \varphi) \}$
 $(0 \leftarrow W \rightarrow X)$

$\text{Met}(\mathcal{C}, \varphi) \subseteq \mathcal{Q}_1(\mathcal{C}, \varphi) \xrightarrow{\text{lb} - \text{"LEFT BOUNDARY"}} (\mathcal{C}, \varphi)$
 $\uparrow$
 "METABOLIC FORMS" $(X' \leftarrow W \rightarrow X) \mapsto X'$

$\rightsquigarrow$ GET Poincaré functor: $\text{Met}(\mathcal{C}, \varphi) \subseteq \mathcal{Q}_1(\mathcal{C}, \varphi) \xrightarrow{\text{lb, rb}} (\mathcal{C}, \varphi) \times (\mathcal{C}, \varphi)$
 $(0 \leftarrow X \rightarrow 0) \uparrow$
 $X \in (\mathcal{C}, \Omega \varphi)$

$$\text{tr} : Q_1(\mathcal{C}, \varphi) \hookrightarrow \text{Met}(\text{Met}(\mathcal{C}, \Sigma \varphi))$$

$$\begin{array}{ccc} \downarrow & \lrcorner & \downarrow \\ (\mathcal{C}, \varphi) & \hookrightarrow & \text{Met}(\mathcal{C}, \Sigma \varphi) \end{array}$$

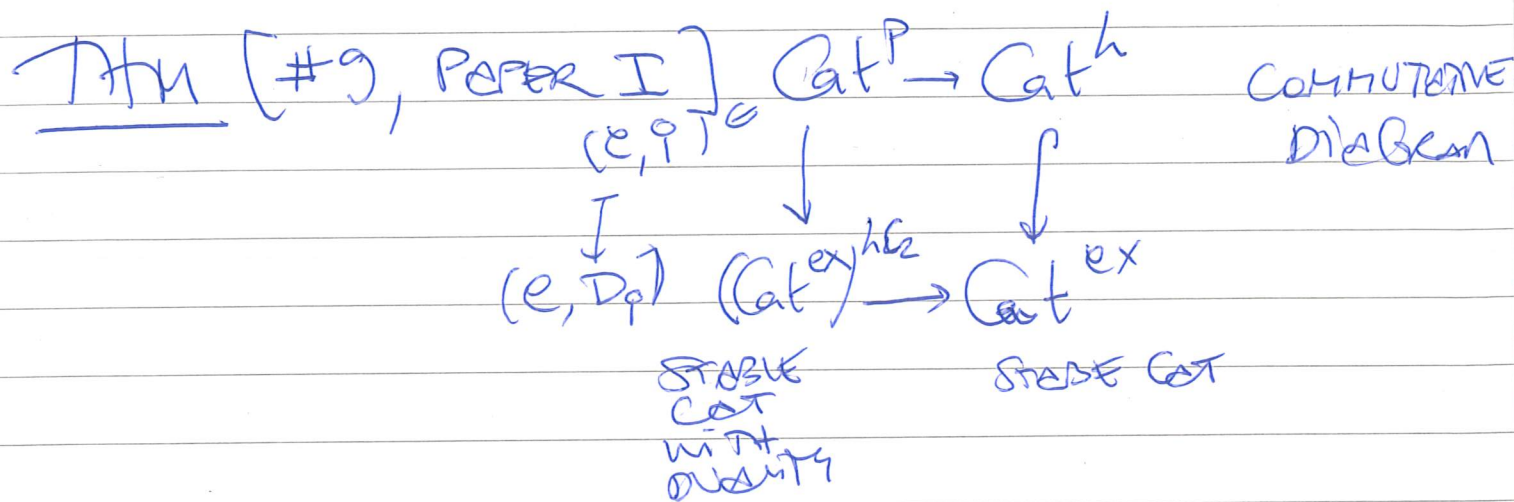
CAN USE THIS TO DESCRIBE $\text{Poin}(Q_1(\mathcal{C}, \varphi))$
IN A DIFF MANNER.

(CORRESPONDING TO SURGERY IS THE
TRACE OF COBORDISM)



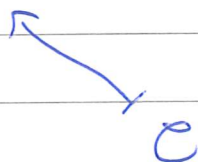
MARKUS LAND LECTURE 2

GROTHENDIECK-WITT & L-THEORY

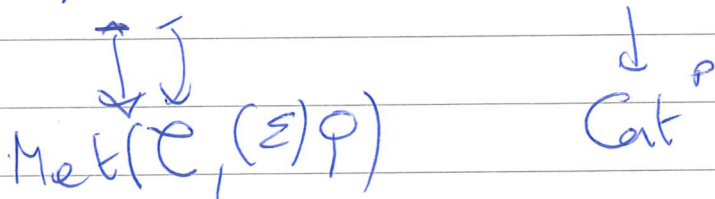


AND ALL FUNCTORS IN THE DIAGRAM HAVE ALL ADJOINTS.

eg. $\text{hyp}(e) = (e \oplus e^{\text{op}}, (X, Y) \mapsto \text{map}(X, Y))$



• (e, i) POINCARÉ in Cat^h



eg: CONSEQUENCE: $\text{Poinc}(\text{Met}(e, (e)_p)) \simeq \text{Form}(e, i)$

• $\text{Cat}^P, \text{Cat}^h$ ARE BICOMPLETE, SEMI-ADDITIVE.

• — ARE CLOSED SYMM. MONOIDAL.

$\leadsto \text{Cat}^h$ IS CAT-ENRICHED VIA THE HORN-TURNER

- Cat^h (co)TENSORED OVER SOME CATEGORY.
- REDUCES TO A (co)TENSORING ON Cat^1 IF I IS
FACE POSET OF A FIN. SIMPL. COMPLEX (LURIE).

DEF: $(\mathcal{C}, \mathcal{P}) \in Cat^1$. Define

$$GN(\mathcal{C}, \mathcal{P}) = \Omega_0 | Cob(\mathcal{C}, \mathcal{P}) |$$

↑
AT THE
0-POINCARÉ
OBJECT.

↑ $\{Obj\} = \text{POINCARÉ OBJECTS}$
in $(\mathcal{C}, \Sigma \mathcal{P})$
MORPH = COBORDISM

$$L(\mathcal{C}, \mathcal{P}) = \Omega_0 | ad(\mathcal{C}, \mathcal{P}) |$$

↑
TO BE DEFINED

ANALOGUE OF "COBORDISM SPACE"

MORE FORMALLY,

$Cob(\mathcal{C}, \mathcal{P}) = \text{CATEGORY ASSOCIATED TO THE}$
COMPLETE SEGAL SPACE

$$[n] \mapsto \text{Poinc}(\Omega_n(\mathcal{C}, \Sigma \mathcal{P}))$$

WITH $\Omega_n(\mathcal{C}, \Sigma \mathcal{P}) \Rightarrow$ CAN EXTEND SUCH AN OBJECT TALKING RUN-POLES

Smallest POINCARÉ CATEGORY

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    graph LR
      X00((X00)) --> X01((X01))
      X01 --> X11((X11))
      X11 --> X12((X12))
      X12 --> Xnn((Xnn))
      X00 --> X11
      X01 --> X12
      X11 --> Xnn
  
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$$\mathcal{P}(\text{ } \swarrow \text{ }) = \lim_{Tw(\Delta^n)} \mathcal{P}(X_{ij})$$

$$= \mathcal{Q}_1(\mathcal{C}, \Sigma \mathcal{P}) \times_{\mathcal{Q}_0(\mathcal{C}, \Sigma \mathcal{P})} \mathcal{Q}_1 \cdots \times \cdots$$

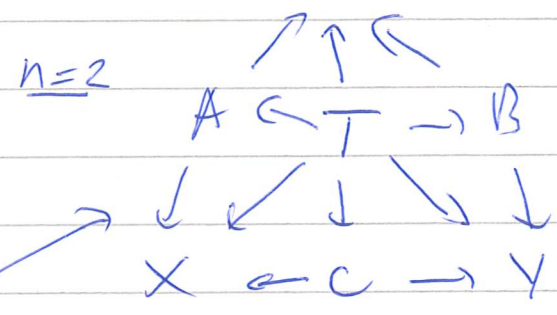
Smiley, ad $(\mathcal{C}, \mathcal{Q}) \in \text{Fun}(\Delta^{\text{op}}, \text{Cat}^P)$

$$[n] \mapsto \text{Poinc}(\mathcal{C}_n(\mathcal{C}, \Sigma \mathcal{Q}))$$

ii $(\mathcal{C}, \Sigma \mathcal{Q})$ $\mathcal{Z}_n = \text{FACE ASSET OF } \Delta^n$

$$n=1 : (X' \in W \rightarrow X)$$

$$w \rightarrow \mathcal{C}_1 = \mathcal{Q}_1$$



"n-AD"

$$\mathcal{Q} \in \mathcal{I}_n = \mathcal{Q}_i(-)$$

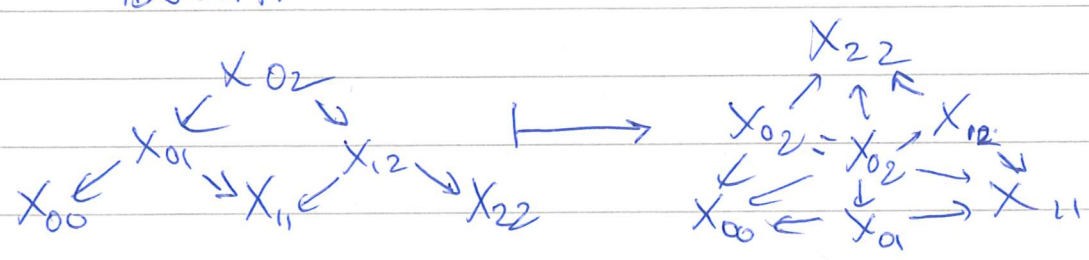
Def: $\mathcal{I}_n \Rightarrow \text{Tw}(\Delta^n) \ni (i \leq j) \quad 0 \leq i, j \leq n$

$$\emptyset \subseteq T \subseteq [n] \mapsto (\min(T), \max(T))$$

$$\mathcal{Q}_n(\mathcal{C}, \Sigma \mathcal{Q}) \mapsto \mathcal{C}_n(\mathcal{C}, \Sigma \mathcal{Q})$$

$n=0, 1$ identity

$n=2$



"DEGENERATE 2-AD"

→ Get a map $gw(\mathcal{C}, \mathcal{I}) \rightarrow \mathcal{L}(\mathcal{C}, \mathcal{I})$

" $\pi_1(gw(\mathcal{C}, \mathcal{I}))$ "
"

(in fact a loop map)
by construction)

OBSERVE: $\pi_0 |\mathcal{C}_b(\mathcal{C}, \mathcal{I})| \xrightarrow{\cong} \pi_0 |\mathcal{L}(\mathcal{C}, \mathcal{I})|$

// BECAUSE ISO ON 0- AND 1-SIMPLES
COHERENCE CLASS OF POINCARÉ OBJECTS.

EXPERIENCE: THIS IS A GROUP.

$\text{Poinc}(\mathcal{C}, \mathcal{I}) \rightarrow \mathcal{Q}_1(\mathcal{C}, \mathcal{I})$

$X \mapsto (0 \leftarrow X \rightarrow 0)$

$\searrow \text{Map}_{\mathcal{C}_b(\mathcal{C}, \mathcal{I})}^{(0,0)} \rightarrow \Omega |\mathcal{C}_b(\mathcal{C}, \mathcal{I})|$

comm.

group
in spaces

" $gw(\mathcal{C}, \mathcal{I})$ "

$\downarrow$
 $\mathcal{L}(\mathcal{C}, \mathcal{I})$

EX: $\text{Unimod}^{eq/eqs}(R) \rightarrow \text{Poinc}(\text{Perf}(R), \mathcal{P}_R^{eq/eqs})$

map of
comm. monoids

Group
completion

$GW^{eq/eqs}(R)$

$GW(\text{Perf}(R), \mathcal{P}_R^{eq/eqs})$

"
(Unimod -)

[HM[HS] THIS IS AN
EQUIVALENCE

CONSEQUENCE: NEED TO USE $\mathcal{P}_R^{eq/eqs} / \mathcal{P}_R^{eqs}$ TO
COMPUTE LHS.

PROOF USING "PARAMETRIZED ALGEBRAIC SURGERY"
 — ALGEBRAIC ANALOGUE OF THE GEOMETRIC
 VERSION BY GALATUS — RANDAL-WILLIAMS.

Def: A Full-back square $(\mathcal{C}, \mathcal{Q}) \rightarrow (\mathcal{C}', \mathcal{Q}')$

$$\begin{array}{ccc} (\ker p, \mathcal{Q}) & \xrightarrow{\quad} & (\mathcal{C}, \mathcal{Q}) \\ \downarrow & \searrow \cong & \downarrow p \\ 0 & \xrightarrow{\quad} & (\mathcal{Q}, \mathcal{Q}) \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & (\mathcal{C}', \mathcal{Q}') \\ & \searrow \cong & \downarrow p' \\ & \xrightarrow{\quad} & (\mathcal{Q}', \mathcal{Q}') \end{array}$$

in Cat^P is called POINCARÉ-VERDIER (PV)-
SQUARE IF LEFT SQUARE + BISEQUARE ARE
 ALSO COCARTESIAN.

• BISPILT PV-SQUARE IF p, p' IN ADDITION HAVE
 LEFT / RIGHT ADDMTR.

• $F : \text{Cat}^P \rightarrow \text{Spc}$ IS CALLED

- ADDITIVE IF IT TAKES BISPILT PV-SQUARES TO CART SQUARES
- LOCALIZING IF PV-SQUARES
- BORDISM INV IF $F(\mathcal{C}, \mathcal{Q}) \xrightarrow[F(\cong)]{F(\cong)} F(\mathcal{Q}, (\mathcal{C}, \mathcal{Q}))$
- GROUPLIKE IF $F(\mathcal{C}, \mathcal{Q}) \in \text{Grp}(\text{Spc})$
 $\forall (\mathcal{C}, \mathcal{Q})$.

Ex: $(\mathcal{C}, \Omega \mathcal{Q}) \rightarrow \text{Met}(\mathcal{C}, \mathcal{Q}) \xrightarrow[\text{RIGHT BORDISM}]{r_b} (\mathcal{C}, \mathcal{Q})$ BISPILT
 PV-SQUARE WITH 0 IN
 LEFT BOTTOM
 CORNER

$X \longmapsto (X \rightarrow 0)$

~~XXXXXXXXXXXXXXXXXXXX~~

$$\begin{array}{c}
 (e, \varphi) \\
 \downarrow \\
 \text{Met}(e, \varphi) \hookrightarrow \mathcal{Q}(e, \varphi) \xrightarrow{\text{lb}} e(\varphi) \\
 (L \mapsto X) \mapsto (0 \leftarrow L \rightarrow X) \downarrow \\
 \searrow \quad \rightarrow \text{Hyp}(e)
 \end{array}$$

• SCR MULTPL

$$\text{Perf}(R \otimes S, \mathcal{O}_R^{\otimes} |_{-}) \hookrightarrow (\text{Perf}(R, \mathcal{O}_R^{\otimes}) \rightarrow (\text{Perf}(R_S, \mathcal{O}_{R_S}^{\otimes}))$$

TYPICAL NON-SPIT PV SEQUENCE.

EXERCISE: F ADDITIVE + GROUP-LIKE. $F: \text{Cat}^P \rightarrow \text{Sp}$. THEN

$$F(\text{Met}(e, \varphi)) \xrightarrow{\sim} F(\text{Hyp}(e))$$

$$\begin{aligned}
 \bullet F \text{ BORDISM INV.} &\Leftrightarrow F(\text{Met}(e, \varphi)) = 0 \quad \forall (e, \varphi) \in \text{Cat}^P \\
 &F(\text{Hyp}(e)) = 0 \quad \forall e \in \text{Cat}^{\text{ex}}
 \end{aligned}$$

Thm [#9, PAPER II] $(e, \varphi) \mapsto |\text{Cob}(e, \varphi)|$
IS ADDITIVE.

MAIN INGREDIENT: $p: (e, \varphi) \rightarrow (\mathcal{Q}, \bar{\varphi})$ AS IN SPIT
PV-SOURCE
 $\Rightarrow \text{Cob}(e, \varphi) \rightarrow \text{Cob}(\mathcal{Q}, \bar{\varphi})$ IS A
BICOEFFICIENT FIBRATION

$\leadsto$ ALLOWS LIFTING.

Thm [Lurie] $(\mathcal{C}, \mathcal{Q}) \rightarrow (\mathcal{A}d(\mathcal{C}, \mathcal{Q}))$ localizing.

Remark: $\mathcal{A}d(p): \mathcal{A}d(\mathcal{C}, \mathcal{Q}) \rightarrow \mathcal{A}d(\mathcal{D}, \bar{\mathcal{Q}})$ KAN FIB.
of simplicial spaces

↪

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Marius Lund LECTURE III

Recall:

$$\text{Poinc}(\mathcal{C}, \mathcal{Q}) \longrightarrow \underset{\Omega_0 |\text{Cob}(\mathcal{C}, \mathcal{Q})|}{\text{GW}(\mathcal{C}, \mathcal{Q})} \longrightarrow \underset{\Omega_0 |\text{ad}(\mathcal{C}, \mathcal{Q})|}{\mathcal{Z}(\mathcal{C}, \mathcal{Q})}$$

Thm • $(\mathcal{C}, \mathcal{Q}) \mapsto |\text{Cob}(\mathcal{C}, \mathcal{Q})|$

$\text{Cat}^P \longrightarrow \text{Spec}$

is ADDITIVE,
GROUPLIKE

• $(\mathcal{C}, \mathcal{Q}) \mapsto |\text{ad}(\mathcal{C}, \mathcal{Q})|$

is LOCALIZING
[WRIE]

Corollary: GW is GROUPLIKE & ADDITIVE

$\mathcal{Z}$ is GROUPLIKE, LOCALIZING,
Bordism INVARIANT

EXERCISE

AND $\mathcal{Z}(\mathcal{C}, \mathcal{Q}) \simeq |\text{Poinc}(\mathcal{C}, \mathcal{Q})|$

$\Omega |\text{Poinc}(\mathcal{C}, \mathcal{Q})|$

CANCEL OUT

Prop: • $\text{Fun}^{(\text{add/loc}) + \text{bord.-inv}}(\text{Cat}^P, \text{Spec}) \xleftarrow[\simeq]{\text{add/loc} + \text{b.mw}} \text{Fun}^{(\text{add/loc}) + \text{b.mw}}(\text{Cat}^P, \text{Sp})$

IN PARTICULAR

$\mathcal{Z} \longleftarrow \mathcal{Z}^{\text{add}}$

• $\exists$ PREFERRED GROUPING OF GW, $\text{GW} \in \text{Fun}^{\text{add}}(\text{Cat}^P, \text{Sp})$

with $\Omega^\infty GW = GW + B(\Omega^\infty GW) = |\text{Cob}(-)|$.

(it is given by the ~~staggered~~ ARE-SPECTRUM of GW in $\text{Fun}^{\text{add}}(\text{Cat}^P, \text{Spc})$).

BLACK BOX: This ARE-SPECTRUM is a SPECTRUM USING AN IMPROVED VERSION OF PREVIOUS THM

CLAIM: IF F ADDITIVE, THEN $|F(\Omega(-))| = \Sigma F$

• THE MAP $GW \rightarrow L$ ~~PROPS~~ TO A MAP $GW \rightarrow L$ SO THAT $B(\Omega^\infty GW) \rightarrow B(\Omega^\infty L)$

$$\begin{array}{ccc} & \parallel & \parallel \\ & |\text{Cob}(-)| & \rightarrow |\text{ad}(-)| \end{array}$$

IS THE SAME MAP AS BEFORE.

PROOF SKETCH / INGREDIENTS:

$\text{Fun}^{\text{add/loc, bord-inv}}(\text{Cat}^P, \text{Spc})$ IS STABLE: Ω ON THIS

CATEGORY IS AN ~~ENRICHMENT~~ ~~ENRICHMENT~~: $(\Omega f)(e, i) = \Omega(f(e, i))$

$$(e, \Omega i) \rightarrow \text{Met}(e, i) \rightarrow (e, i) \quad \exists \text{ SPLIT PV}$$

$$\leadsto \begin{array}{ccccc} f(e, \Omega i) & \rightarrow & f(\text{Met}(e, i)) & \rightarrow & f(e, i) \end{array} \quad \begin{array}{l} \text{FIB. SEQU.} \\ (f \text{ ADD}) \end{array}$$

$$\Omega f(e, i) \leftarrow \underbrace{f(\text{Met}(e, i))}_{\substack{\simeq * \\ \text{BECAUSE } f \\ \text{BORDISM INVARIANT}}}$$

Thm [9, Paper II] $\Rightarrow$ Fiber sequence/square

$$\begin{array}{ccccc} K(\mathcal{C}, \mathcal{Q})_{h\mathcal{C}_2} & \xrightarrow{\text{hyp}} & GW(\mathcal{C}, \mathcal{Q}) & \longrightarrow & L(\mathcal{C}, \mathcal{Q}) \\ & \searrow \text{Norm} & \downarrow & & \downarrow \\ & & K(\mathcal{C}, \mathcal{Q})^{h\mathcal{C}_2} & \longrightarrow & K(\mathcal{C}, \mathcal{Q})^{t\mathcal{C}_2} \end{array}$$

Proof: $(\mathcal{C}, \mathcal{Q}) \xrightarrow{\text{Forget}} \mathcal{C} \xrightarrow{\text{Hyp}} \text{Hyp}(\mathcal{C})$

$$\begin{array}{ccccc} \mathcal{C}at^P & \xrightarrow{\text{Forget}} & \mathcal{C}at^{ex} & \xrightarrow{\text{Hyp}} & \mathcal{C}at^P \\ & & \uparrow & & \nearrow \\ & & (\mathcal{C}at^{ex})^{h\mathcal{C}_2} & & \end{array}$$

$\text{Hyp}(\mathcal{C}) \in (\mathcal{C}at^+)^{BC_2} \longleftrightarrow$ object with $\mathcal{C}_2$ -action

$$\begin{array}{c} (X, \mathcal{Q}) \longmapsto (DY, DX) \\ \uparrow \\ \mathcal{C} \oplus \mathcal{C}^{op} \end{array}$$

General Categorical Bialgebra (or "Argument")

$$\begin{array}{c} \leadsto \text{Hyp}(\mathcal{C}) \longrightarrow (\mathcal{C}, \mathcal{Q}) \text{ } \mathcal{C}_2\text{-Equiv} \\ (X, Y) \longmapsto X \oplus DY \end{array}$$

EX (below)

$$K(\mathcal{C}, \mathcal{Q})_{h\mathcal{C}_2} \cong GW(\text{Hyp}(\mathcal{C}))_{h\mathcal{C}_2} \xrightarrow{\text{hyp}} GW(\mathcal{C}, \mathcal{Q})$$

$$\begin{array}{ccc} \downarrow & \parallel & \downarrow \\ 0 = L(\text{Hyp}(\mathcal{C}))_{h\mathcal{C}_2} & \longrightarrow & L(\mathcal{C}, \mathcal{Q}) \end{array} \quad (*)$$

CONNECTIVE K-THEORY SPECTRUM

$\uparrow$ L BORDISM INVARIANT

EXERCISE: $\mathcal{Q}(\text{Hyp}(e, i)) = \text{Hyp}(\mathcal{Q}(e, i))$

Point $\text{---} = \text{Conf}(\mathcal{O}_0(p, q))$

→ DEF. of $K(\mathcal{E})$

Claim: $(*)$ is a push-out.

Get a map $\text{cofib}(\text{hyp}) \rightarrow L$

① $\text{cofib}(h_{\text{gp}})$ is bordism invariant
(check by applying to $h_{\text{gp}}(e)$)

+ ADDITIVE

② $\pi_{-1} \text{Cofib}(h_g) \xrightarrow{\cong} \pi_{-1} L$

$K(e)$ her
CONNECTIVE

 $\pi_{-1}GW$

COFIB
BORDIN

$$\begin{array}{ccc} \rightarrow \pi_2 \operatorname{cofib}(\operatorname{hyp}(c, q)) & \xrightarrow{\sim} & \pi_1 \operatorname{cofib}(\operatorname{hyp}(c, \sigma^{-1} q)) \\ \downarrow \cong & \swarrow \searrow & \downarrow \cong \\ \pi_2 L(\text{---}) & \xrightarrow{\sim} & \pi_1 L(\text{---}) \end{array}$$

Cor. GW is localizing (L is localizing, K is localizing)

$$\bullet \text{ } \mathcal{O}W(e, 1) \left[\frac{1}{2} \right] \simeq K(e, 1)_{hC_2} \left[\frac{1}{2} \right] \oplus L(e, 1) \left[\frac{1}{2} \right]$$
$$(\text{on } \pi_{\downarrow}(K(e, i)_{hG_2}[\frac{1}{2}])) = \pi_{\downarrow}(K(e, i)_{hG_2}[\frac{1}{2}])_{hG_2}$$
$$= \left(\pi_{\mathbb{A}} K(\mathcal{C}, \mathcal{P}) \left[\frac{1}{2} \right] \right) c_2$$

$$\Rightarrow \mathrm{GW}^{\mathrm{eg}/\mathrm{es}}(\mathbb{Z})[\tfrac{1}{2}] \simeq \Omega^\infty \mathrm{GW}(\mathrm{Perf}(\mathbb{Z}), -)$$

$$\langle \mathrm{GW}(\mathrm{Perf}(\mathbb{Z}), \mathrm{GW}) \rangle = k(\mathbb{Z})[\tfrac{1}{2}]_{h_2} \oplus L^{\mathrm{eg}/\mathrm{es}}(\mathbb{Z})[\tfrac{1}{2}]$$

LEMMA: $\mathbb{G}_2$ action on $\pi_{2n-1} k(\mathbb{Z})$
 $+ \pi_{2n-2} k(\mathbb{Z})$

is MULT. BY $(-1)^n$.

Fact: $\begin{cases} KO[\tfrac{1}{2}] \varepsilon = 1 \\ \Omega^2 KO \varepsilon = -1 \end{cases}$

MARKUS LAND LECTURE 4 $GW(\mathbb{Z})$

RECALL: $GW(\mathcal{C}, \mathcal{I}) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = K(\mathcal{C}, \mathcal{I})|_{h\mathbb{Z}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \oplus L(\mathcal{C}, \mathcal{I}) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

FACT: AT PRIME 2, THE FIBER SEQUENCE

$$K(\mathcal{C}, \mathcal{I})|_{h\mathbb{Z}} \rightarrow GW(\mathcal{C}, \mathcal{I}) \rightarrow L(\mathcal{C}, \mathcal{I})$$

SUFFERS FROM THE FACT THAT THE 2-TORSION OF THE BASE AND THE FIBER CANCEL-OUT AND THIS NEEDS TO BE ANALYZED...

BETTER THING (THORARON HUP):

$$\begin{array}{ccc} GW(\mathcal{C}, \mathcal{I}) & \xrightarrow{\sim} & K(\mathcal{C}, \mathcal{I})|_{h\mathbb{Z}} \\ \downarrow & & \nearrow \textcircled{2} \\ & GW(\mathcal{C}, \mathcal{I}_D^S) & \end{array}$$

FACTORS AS TARGET ONLY DEPENDS ON THE EVENT.

$$\begin{array}{ccc} \text{COR: } GW(\mathcal{C}, \mathcal{I}) & \rightarrow & GW(\mathcal{C}, \mathcal{I}_D^S) \\ \downarrow & \dashv & \downarrow \\ L(\mathcal{C}, \mathcal{I}) & \rightarrow & L(\mathcal{C}, \mathcal{I}_D^S) \end{array}$$

AS THE FIBERS ARE K-THEORY STUFF THAT DON'T SEE THE DIFFERENCE.

THM (#9, Paper III) LET R BE A RING OF INTEGERS IN A NUMBER FIELD (eg $R = \mathbb{Z}$)

a) $GW(\text{Perf}(R), \mathcal{O}_R^{\text{eg}}) \rightarrow GW(\text{Perf}(R), \mathcal{O}_R^{\text{es}})$ ISO ON π_* FOR $* \geq 2$ $E=1$
 $GW(\text{Perf}(R), \mathcal{O}_R^{\text{eg}}) \xrightarrow{\tau \geq 0} \text{---} //$ $* \geq 4$ $E=-1$

b) $GW(\text{Perf}(R), \mathcal{O}_R^{\text{es}}) \rightarrow GW(\text{Perf}(R[\frac{1}{2}]), \mathcal{O}_{R[\frac{1}{2}]}^{\text{es}})$ 2 LOCAL EQUIV ON π_* $* \geq 1$

c) $GW(\text{Perf}(R), \mathcal{O}_R^{\text{es}}) \xrightarrow{\sim} K(\text{Perf}(R), \mathcal{O}_R^{\text{es}})^{h_G}$ 2-ADIC EQUIV.

INGREDIENTS: a) EXPLICIT COMPUTATION USING ALGEBRAIC SURGERY.

b) $(\text{Perf}(R \text{ on } 2), \mathcal{O}_R^{\text{es}}) \rightarrow (\text{Perf}(R), \mathcal{O}_R^{\text{es}}) \rightarrow (\text{Perf}(R[\frac{1}{2}]), \mathcal{O}_{R[\frac{1}{2}]}^{\text{es}})$
 $\uparrow$ (2-TORSION CPXS OVER R)

PV FREQUENCIES

$\oplus_{p \in \text{Spec}(R)} (\text{Perf}(R/p), \mathbb{Z} \mathcal{O}_R^{\text{es}})$
 LIES OVER 2

THM THIS IS A GW-ISO.

PF: PV FREQUENCIES WITH K -THEORY (GILBERT'S DEVIANCE) $\checkmark$
 L -THEORY $\checkmark$

(VIA ALG. SURGERY)

DEVIANCE FOR SIM. WITH GROUPS OF ACTION

(KNEIBUCH ET AL.)

GW OF THIS NEEDS TO BE SHOWN

$$\begin{array}{ccccc}
 \mathrm{GW}(\mathbb{F}_q, \Omega q^{\mathrm{es}}) & \longrightarrow & K(\mathbb{F}_q, \Omega q^{\mathrm{es}})^{hG_2} \\
 \downarrow & & \downarrow & & \downarrow \\
 L(\mathbb{F}_q, \Omega q^{\mathrm{es}}) & \longrightarrow & K(\mathbb{F}_q, \Omega q^{\mathrm{es}})^{tG_2} & \longrightarrow & \mathbb{Z}^{tG_2}
 \end{array}$$

↑
CALCULATE BOTH SIDES

- GET RID OF Ω
- $L(\mathbb{F}_q, q^{\mathrm{es}}) \rightarrow K(\mathbb{F}_q, q^{\mathrm{es}})$ RING MAP

...

CAN USE b) TO PROVE c) USING THE EXISTING LITERATURE FOR $R[\frac{1}{2}]$.

MORE DIRECT COMPUTATION WHEN $R = \mathbb{Z}$:

THM [BOKSTADT, ROGNER-WEIBEL]

$$\begin{array}{ccccc}
 K(\mathbb{Z}[\frac{1}{2}])_2^1 & \longrightarrow & KO_2^1 & \cong & \text{TRIVIAL } G_2\text{-ACTION} \\
 \swarrow \checkmark & \downarrow & \downarrow & \downarrow & \\
 K(\mathbb{F}_3)_2^1 & \xleftarrow{\cong} & K(\mathbb{Z}_3)_2^1 & \longrightarrow & KU_2^1
 \end{array}$$

$$\mathbb{Z}_3 \subseteq \mathbb{C}_3 \cong \mathbb{C}$$

PROP: $L(\mathbb{Z}[\frac{1}{2}]) \rightarrow L\mathbb{R}$

$$\begin{array}{ccc}
 & \downarrow & \downarrow \\
 & L(\mathbb{Z}_3) & \rightarrow L\mathbb{C}
 \end{array}$$

IF: CALCULATE.

INTERESTING ENOUGH MAP.

$$\begin{array}{ccc} \text{TO SEE THAT } & \text{GW}(\mathbb{Z}[\frac{1}{2}]) & \rightarrow K(\mathbb{Z}[\frac{1}{2}])^{hG_2} \\ & \downarrow & \downarrow \\ & L(\mathbb{Z}[\frac{1}{2}]) & \rightarrow K(\mathbb{Z}[\frac{1}{2}])^{tG_2} \end{array}$$

REDUCES TO CASE $\mathbb{Z}_3, \mathbb{R}, \mathbb{C}$.

EXERCISE: $E = \{L_2^1, K(-)^{tG_2}\}$

$$E_*(\mathbb{Z}_3) = \mathbb{Z}/4\mathbb{Z}[b^{\pm 1}]$$

$$E_*(\mathbb{R}) = \mathbb{Z}_2[b^{\pm 1}]$$

$$|b| = 4$$

$$E_*(\mathbb{C}) = \mathbb{Z}_2[b^{\pm 1}]$$

CONSTRUCTION: $\text{GW}^{\text{es}}(\mathbb{Z}[\frac{1}{2}]) \rightarrow K^0$ $(E=1)$

$$\begin{array}{ccccc} & \downarrow k_0 & & & \\ & \text{GW}^{\text{es}}(\mathbb{Z}[\frac{1}{2}]) & \rightarrow & K^0 & \xrightarrow{hG_2} K^0 \oplus K^0 \\ & \downarrow & & \downarrow & \text{UP TO COMPARISON} \\ K(\mathbb{Z}_3) & \xrightarrow{hG_2} & K\mathbb{A} & \xrightarrow{hG_2} & K\mathbb{U} \\ & & & \searrow \psi_{-1}^3 & \downarrow hG_2 \\ & & & & K\mathbb{U} \\ & & & & \downarrow hG_2 \\ & & & & K\mathbb{O} \\ & & & & \downarrow hG_2 \\ & & & & K\mathbb{O} \end{array}$$

pull-back
UP TO π_0, π_1 ISUVER ...

ADAMS OBSERVATION

THM [...]: $\Omega_0^\infty \text{GW}^s(\mathbb{Z})_2^1 = \Omega_0^\infty (K^0 \oplus \text{fib}(\psi_{-1}^3; K^0 \rightarrow \tau_{\geq 2} K^0))$

In Sp^0 : $\Omega_0^\infty \text{GW}^q(\mathbb{Z})_2^1 = \Omega_0^\infty (K^0 \oplus \text{fib}(\psi_{-1}^3; K^0 \rightarrow \tau_{\geq 4} K^0))$

$\Omega_0^\infty \text{GW}^{-s}(\mathbb{Z})_2^1 = \Omega_0^\infty (\text{fib}(\psi_{-1}^3; K\mathbb{U} \rightarrow \tau_{\geq 1} K\text{Sp}))$

$\Omega_0^\infty \text{GW}^{-q}(\mathbb{Z})_2^1 = \mathbb{Z}/4\mathbb{Z} \times \text{fib}(\Omega_0^\infty \text{GW}^{-s}(\mathbb{Z})_2^1 \rightarrow K(\mathbb{Z}_2/2) \times K(\mathbb{Z}_2/2))$

→ OPEN OLD BOOKS TO GET CONCRETE COMPUTATIONS --

Car [HABERSTADT-L-MANOW]

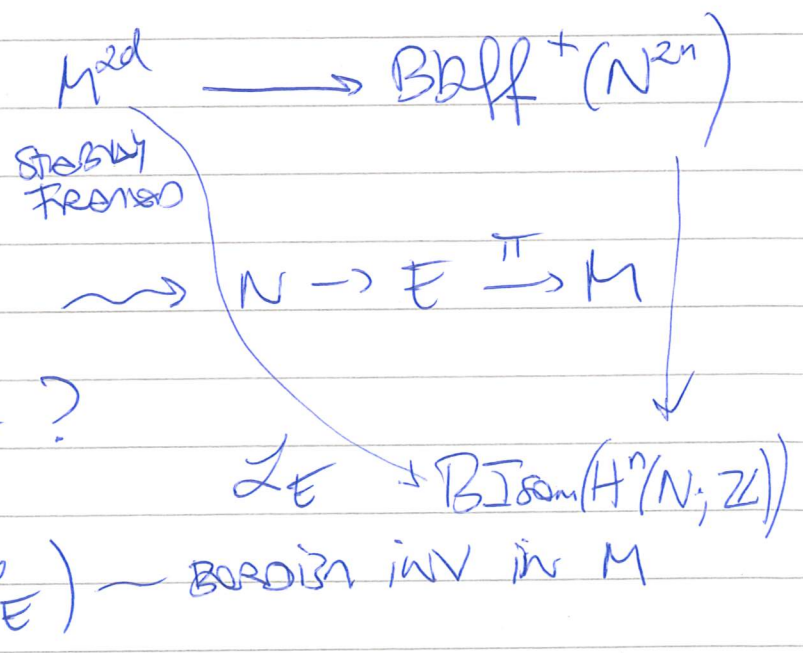
$$H^*(Sp_{\infty}(\mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2[c_i]_{i \geq 1} \otimes \mathbb{F}_2[b_i]_{i \geq 1}$$

$(\hookrightarrow qg(-s)(\mathbb{Z}) \dots)$
 $|c_i| = 2i$
 $|b_i| = 4i - 1$

$$H^*(Sp_{\infty}^q(\mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2[u, x, z_j, c_i]_{i \geq 2, j \geq 1} \otimes \mathbb{F}_2[v, y, b_i]_{i \geq 2}$$

$(\hookrightarrow q^{g(-q)}(\mathbb{Z}))$
 $|u| = 2$ $|z_j| = 2j + 1$ $|x| = 4$
 $|v| = 1$ $|c_i| = 2i$ $|y| = 5$
 $|b_i| = 4i - 1$

ANOTHER APPLICATION:



QUESTION: $\text{sign}(E) = ?$

$\text{sign}(M, \mathbb{Z}_E) \sim$ BORDISM INV in M

COMPUTABLE VIA

$$\begin{aligned} \Omega_{2d}^{\text{fr}}(B\text{Isom}(H^*(N; \mathbb{Z}))) &\xrightarrow{\text{fr}} \mathbb{Z} \xrightarrow{\text{sign}(E)} \\ \downarrow & \\ \Omega_{2d}^{\text{fr}}(\Omega^u \text{GW}^{es}(\mathbb{Z})) &\xrightarrow{\text{fr}} \pi_{2d} \text{GW}^{es}(\mathbb{Z}) \xrightarrow{\text{fr}} \pi_{2d} L_{\sigma}^{es}(\mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$

Thm [L] $\varepsilon=1$ IMAGE of sign: $\pi_{2d}^{GW}(\mathbb{Z}) \xrightarrow{\sigma} \mathbb{Z}$

$$\bar{B} \begin{cases} 2^{d+1} \mathbb{Z} \\ 2^{d+2} \mathbb{Z} \end{cases}$$

$$d=0,1,3 \quad (4)$$

$$d=2 \quad (4)$$

MARKUS LAND: EXTRA LECTURE

GW + L-Theory of spaces

$$\text{Man}_d^{C, \approx} \longrightarrow \text{Sp}_{PD}^{\approx}$$

spaces
with Poncaré
duality

$$? \longmapsto X$$

RECAP/DEF: IF $X \in \text{Sp}^w$ CPT SPACE

THEN

$$\text{Sp}^X \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{\text{constant}} \\ \xrightarrow{\Gamma^*} \end{matrix} \text{Sp}$$

Γ_*

Since X correct
↓

(MURITA THEOREM)

$$\text{Sp}^X \xrightarrow{\cong} \text{Fun}^L(\text{Sp}^X, \text{Sp}) \ni \Gamma_*$$

$$E \longmapsto [F \rightarrow \Gamma_!(E \otimes F)]$$

$$\Leftrightarrow \text{For } X \in \text{Sp}^w \exists! D_X \in \text{Sp}^X \text{ dual CPT s.t.}$$

$$\Gamma_*(F) = \Gamma_!(D_X \otimes F)$$

LEMMA: $X \in \text{Sp}^w \Rightarrow$

$$\text{Fun}(X, \text{Sp})^w \supseteq \text{Fun}(X, \text{Sp}^w)$$

POINTWISE
CORRECT

$$\text{Pic}(S) \subset \text{Sp}^{\approx}$$

$$\simeq \left(\coprod_{n \geq 0} BG(n) \right)^{\text{gr}} \simeq_{\text{SPACE}} \mathbb{Z} \times BG$$

$$\text{Fun}(X, \text{Pic}(S))$$

POINTWISE
INVERTIBLE

DEF: $X \in \text{Spc}^w$ is called ∇D if $D_X \in \text{Fun}(X, \text{Pic } S)$

EX: M closed MFD $\xrightarrow{w_M} M \longrightarrow \mathbb{Z} \times BG$
 $\Downarrow$ $\xrightarrow{\quad} \mathbb{Z} \times BG$

$$\text{Pic}(S) \longrightarrow \text{Cat}^P = \text{POINCARÉ CATEGORY}$$

$$\downarrow \quad \downarrow$$

$$S^d \longmapsto (S^w, \Omega^d \mathcal{P}^u)$$

$$\Rightarrow X \xrightarrow{\zeta} \text{Pic}(S) \longrightarrow \text{Cat}^P$$

SPHERICAL
FIBRATION

FACT: Cat^P HAS ALL LIMITS
& COLIMITS

"VISIBLE"
[WEISS]

DEFINE: $((S^w)^w, \mathcal{P}_{\zeta}^w) = \underset{X}{\text{colim}} (\downarrow)$

$$1) \mathcal{P}_{\zeta}^w(E) \longrightarrow \text{map}(E, \zeta)$$

$$\downarrow \quad \searrow$$

$$\text{map}_{X \times X}(E \boxtimes E, \Delta_!(\zeta)) \xrightarrow{hG_2} \text{map}_{X \times X}(E, \Delta^* \Delta_!(\zeta) \boxtimes E)$$

$$D_{\mathcal{P}_{\zeta}^w}(E) = D_{CW} \otimes \zeta^{\pm 1} = \text{map}_X(E, \Delta_!(\zeta))$$

CONSTANTNESS
- MAVER AVAILTY

$$L^q(\mathcal{S}) \longrightarrow L^u(\mathcal{S}) \longrightarrow \mathcal{S} \oplus \text{MTO}(1)$$

$\Downarrow$

$\downarrow$

$$L^q(\mathcal{S}) \longrightarrow L^s(\mathcal{S}) \longrightarrow (\mathcal{S}_2' \otimes_{\mathcal{S}} \mathcal{S}_2') \oplus \text{MTO}(1)$$

$\uparrow$
OBSCURE
OBJECT ? ...

REM: $\text{colim}_{(X, \mathcal{Z})} (\text{Perf}(\mathbb{Z}), \varphi^{\text{aff/5.11.11}}) \Leftrightarrow \text{"ALGEBRAIC VISIBILITY THEORY"}$

$$\text{colim}_{(X, \mathcal{Z})} (\mathcal{S}_p^w, \varphi^q) \simeq (\mathcal{S}_p^X)^w, \varphi^q_{\mathcal{Z}}$$

FACT: $X \in \mathcal{S}pc_{\text{AD}}^w \Rightarrow D_X \in \text{Poinc}^{\text{can}}((\mathcal{S}_p^X)^w, \varphi^v_{D_X})$

REASON ← compute $\varphi^v_{D_X}(D_X) \rightarrow \Gamma_! D_X$ (as then spect. (NORMAL BR))

$$\begin{array}{ccc} \downarrow & & \downarrow \\ (\Gamma_! D_X)^{+G_2} & \xrightarrow{h_{G_2}} & (\Gamma_! D_X)^{+G_2} \end{array}$$

FACT: $D_{\text{CW}}(\Gamma^* \mathcal{S}) \simeq D_X \rightarrow D_{D_X}(D_X) \simeq D_X$

OR: $(X, \mathcal{Z}) \rightsquigarrow X \rightsquigarrow \underbrace{\lim_X (\mathcal{S}_p^w, \varphi^u)}_{\text{ANALYTIC } (E \mapsto E^V \text{ POINTWISE ANAL})} = (\mathcal{S}_p^w)^X, \lim_X \varphi^u(-)$

$$\text{Poinc} \left(\varinjlim_X (\mathcal{F}_p^w, \mathcal{Q}^u) \right) = \varinjlim_X \text{Poinc} (\mathcal{F}_p^w, \mathcal{Q}^u) \ni r^*(S)$$

Fact: $\varinjlim_X (\mathcal{F}_p^w, \mathcal{Q}^u) \xrightarrow{i \mapsto i \otimes D_X} \text{colim}_X ((\mathcal{F}_p^x)^w, \mathcal{Q}_{D_X}^v)$

When X is PD, FURTHER FUNDAMENTAL POINCARÉ EMBEDDING.

$$\Rightarrow [D_X] \in \text{Poinc} \left(\varinjlim_X (\mathcal{F}_p^w, \mathcal{Q}^u) \right)$$

AND HENCE AN ELEMENT IN L -THEORY AND GW

Claim: a) Given $[D_X] \in L^V(X, D_X)$

b) UPGRADES TO A MAP OF SPACES

$$[X \xrightarrow{\text{id}} X] \in \Omega^{\text{PD}}(X) \longrightarrow L^V(X, D_X)$$

"FUNDAMENTAL CLASS"



← COFIBER OF THE MAP FROM

$$\Omega^N(X) \longrightarrow L^N(X, D_X) \quad L^q(X, D_X)$$

"NORMAL BORDISM"

⚠ $\Omega^{\text{PD}}(-)$ is NOT A Homology Theory.

(Homology theory for $\text{Spec } \mathbb{Z}$ stop)

IF X MFD,

$$\Omega^{\text{mfd}}(X)$$

$$\longrightarrow \text{colim}_{(X, \mathcal{F})} L(\mathcal{F}^w, \mathcal{Q}^u)$$



$$\Omega^{\text{PD}}(X) \longrightarrow L(\text{colim}_{(X, \mathcal{F})} \dots) \ni [D_X]$$

DEF $\tilde{\Sigma}_{\text{per}}^{\text{top}}(X) = \text{cofib} \left(\text{ad}_X L(-) \rightarrow L(\text{coll}_X -) \right)$

RELATED
TO BLOW-UP

$= \text{cofib}_X \text{ OF ASSEMBLY: } \text{col}_X L^9(\mathbb{Z}) \rightarrow L^9_{\text{col}_X \mathbb{Z}}$

~~OR PD~~

OR PD $\rightarrow (X, \mathbb{Z})$
 $X_+ \otimes L^9(\mathbb{Z}) \xrightarrow{\text{ass}} L^9(\mathbb{Z} \pi_1(X))$

$[D_X] \mapsto \boxed{\uparrow} \in \tilde{\Sigma}_{\text{per}}^{\text{top}}(X)$

ADD TO THE OBSTRUCTION TO HTPY
 EQUIV TO A DEP. MFD.

LIFTING $[D_X]$ TO ELEMENTS IN $\Omega^d(X_+ \otimes L^S(\mathbb{Z}))_{\geq 0}$

GIVE AN ELEMENT IN $\pi_0(\Omega^d(X_+ \otimes \mathbb{Z})) \stackrel{\pi_0}{\cong} \mathbb{Z}$
 OR.

LEMMA: THIS INTEGER LIES IN $1+8\mathbb{Z}$.

