Sander Kupers :

Lecture 1: Calculus of embeddings

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(lecture notes on webpage)

1) Spaces of embeddings

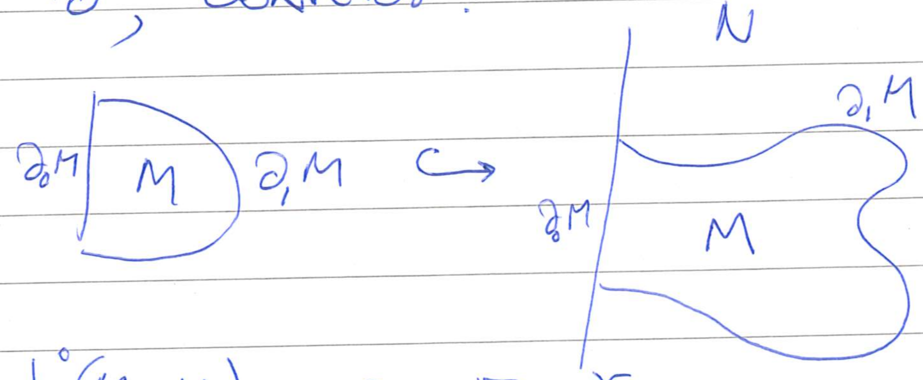
M, N d-dim, smooth, without ∂

$M \hookrightarrow N$ smooth embedding $\iff$ homeo onto image + locally (w.r.t. charts) $\mathbb{R}^d \xrightarrow[id]{} \mathbb{R}^n$

Def: $\mathrm{Emb}^{\partial}(M,N) \subset C^\infty(M,N)$ smooth maps with C^∞-top.

Can also have ∂, corners.

eg triads



Ex: . $\mathrm{Diff}(M) \subset \mathrm{Emb}^{\partial}(M,M)$ equality if M closed

. $O(d) \hookrightarrow GL_d(\mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{R}^d) \hookrightarrow \mathrm{Emb}(\mathbb{R}^d, \mathbb{R}^d)$ derivative at 0

are all equivalences.

WHY? EMBED MFDS INTO M OR LET $\mathrm{Diff}(M)$ ACT ON $\mathrm{Emb}(P,M)$.

$$\mathrm{Diff}_\partial(M \setminus \mathrm{int}\, e(P)) \longrightarrow \mathrm{Diff}(M) \longrightarrow \mathrm{Emb}^0(P,M)_{\ni e}$$

EXTENSION BY id

SERRE FIBRATION IF P COMPACT

(2) TOPOLOGICAL EMBEDDINGS

DEFINITION MAKES SENSE TOPOLOGICALLY.

DEF: $\mathrm{Emb}^t(M,N) \subset C^0(M,N)$ CONTINUOUS MAPS WITH COMPACT-OPEN TOPOLOGY

EX: $\mathrm{Top}(d) \xhookrightarrow{\simeq} \mathrm{Homeo}(\mathbb{R}^d) \xhookrightarrow{\simeq} \mathrm{Emb}^t(\mathbb{R}^d, \mathbb{R}^d)$ [KISTER].

$\mathrm{Top}(d)$ = GERMS OF HOMEO NEAR 0.

2. CALCULUS OF EMBEDDINGS

GOAL: GIVE A HOMOTOPY-THEORETIC DESCRIPTION OF $\mathrm{Emb}^c(M,N)$, $c \in \{0,t\}$

2.1) SET-UP: $\mathrm{Man}_d^c = \begin{cases} \text{OBJ} = d\text{-di MFDS } M \\ \text{MAPPING SPACES} = \text{EMBEDDINGS} \end{cases}$

THINK OF THIS AS ∞-CATEGORY VIA COHERENT NERVE, OR TOPOLOGICALLY ENRICHED.

M d-MFD $\rightsquigarrow$ PRESHEAF $(\mathrm{Man}_d^c)^{op} \xrightarrow{E_M} \mathcal{S}pc$

$$P \longmapsto \mathrm{Emb}^c(P, M).$$

$$\mathrm{Man}_d^c \overset{y}{\hookrightarrow} \mathrm{PSh}(\mathrm{Man}_d^c) \quad \text{YONEDA EMB}$$

$$M \longmapsto E_M := y(M).$$

IDEA OF EMBEDDING CALCULUS: RECOVER E_M FROM RESTRICTION TO SUBCAT OF DISCS IN Man_d^c.

$\mathrm{Disc}_d^c \underset{i}{\subset} \mathrm{Man}_d^c$ FULL SUBCATEGORY ON $S \times \mathbb{R}^d$ FOR S FINITE SET.

$$\mathrm{PSh}(\mathrm{Disc}_d^c) \underset{i^*}{\longleftarrow} \mathrm{PSh}(\mathrm{Man}_d^c)$$

EX: $(i^* E_M)(S \times \mathbb{R}^d) = E_M(S \times \mathbb{R}^d) = \mathrm{Emb}^c(S \times \mathbb{R}^d, M)$

$$\downarrow \simeq$$

$$\mathrm{Conf}_S(M) \underset{\Sigma_n}{\times} \mathrm{Fr}_r^c(TM)$$

WHERE $\mathrm{Fr}_r^c(TM)$ = GERMS OF EMB. $\mathbb{R}^d \to M$

Def: (EMBEDDING CALCULUS APPROX)

$$\mathrm{Emb}^c(M,N) \longrightarrow T_\infty \mathrm{Emb}^c(M,N) := \mathrm{Map}_{\mathrm{PSh}(\mathrm{Disc}_d^c)}(i^*E_M, i^*E_N)$$

THE MAP COMES FROM THE FUNCTORIALITY OF

$$\mathrm{Man}_d^c \xhookrightarrow[y]{} \mathrm{PSh}(\mathrm{Man}_d^c) \xrightarrow[i^*]{} \mathrm{PSh}(\mathrm{Disc}_d^c)$$

2.2) UNIVERSAL PROPERTY

MORE GENERALLY, $E_M \rightsquigarrow F \in \mathrm{PSh}(\mathrm{Man}_d^c)$

$$T_\infty \mathrm{Emb}^c(M,N) \rightsquigarrow T_\infty F(M) = \underbrace{\mathrm{Map}_{\mathrm{PSh}(\mathrm{Disc}_d)}(E_M, F)}_{\text{FORMULA RIGHT KAN EXT OF } F \text{ ALONG } i:\, \mathrm{Disc}_d^c \hookrightarrow \mathrm{Man}_d^c.}$$

2.3) DEF: $P \hookrightarrow Q$ EMB IS A EQUIV OF TANGENTIAL k-TYPES IF

$\exists$ FACTORIZATION

$$\begin{tikzcd}
P \arrow[r] \arrow[ddr, "k\text{-connected}"'] & Q \arrow[r, "T^s Q"] \arrow[dd, "k\text{-connected}"] & BO / BTop \\
 & & \\
 & B \arrow[uur] &
\end{tikzcd}$$

THM (GOODWILLIE-KLEIN-WEISS, KRANNICH-K)
if $d \geq 5$, M COMPACT, $\partial M \hookrightarrow M$ EQUIV OF TANGENTIAL 2-TYPES, THEN

$$\mathrm{Emb}^\partial(M,N) \xrightarrow{\simeq} T_\infty \mathrm{Emb}^\partial(M,N).$$

EX: HYPOTHESIS SATISFIED IF $\mathrm{hdim}(M) \leq d-3$
OR IF $(\partial M, M)$ 1-cted SPIN.

QUESTION · TOPOLOGICAL CASE? $\longrightarrow$ LECTURE 3

· OPTIMAL? · NO — LOWER DIMENSIONS ...
· BASICALLY YES IN $\dim \geq 5$.

2.4) TOWER AND LAYERS

$$\mathrm{Disc}^c_{\leq 1} \hookrightarrow \cdots \underbrace{\mathrm{Disc}^c_{\leq k}}_{\substack{\text{FULL SUBCAT} \\ \text{ON } S \times \mathbb{R}^d \\ |S| \leq k}} \hookrightarrow \cdots \hookrightarrow \mathrm{Disc}^c$$

$$\mathrm{PSh}(\mathrm{Disc}^c_{\leq 1}) \cdots \leftarrow \mathrm{PSh}(\mathrm{Disc}^c_{\leq k}) \leftarrow \cdots \leftarrow \mathrm{PSh}(\mathrm{Disc}^c)$$

$$T_1 \mathrm{Emb}^c(M,N) \cdots \leftarrow T_k \mathrm{Emb}^c(M,N) \leftarrow \cdots \leftarrow T_\infty \mathrm{Emb}^c(M,N)$$

$$T_k \mathrm{Emb}^c(M,N) = \mathrm{Map}_{\mathrm{PSh}(\mathrm{Disc}^c_{\leq k})}(E_M, E_N) \qquad T_\infty \mathrm{Emb}^c(M,N) = \mathrm{Map}_{\mathrm{PSh}(\mathrm{Disc}^c_d)}(E_M, E_N)$$

EMBEDDING CALCULUS TOWER

TO USE THIS: FIRST STAGE: $T_1 \mathrm{Emb}^c(M,N)$

LAYERS: $\mathrm{fib}_x \left(T_k \mathrm{Emb}^c(M,N) \longrightarrow T_{k-1} \mathrm{Emb}^c(M,N) \right)$
$x \in T_{k-1} \mathrm{Emb}^c(M,N)$ —— LECTURE 3.

FIRST STAGE: OBSERVE: $E_M(\emptyset) \simeq *$ $\rightsquigarrow$ IT IS A "REDUCED" PRESHEAF.

$$PSh(Disc^c_{\leq 1}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} PSh(Disc^c_{=1}) \simeq \begin{cases} Spc_{/BO(d)} & \text{if } c=0 \\ Spc_{/BTop(d)} & \text{if } c=t \end{cases}$$

i_* - FULLY FAITHFUL EMBEDDING INTO $PSh^{red}(Disc^c_{\leq 1})$

$$\Rightarrow T_1 Emb^c(M,N) \simeq \begin{cases} Map^{/BO(d)}(M,N) \\ Map^{/Top(d)}(M,N) \end{cases}.$$

3) OPERADIC CALCULUS

OPERAD E^c_d (TOPOLOGICALLY) FRAMED LITTLE d-DISC OPERAD

k-ARY OPERATIONS: $E^c_d(k) = Emb^c(\coprod_k \mathbb{R}^d, \mathbb{R}^d)$

COMPOSITION = COMPOSITION OF EMBEDDINGS

$$E^c_d(k) \times \prod_{i=1}^{k} E^c_d(l_i) \longrightarrow E^c_d(l_1 + \dots + l_k)$$

SYMM. MONOIDAL CAT $(\mathcal{C}, \otimes) \rightsquigarrow$ OPERAD $\mathcal{C}^{\otimes}$

WITH COLOURS = OBJECTS OF $\mathcal{C}$

AND OPERATIONS $Mul^{\mathcal{C}}((c_1, \dots, c_k); c) = Map_{\mathcal{C}}(c_1 \otimes \dots \otimes c_k, c)$

GIVES $CMon(Cat) \underset{Env.}{\overset{}{\rightleftarrows}} Opd$ with $\perp$

$$\mathrm{Env}(\mathcal{O}) = \begin{cases} \text{Obj}: \text{COLLECTION OF COLORS } (c_s)_{s\in S} \quad S \text{ FINITE} \\ \text{MORPH}: \mathrm{Map}_{\mathrm{Env}(\mathcal{O})}\left((c_s)_{s\in S}, (c'_t)_{t\in T}\right) \\ \quad = \coprod_{\varphi: S\to T} \prod_{t\in T} \mathrm{Mul}^{\mathcal{O}}\left((c_s)_{s\in\varphi^{-1}(t)}, c'_t\right) \end{cases}$$

PROP: $\mathrm{Env}(E_d^c) \simeq \mathrm{Disc}_d^c$

3.2 CALCULUS FOR RIGHT MODULES

$$E_d^c \rightsquigarrow \mathcal{O} = \text{OPERAD}$$

$$\mathrm{PSh}(\mathrm{Disc}_d^c) \rightsquigarrow \mathrm{RMod}(\mathcal{O}) := \mathrm{PSh}(\mathrm{Env}(\mathcal{O}))$$

AND DEVELOP CALCULUS IN THAT GENERAL SETTING.

— WILL SIMPLIFY SOME THINGS… —

DESIDERATA: (A) TOWER

(B) DESCRIPTION OF FIRST STAGE

(C) DESCRIPTION OF LAYERS

(D) ALL OF ABOVE AS NATURAL AS POSSIBLE.

3.3 A NEW VARIANT OF E_d

$\mathcal{O}$ OPERAD. $B \xrightarrow{\mathcal{O}} \mathrm{BAlt}_{\mathrm{Opd}}(\mathcal{O}) \hookrightarrow \mathrm{Opd}$

$\rightsquigarrow \quad \mathcal{O}^{\theta} = \operatorname{colim}_{B} \mathcal{O}$ "SEMI DIRECT PRODUCT OPERAD"

$$\mathcal{O}^{\theta}(k) = \mathcal{O}(k) \times (\Omega B)^{k} \quad (B \text{ ctd})$$

<u>EX</u>: $E_d^{o} = (E_d)^{o}$ FOR $o : BO(d) \longrightarrow BAut(E_d)$

$E_d^{t} = (E_d)^{t}$ FOR $t : BTop(d) \longrightarrow BAut(E_d)$

E_d = LITTLE DISC OPERAD

$E_d^{p} = (E_d)^{id_{BAut(E_d)}}$ "<u>P</u>ARTICLE"

$$E_d(k) = \operatorname{Emb}^{fr}\left(\coprod_{k} \mathbb{R}^d, \mathbb{R}^d\right)$$

$$= \operatorname{fib}\left(\begin{array}{c} E_d^{c}(k) = \operatorname{Emb}^{c}(\sqcup \mathbb{R}^d, \mathbb{R}^d) \\ \downarrow \\ \prod_{k} \underbrace{\operatorname{Emb}^{c}(\mathbb{R}^d, \mathbb{R}^d)}_{\simeq \; O(d),\ Top(d)} \end{array} \right)$$

SANDER KUPERS : LECTURE 2

II BOUNDARY CONDITIONS AND BORD(B)$_n$ CAT

0) RECOLLECTION

EMBEDDING CALCULUS IN
$$\mathrm{Man}_d^\theta \longrightarrow \mathrm{RMod}(E_d^\theta) = \mathrm{PSh}(\mathrm{Env}(E_d^\theta))$$

$$\theta = \{*, 0, t, p = id\}$$
$$\theta : B \longrightarrow BAut(E_d)$$

MAKES SENSE FOR OPERAD $\mathcal{O}$

WANTED CALCULUS $\mathrm{RMod}(\mathcal{O})$ WITH

A) TOWER

B) FIRST STAGE

C) LAYERS

D) NATURAL.

A) & B)

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[d, "\mathrm{Env}"] & \mathrm{Com} = \text{TERMINAL OPERAD} \\
\mathrm{Env}(\mathcal{O}) \arrow[r] & \mathrm{Env}(\mathrm{Com}) = \mathrm{Fin} = \text{FINITE SETS} \\
\mathrm{Env}(\mathcal{O})_{\leq k} = \mathrm{Env}(\mathcal{O}) \times_{\mathrm{Fin}} \mathrm{Fin}_{\leq k} \arrow[u] \arrow[r] & \mathrm{Fin}_{\leq k} = (\text{CARD} \leq k) \arrow[u, hook]
\end{tikzcd}$$

$$\mathrm{Env}(\mathcal{O})_{\leq 1} \longrightarrow \cdots \longrightarrow \mathrm{Env}(\mathcal{O})$$
$$\mathrm{PSh}(\mathrm{Env}(\mathcal{O})_{\leq 1}) \longleftarrow \cdots \longleftarrow \mathrm{RMod}(\mathcal{O})$$
$$= \mathrm{RMod}_1(\mathcal{O})$$

B) $\mathrm{RMod}_1(\mathcal{O}) := \mathrm{PSh}(\mathrm{Env}(\mathcal{O}_{\leq 1}))$

$\mathcal{O}$ UNITAL IF 0-ARY OPERATIONS ARE CONTRACTIBLE

$$\begin{tikzcd}
\mathrm{RMod}_1(\mathcal{O}) \arrow[r, phantom, ":="] & \mathrm{PSh}(\mathrm{Env}(\mathcal{O}_{\leq 1})) \arrow[d, shift left] \\
\mathrm{RMod}_1^{red}(\mathcal{O}) \arrow[u] \arrow[r, phantom, "\simeq"] & \mathrm{PSh}(\mathrm{Env}(\mathcal{O}_{=1})) \arrow[u, shift left]
\end{tikzcd}$$

$\mathrm{Spc}/_{grp} \simeq$ (IF) $\supseteq$

$\mathcal{O}^{col} = \left\{ \begin{array}{l} \text{OBJ} = \text{COLORS} \\ \text{MORPH} = \text{UNARY OP.} \end{array} \right)$ GROUPOID

1) DAY CONVOLUTION

DAY CONVOLUTION ON PRESHEAVES

$$(\mathcal{C}, \otimes) \underset{\text{LURIE}}{\rightsquigarrow} (\mathrm{PSh}(\mathcal{C}), \otimes_{Day})$$

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[r, "F \times G"] \arrow[d, "\otimes"] & \mathrm{Spc} \\
\mathcal{C}^{op} \arrow[ur, dashed, "F \otimes G"'] &
\end{tikzcd}$$

Ex: FOR $\mathcal{C} = \mathrm{Env}(\mathcal{O})$, THIS SIMPLIFIES TO

$$(F \otimes G)(S) = \coprod_{S = S_1 \sqcup S_2} F(S_1) \times G(S_2).$$

Rem: FORMULA ALSO MAKES SENSE FOR

$$\begin{tikzcd}
\mathrm{RMod}_k(\mathcal{O}) = \mathrm{PSh}(\mathrm{Env}(\mathcal{O}_{\leq k})). \arrow[d, shift right, "i_*"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{PSh}(\mathrm{Env}(\mathcal{O})) \arrow[u, shift right, "i^*"']
\end{tikzcd}$$

RECOGNIZE THE FORMULA FOR $F, G \in \mathrm{RMod}_k(\mathcal{O})$

AS $i^*((i_* F) \otimes (i_* G))$

EXAMPLE: $k=1$ RESTRICT TO $\mathrm{RMod}_1^{red}(\mathcal{O})$

THEN $(F \otimes G)(\underline{1}) = F(\underline{1}) \times G(\phi) \sqcup F(\phi) \times G(\underline{1})$

$\simeq F(1) \sqcup G(1)$ AS F, G REDUCED

$\rightsquigarrow \mathrm{RMod}_1^{red}(\mathcal{O}) \simeq \mathcal{S}_{/\mathcal{O}^{col}}$ IF $\mathcal{O}^{col}$ GROUPOID

UPGRADES TO AN EQUIVALENCE OF SYM. MONOIDAL CATEGORIES

$\otimes \longrightarrow \sqcup$.

PROP: THERE IS A TOWER OF SYM. MON CATS. & FUNCTORS

$(\mathrm{RMod}_1(\mathcal{O}), \otimes_{DAY}) \leftarrow (\mathrm{RMod}_2(\mathcal{O}), \otimes) \leftarrow \cdots \leftarrow (\mathrm{RMod}(\mathcal{O}), \otimes_{DAY})$

1.2) PRESHEAVES OF DISJOINT UNIONS

$M, M' \in \mathrm{Man}_d^c \rightsquigarrow M \sqcup M' \in \mathrm{Man}_d^c$

$E_M, E_{M'}, E_{M \sqcup M'} \in \mathrm{RMod}(E_d^c)$.

$$E_{M \sqcup M'}(S \times \mathbb{R}^d) = \mathrm{Emb}^c(S \times \mathbb{R}^d, M \sqcup M')$$

$$= \coprod_{S = S_1 \sqcup S_2} \mathrm{Emb}^c(S_1 \times \mathbb{R}^d, M) \times \mathrm{Emb}^c(S_2 \times \mathbb{R}^d, M')$$

$$= (E_M \otimes E_{M'})(S \times \mathbb{R}^d).$$

$$\rightsquigarrow \begin{tikzcd} (\mathrm{Man}_d^c, \amalg) \arrow[r, "y"', "\text{YONEDA}"] \arrow[dr, bend right, "E"'] & (\mathrm{PSh}(\mathrm{Man}_d^c), \otimes_{\mathrm{Day}}) \arrow[d, "i^*", "\text{RESTRICT}"'] \\ & (\mathrm{RMod}(E_d^c), \otimes_{\mathrm{Day}}) \end{tikzcd}$$

EMBEDDING CALCULUS.

SYMM. MONOIDAL FUNCTORS

2. CALCULUS OF EMBEDDINGS WITH BDRY

DEF: SUPPOSE GIVEN

$e_\partial : \partial_0 M \hookrightarrow \partial N$

THEN $\mathrm{Emb}^c_{\partial_0}(M,N) \subset \mathrm{Emb}^c(M,N)$

of e st. $e|_{\partial_0 M} = e_{\partial_0}$

— THESE APPEAR WHEN YOU DO EXTENSION BY THE IDENTITY.

2.1) ALGEBRAS & MODULES

$(\mathcal{C}, \otimes)$ SYM. MON $\rightsquigarrow$ CAN DEFINE (ASS) ALGEBRAS, (LEFT) MODULES.

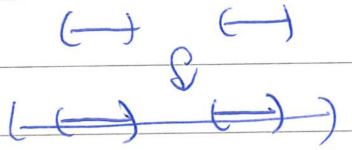
E_1 $\quad E_1(k) = \mathrm{Emb}^{rec}(\amalg_k I, I)$

is

Ass

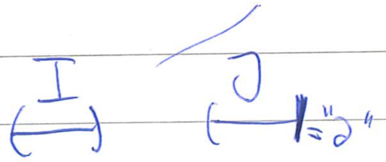
$E_1 \longrightarrow \mathcal{C}^{\otimes}$ MAP OF OPERADS
(OR EQUIV $\mathrm{Env}(E_1) \to \mathcal{C}$) SYM. MON.

$I \longmapsto A$ $=:$ ALGEBRA



$\longmapsto$ $A \otimes A \to A$

$LE_1 \longrightarrow \mathcal{C}^{\otimes}$ MAP OF OPERADS

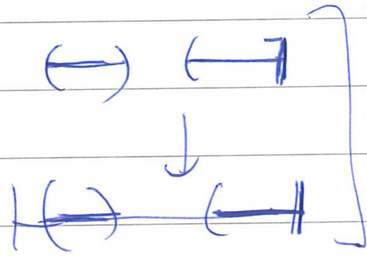


$I \longmapsto A$

$\partial \longmapsto M$

$\mathrm{Mul}^{LE_1}(\amalg_k I \sqcup J, J)$

$= \mathrm{Emb}^{rec}_{\partial \to \partial}(\amalg_k I \sqcup J, J)$



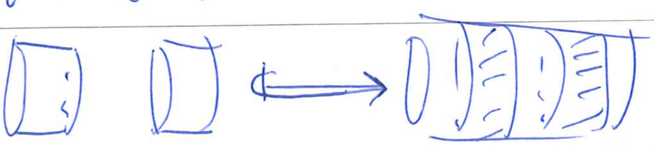
$\longmapsto$ $A \otimes M \xrightarrow{\alpha} M$

2.2) ALGEBRAS & MODULES IN MANIFOLDS

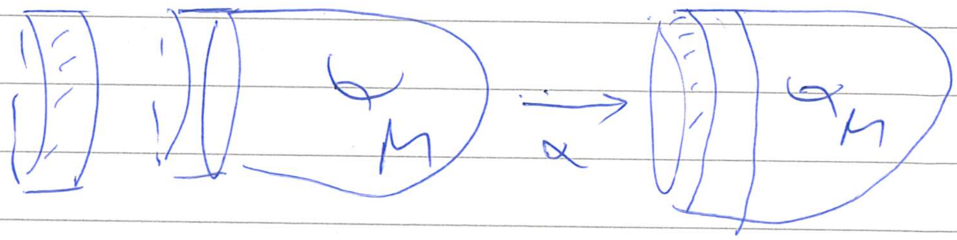
$(\mathcal{C}, \otimes) = (\mathrm{Man}^c_d, \sqcup)$ $\quad I$ is TAUTOLOGICALLY AN ALGEBRA in Man^c_d.

$\downarrow \partial_0 M \times (\text{—})$

$\partial_0 M \times (\text{—})$ is AN ALGEBRA



IF $(M, \partial_0 M, \partial_1 M)$ TRIAD AS ABOVE, FIXING A COLOR, M BECOMES A $\partial_0 M \times I$-MODULE IN $(\mathrm{Man}_d^c, \sqcup)$:



2.3) BOUNDARY CONDITIONS IN EMBEDDING CALCULUS

1) TRIAD $M \rightsquigarrow$ ALGEBRA $\partial_0 M \times I$ + $\partial_0 M \times I$ - MODULE M

2) $\mathrm{Man}_d^c \xrightarrow[E]{} \mathrm{RMod}(E_d^c)$ SYM. MON

$\Longrightarrow$ ALGEBRA $E_{\partial_0 M \times I}$ IN $(\mathrm{RMod}(E_d^c), \otimes_{DAY})$

AND $E_{\partial_0 M \times I}$ - MODULE E_M.

$M \underset{e}{\hookrightarrow} M$ FIXING $\partial_0 M$.

$\rightsquigarrow$ $\partial_0 M \times I$ - MODULE MAP $M \longrightarrow M$

P $(d-1)$-MFDS. Man_P^c = CATEGORY OF TRIADS $(M, \partial_0 M \cong P, \partial_1 M)$

$$\mathrm{Man}_P^c \longrightarrow \mathrm{Mod}_{P \times I}(\mathrm{Man}_d^c) \xrightarrow[E]{} \mathrm{Mod}_{E_{P \times I}}(\mathrm{RMod}(E_d^c))$$

<u>DEF</u>: EMBEDDING CALCULUS WITH BDRY CONDITION

$$\mathrm{Emb}^c_{\partial_0}(M,N) \longrightarrow T_\infty \mathrm{Emb}^c_{\partial_0}(M,N)$$
$$:= \mathrm{Map}_{\mathrm{Mod}_{E_{P\times I}}(\mathrm{RMod}(E_d^c))}(E_M, E_N)$$

2.4) WHY IS THIS A GOOD DEFINITION?

1) <u>THM</u> IF $d \geq 5$, M COMPACT, $\partial_1 M \hookrightarrow M$ IS AN EQUIVALENCE ON TANGENTIAL 2-TYPES

$$\mathrm{Emb}^0_{\partial_0}(M,N) \xrightarrow{\simeq} T_\infty \mathrm{Emb}^0_{\partial_0}(M,N)$$

2) MODEL GLUING OF MANIFOLDS

$$E_M \underset{E_{P\times I}}{\otimes} E_N \simeq E_{M \underset{P}{\cup} N}.$$

3) <u>BORDISM CATEGORIES</u>

WE WANT COHERENT AND COMPATIBLE ANALOGUES FOR EMBEDDING CALCULUS OF GEOMETRIC CONSTRUCTIONS FOR EMBEDDINGS

- COMPOSE
- GLUE
- DISJOINT UNION

ENCODED FOR EMBEDDING BY THE EXISTENCE

OF A SYM. MON. DOUBLE CATEGORY

$nc\mathrm{Bord}_d^c$ $\qquad$ $\mathrm{CMon}(\mathrm{Cat}(\mathrm{Cat}))$

$$D : \Delta^{op} \longrightarrow \mathrm{Cat}$$

$\hookrightarrow$ $D_0 =$ "CAT OF OBJECTS"

$D_1 =$ "CAT OF MORPH"

$$D_n \xrightarrow{\simeq} D_1 \underset{D_0}{\times} \cdots \underset{D_0}{\times} D_1 .$$

SEGAL COND

[NOT NECESSARILY COMPACT!]

$(nc\mathrm{Bord}_d^c)_0 = \Big\{$ $(d-1)$ MANIFOLDS EMBEDDING

$(nc\mathrm{Bord}_d^c)_1 = \Big\{$ d-dim COBORDISMS EMBEDDING

$(\quad)_n =$ COMPOSABLE COBORDISMS.

$P, Q \in \mathrm{OBJ}\ nc(\mathrm{Bord}_d^c)_0$

$$nc\mathrm{Bord}_d^c(P,Q) = \mathrm{fib}_{(P,Q)}\left((nc\mathrm{Bord}_d^c)_1 \to (\ )_0 \times (\ )_0\right)$$

SEGAL CONDITION GIVES COMPOSITION = GLUE

ON THE EMBEDDING CALCULUS SIDE, THIS STRUCT IS ENCODED BY SYM. MON DOUBLE CAT OF ALGEBRAS, BIMODULES "MORITA CATEGORY"

$\mathrm{ALG}(\mathcal{C})_0 = \begin{cases} \text{ALG IN } \mathcal{C} \\ \text{ALGEBRA MAPS} \end{cases}$

$\mathrm{ALG}(\mathcal{C})_1 = \begin{cases} \text{TRIPLES } (A, M, B) \text{ OF ALGEBRAS } A, B \\ \qquad\qquad + (A,B)\text{-BIMODULE } M \\ \text{MAPS OF SUCH} \end{cases}$

<u>THM</u> (KRANNICH-K) $\exists$ MAP OF SYM. MON DOUBLE CAT

$$\mathrm{ncBord}_d^c \longrightarrow \mathrm{ALG}(\mathrm{RMod}(E_d^c))$$

<u>EXAMPLE</u>:

$\partial_0 W = P \sqcup Q$

$\partial_0 W' = Q \sqcup R$

$$\begin{tikzcd}
\mathrm{Emb}^c_{\partial_0}(W,W) \times \mathrm{Emb}^c_{\partial_0}(W',W') \arrow[r] \arrow[d, "\text{GLUE}"] & T_\infty \mathrm{Emb}(\ldots) \cdots \times T_\infty \cdots \arrow[d] \\
\mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W') \arrow[r] & T_\infty \mathrm{Emb}^c_{\partial_0}(W \cup W', \ldots
\end{tikzcd}$$

SANDER KUPERS, LECTURE III

TOOLS IN EMBEDDING CALCULUS

RECOLLECTION

EMBEDDING CALCULUS WITH BDRY COND.

$$\begin{tikzcd}
 & nc\,\mathrm{Bord}_d^c \arrow[r] & \mathrm{ALG}(\mathrm{RMod}^{red}(E_d^c)) \arrow[d] \\
 & & \vdots \arrow[d] \\
 & & \mathrm{ALG}(\mathrm{RMod}_2^{red}(E_d^c)) \arrow[d] \\
 & \mathrm{COSPAN}\left(\mathrm{Spc}/_{\substack{BO(d) \\ BTop(d)}}\right) \arrow[r, phantom, "\simeq"] & \mathrm{ALG}(\mathrm{RMod}_1^{red}(E_d^c))
\end{tikzcd}$$

(MORE GENERALLY FOR GROUPOID-COLORED OPERADS)

LEFT TO DO: C) DESCRIPTION OF LAYERS

D) NATURALITY IN OPERAD.

1) LAYERS $\quad M, N \in \mathrm{Man}_d^c$

$$\mathrm{fib}_x\left[T_k \mathrm{Emb}^c(M,N) \to T_{k-1}^c(M,N)\right]$$

A DESCRIPTION OF THE LAYERS WILL FOLLOW

FROM A PULL-BACK SQUARE $2 \leftarrow 1 \leftarrow 0$

$$\begin{tikzcd}
\mathrm{RMod}_k(\mathcal{O}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spc}^{[2]} \arrow[d, "d^1"] \\
\mathrm{RMod}_{k-1}(\mathcal{O}) \arrow[r] & \mathrm{Spc}^{[1]}
\end{tikzcd}$$

1.1) GENERAL RECOLLEMENT THM

$$\mathcal{C}_0 \overset{i}{\hookrightarrow} \mathcal{C}$$

FULL SUBCATEGORY

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}_0) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathrm{PSh}(\mathcal{C}) \arrow[l, "i^*"']
\end{tikzcd}$$

$$i_! i^* \longrightarrow \mathrm{id}_{\mathrm{PSh}(\mathcal{C})} \longrightarrow i_* i^*$$

WHOSE COMPOSITION AGREES WITH PRECOMP. BY i^*: $\quad i_! \xleftarrow{\simeq} i_! i^* i_* \longrightarrow i_*$

(*)
$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}) \arrow[r, "i_! i^* \to \mathrm{id} \to i_* i^*"] \arrow[d, "i^*"] & \mathrm{PSh}(\mathcal{C})^{[2]} \arrow[r, "j^*"] \arrow[d, "d^1"] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[2]} \arrow[d] \\
\mathrm{PSh}(\mathcal{C}_0) \arrow[r, "i_! \to i_*"] & \mathrm{PSh}(\mathcal{C})^{[1]} \arrow[r, "j^*"'] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[1]}
\end{tikzcd}$$

FOR $j: \mathcal{C}_{\neq 0} \hookrightarrow \mathcal{C}$ THE SUBCATEGORY ON OBJ. NOT IN $\mathcal{C}_0$, AND MORPHISMS NOT FACTORING THROUGH $\mathcal{C}_0$

WILL SUPPOSE IT IS A CAT.

THM: IF $\mathcal{C}_{\neq 0}$ IS A SUBCATEGORY AND

$\forall\, c \in \mathcal{C}_{\neq 0}$ $\quad i_! i^* . y_{\mathcal{C}}(c) \longrightarrow y_{\mathcal{C}}(c)$

IS AN EQUIV. ONTO SUBPRESHEAF OF THOSE COMP. THAT FACTOR THROUGH $\mathcal{C}_0$

THEN THE OUTER SQUARE IN (*) IS A PULL-BACK.

MOREOVER, IF $\mathcal{C}_{\neq 0}$ IS A GROUPOID, THEN ~~[illegible]~~ THE SQUARES

$$\begin{array}{ccccc} PSh(\mathcal{C}) & \longrightarrow & PSh(\mathcal{C}_{\neq 0})^{[2]} & \xrightarrow{\;\mathrm{coli}^{[2]}\;} & S_{pc}^{[2]} \\ \downarrow & & \downarrow d' & & \downarrow \\ PSh(\mathcal{C}_0) & \longrightarrow & PSh(\mathcal{C}_{\neq 0})^{[1]} & \xrightarrow[\mathrm{coli}^{[1]}]{} & S_{pc}^{[1]} \end{array}$$

1.2 RECOLLEMENT FOR RIGHT MODULES

$$\mathcal{C}_0 \longrightarrow \mathcal{C} \hookleftarrow \mathcal{C}_{\neq 0}$$

$$\mathrm{Env}(\mathcal{O})_{\leq k-1} \hookrightarrow \mathrm{Env}(\mathcal{O})_{\leq k} \hookleftarrow \underbrace{\mathrm{Env}(\mathcal{O})_{\Sigma_k}}_{\substack{\text{GROUPOID} \\ \Updownarrow \\ \mathcal{O}\text{ GROUPOID-COLOURED}}} = \left\{ \begin{array}{l} \mathrm{OBJ} := (S)_{S \in S},\ |S| = k \\ \text{MORPH THAT ARE BIJ. ON SET OF COLORS} \end{array} \right.$$

LATER: THE HYPOTHESIS OF THE THM APPLIES FOR THIS CHOICE OF $\mathcal{C}, \mathcal{C}_0, \mathcal{C}_{\neq 0}$ WITH $\mathcal{O} = E_d^c$

COR:
$$\begin{tikzcd} \mathrm{RMod}_k(\mathcal{O}) \arrow[r, "\Omega"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spc}^{[2]} \arrow[d] \\ \mathrm{RMod}_{k-1}(\mathcal{O}) \arrow[r, "\Lambda"] & \mathrm{Spc}^{[1]} \end{tikzcd}$$

WHERE $\longrightarrow$ 1.3 Ω, Λ FOR E_d^c

$$\Omega = \left(\mathrm{colim}\, j^* i_! i^* \longrightarrow \mathrm{colim}\, j^* \longrightarrow \mathrm{colim}\, j^* i_* i^* \right)$$

$$\mathrm{RMod}_k(E_d^c) \ni E_M \longmapsto j^* E_M \in \mathrm{PSh}(\mathrm{Env}(E_d^c)_{\Sigma_k}) \underbrace{}_{\Sigma_k \wr O(d)}^{k}$$

$$\simeq \mathrm{Conf}_k(M) \times_{M^k} \mathrm{Fr}^\circ(TM)^k$$

ORDERED CONF.

PROP:
$$\begin{tikzcd} \mathrm{colim}\, j^* i_! i^* E_M \simeq \partial \overline{C}_k(M) \arrow[d] \\ \mathrm{colim}\, j^* E_M \simeq \overline{C}_k(M) \arrow[d] \\ \mathrm{colim}\, j^* i_* i^* E_M \simeq \left(\lim_{I \subsetneq \{1,\dots,k\}} \mathrm{Conf}_I(M) \right) / \Sigma_k \end{tikzcd}$$

$$\mathrm{colim}\, j^* E_M \simeq C_k(M)$$

" $\simeq$ UNORDERED CONFIGURATIONS

$\overline{C}_k(M)$ " FULTON-MACPHERSON COMPACT.

(OR PLAYING WITH AN INNER-COLLAR IN THE TOPOLOGICAL CASE)

THM: THE LAYERS ARE GIVEN BY

$$\mathrm{fib}_x\left[T_k \mathrm{Emb}^\circ(M,N) \longrightarrow T_{k-1}\mathrm{Emb}^\circ(M,N)\right]$$

$\simeq$ SPACE OF FILLERS IN

$$\begin{tikzcd}
\partial\bar{C}_k(M) \arrow[r, "x"] \arrow[d] & \partial\bar{C}_k(N) \arrow[d] \\
\bar{C}_k(M) \arrow[r, dashed, "\boxed{}"] \arrow[d] & \bar{C}_k(N) \arrow[d] \\
\left(\lim\limits_{I} \mathrm{Conf}_I(M)\right)/\Sigma_k \arrow[r] & \left(\lim\limits_{I} \mathrm{Conf}_I(N)\right)/\Sigma_k
\end{tikzcd}$$

" T_{k-1} APPROX OF $\bar{C}_k$ "

(COULD ALTERNATIVELY BE WRITTEN AS A SPACE OF SECTIONS)

2. SMOOTHING THEOREM

OBSERVE $BO(d)$ DOES NOT APPEAR

PROP: GIVEN

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\
& BAut(E_d) &
\end{tikzcd}$$

$\rightsquigarrow$ $E_d^\alpha \xrightarrow{\varphi} E_d^\beta$ MAP OF OPERADS

AND

$$\begin{tikzcd}
RMod_k(E_d^\alpha) \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & RMod_k(E_d^\beta) \arrow[d] \\
RMod_{k-1}(E_d^\alpha) \arrow[r, "\varphi"] & RMod_{k-1}(E_d^\beta)
\end{tikzcd}$$

PASTING THESE DIAGRAMS TOGETHER $\forall k$, AND TAKING A LIMIT, GET

THM:
$$\begin{tikzcd} \mathrm{RMod}^{red}(E_d^\alpha) \arrow[r] \arrow[d] & \mathrm{RMod}^{red}(E_d^\beta) \arrow[d] \\ \mathrm{Spc}_{/A} \arrow[r] & \mathrm{Spc}_{/B} \end{tikzcd}$$

"SMOOTHING THM"
(e.g. $A = O(d)$, $B = \mathrm{Top}(d)$)

$\Big\}$ ALG

$$\begin{tikzcd} \mathrm{ALG}(\mathrm{RMod}^{red}(E_d^\alpha)) \arrow[r] \arrow[d] & \mathrm{ALG}(\mathrm{RMod}^{red}(E_d^\beta)) \arrow[d] \\ \mathrm{COSPAN}(\mathrm{Spc}_{/A}) \arrow[r] & \mathrm{COSPAN}(\mathrm{Spc}_{/B}) \end{tikzcd}$$

EXAMPLE: $\partial_0 M \overset{M}{\rightsquigarrow} \emptyset$

$$\begin{tikzcd} \mathrm{Mod}_{E_{\partial_0 M\times I}}(\mathrm{RMod}(E_d^\alpha)) \arrow[r] \arrow[d] & \mathrm{Mod}_{E_{\partial_0 M\times I}}(\mathrm{RMod}(E_d^\beta)) \arrow[d] \\ (\mathrm{Spc}_{/A})^{\partial_0 M/} \arrow[r] & (\mathrm{Spc}_{/B})^{\partial_0 M/} \end{tikzcd}$$

3. Convergence of Topological Embedding Calculus

$o = \alpha$ $\quad$ $t = \beta$

$$\begin{tikzcd}
A = BO(d) \arrow[d] \arrow[r, "o"] & BAut(E_d) \\
B = BTop(d) \arrow[ur, "t"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Emb}^o_{\partial_0}(M,N) \arrow[r] \arrow[d, "(1)"] & \mathrm{Emb}^t_{\partial_0}(M,N) \arrow[d, "(2)"] \\
T_\infty \mathrm{Emb}^o_{\partial_0}(M,N) \arrow[r, "(3)"] \arrow[d] & T_\infty \mathrm{Emb}^t_{\partial_0}(M,N) \arrow[d] \\
\mathrm{Map}^{/BO(d)}_{\partial_0}(M,N) \arrow[r] & \mathrm{Map}^{/BTop(d)}_{\partial_0}(M,N)
\end{tikzcd}$$

<u>THM</u> (LASHOF) IF $d \geq 5$, THE OUTER SQUARE IS A PULL-BACK.

$\Rightarrow$ IF SMOOTH EMBEDDING CALCULUS CONVERGES, (I.E. (1) IS AN EQUIVALENCE) THEN TOPOLOGICAL EMB. CALC. DOES IN THE SENSE THAT (2) IS AN EQUIV ON THOSE COMPONENTS HIT BY (3)

<u>REM</u>: BY VARYING THE SMOOTH STRUCT OF M, CAN HIT ALL COMPONENTS:

$$\begin{tikzcd}
M \arrow[r] \arrow[dr, dashed] \arrow[ddr] & N \arrow[d] \\
& BO(d) \arrow[d] \\
& BTop(d)
\end{tikzcd}$$
(N SMOOTH)

RELEVANT SMOOTH STRUCT ON M

THM (KRANNICH–K) IF $d \geq 5$, M COMPACT, $\partial_1 M \to M$ IS AN EQUIV CLASS ON TANGENTIAL 2-TYPES AND N SMOOTHABLE, THEN $\mathrm{Emb}^t_{\partial_0}(M,N) \xrightarrow{\simeq} T_\infty \mathrm{Emb}^t_{\partial_0}(M,N)$.

SANDER KUPERS: LECTURE 4

THE DISC STRUCTURE SPACE

1) STRUCTURE SPACES

1.1) CLASSICAL SITUATION $\quad c \in \{0, t\}$

$$\mathrm{Man}_d^c \xrightarrow[\text{UNDERLYING SPACE}]{} \mathcal{S}pc$$

MANIFOLDS AND INVERTIBLE EMBEDDINGS

$$(\mathrm{Man}_d^c)^{\simeq} \longrightarrow \mathcal{S}pc^{\simeq} \leftarrow \text{HTPY CAT.}$$

$$\cup$$

$\mathrm{Man}_d^{c,\simeq}$

SUBCATEGORY OF CLOSED MANIFOLDS

__Def__: For $X \in \mathcal{S}pc$,

$$\mathcal{S}^{\mathcal{S}pc,c}(X) := \mathrm{fib}_X\left[\mathrm{Man}_d^{c,\simeq} \longrightarrow \mathcal{S}pc^{\simeq}\right]$$

- $\pi_0 \mathcal{S}^{\mathcal{S}pc,c}(X) = \left\{M, M \xrightarrow[\simeq]{f} X\right\} \Big/$
\begin{tikzcd} & N \arrow[dr, "g"] & \\ M \arrow[ur, "\cong", "\varphi"'] \arrow[rr, "f"'] & & X \end{tikzcd}

- $\mathcal{S}^{\mathcal{S}pc,c}(X) \neq \emptyset \iff X \simeq M$, M closed d-MFD.

- $\mathrm{fib}\left(B\mathrm{Diff}(M) \longrightarrow Bh\mathrm{Aut}(M)\right) = h\mathrm{Aut}(M) \big/ \mathrm{Diff}(M)$

PATH COMPONENTS $\subseteq \mathcal{S}^{\mathcal{S}pc,0}(X)$

CLASSICALLY,

$$\mathrm{Man}_d^{c,\cong} \xrightarrow{②} \widetilde{\mathrm{Man}}_d^{c,\cong} \xrightarrow{①} \underbrace{\mathrm{Spc}_{PD,s}^{\simeq}}_{\text{POINCARÉ DUALITY SPACES WITH SPECIFIED SIMPLE HTPY TYPE.}} \longrightarrow \mathrm{Spc}^{\simeq}$$

CLOSED d-di MFDS & BLOCK DIFFEO/HOMEO

$$\pi_2 \widetilde{\mathrm{Diff}}(M) \cong \pi_0 \mathrm{Diff}_\partial(M \times D^2).$$

SURGERY THEORY IS ABOUT ①:

$$\mathrm{fib}_X \left[\widetilde{\mathrm{Man}}_d^{c,\cong} \xrightarrow{①} \mathrm{Spc}_{PD,s}^{\simeq} \right]$$

MAIN THM OF SURGERY THEORY: THIS FITS IN A FIBRE SEQUENCE WITH TWO INFINITE LOOP SPACES $N^c(X)$, $L(X)$.
(NOVIKOV, BROWDER, SULLIVAN, ...)

PSEUDOISOTOPY THEORY IS ABOUT ②:

$$\mathrm{fib}_M \left[\mathrm{Man}_d^{c,\cong} \xrightarrow{②} \widetilde{\mathrm{Man}}_d^{c,\cong} \right]$$

MAIN RESULT:

$$\Big\downarrow \sim \tfrac{d}{3}\text{-CONNECTED}$$

INFINITE LOOP SPACE OF $\Sigma^{-1} \mathrm{Wh}^c(X)_{C_2}$

[WALDHAUSEN, ...]

ALL OF THIS WORKS WITH BOUNDARY: FIX A

CLOSED $(d-1)$-MFD P

$\mathrm{Man}_d^{c,\cong} \rightsquigarrow \mathrm{Man}_P^{c,\cong}$

$\mathcal{S}_{pc} \rightsquigarrow \mathcal{S}_{pc}^{P/}$ ETC.

1.2) DISC-STRUCTURE SPACE

$$\mathrm{Man}_d^{c,\cong} \longrightarrow \mathrm{RMod}(E_d^c)^{\simeq} \longrightarrow \mathcal{S}_{pc}^{\simeq}$$

OR $$\mathrm{Man}_P^{c,\cong} \xrightarrow{3} \mathrm{Mod}_{E_{P\times I}}(\mathrm{RMod}(E_d^c))^{\simeq} \longrightarrow \mathcal{S}_{pc}^{\simeq P/}$$

DEF: FOR $E_{P\times I}$-MODULE X

$$\mathcal{S}_\partial^{\mathrm{DISC},c}(X) = \mathrm{fib}_X\left[\mathrm{Man}_P^{c,\cong} \xrightarrow{3} \mathrm{Mod}_{E_{P\times I}}(\mathrm{RMod}(E_d^c))^{\simeq}\right]$$

EX: $\pi_0 \mathcal{S}_\partial^{\mathrm{DISC},c}(X) = \{M, E_M \xrightarrow[\simeq]{\varphi} X\}/$

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \\ E_M \arrow[r, "E\varphi"] \arrow[dr, "\simeq"'] & EN \arrow[d, "\simeq"] \\ & X \end{tikzcd}$$

$\Rightarrow$ $\mathcal{S}_\partial^{\mathrm{DISC},c}(X) \neq \emptyset$

$\iff \exists M$ d-MFD WITH $\partial M = P$ s.t. $E_M \simeq X$.

- $\mathrm{fib}(B\mathrm{Diff}_\partial(M) \to BT_\infty \mathrm{Emb}_\partial^\circ(M,M)^\times) \subset \mathcal{S}_\partial^{\mathrm{DISC},c}(X)$

$$= \frac{T_\infty \mathrm{Emb}_\partial^\circ(M,M)^\times}{\mathrm{Diff}_\partial(M)}$$

THM (K-K) FOR $d \geq 8$, HAVE

(A) $S_\partial^{\text{DISC}}(M)$ IS AN INFINITE LOOP SPACE

(B) " " ONLY DEPENDS ON TANGENTIAL 2-TYPE

(C) " " DOES NOT DEPEND ON SMOOTH STRUCTURE

(D) " " IS NON-TRIVIAL IF M 1-CONN. SPIN.

(A),(B) SUGGEST THIS SPACE IS RELATED TO K-/L-THEORY.

REM: ISOTOPY ~~EXTENSION~~ FOR EMBEDDING CALCULUS $\xrightarrow[\text{SURGERY TRICK}]{}$ B) $\xrightarrow[\text{OPERADS WITH HR-STABILITY}]{}$ A)

GOAL FOR THIS LECTURE: PREVIOUS LECTURES $\rightarrow$ C) AND D)

2) INDEPENDENCE OF SMOOTH STRUCTURE

~~[illegible]~~

$$\begin{tikzcd}
S_\partial^{Disc,0}(X) \arrow[r, dashed] \arrow[d] & Man_p^{0,\simeq} \arrow[r] \arrow[d] & Mod_{E_{p\times I}}(RMod(E_d^0))^\simeq \arrow[d] \\
S_\partial^{Disc,t}(X) \arrow[r, dashed] & Man_p^{t,\simeq} \arrow[r] & Mod_{E_{p\times I}}(RMod(E_d^t))^\simeq
\end{tikzcd}$$

WANT TO SHOW THIS SQUARE IS A PULLBACK.

PROOF:

$$\begin{tikzcd}
 & ncBord_d^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & Alg(RMod(E_d^0)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & COSPAN(Spc/BO(d)) \arrow[d] \\
Bord_d^0 \arrow[ur] \arrow[d] & ncBord_d^t \arrow[r] & Alg(RMod(E_d^t)) \arrow[r] & COSPAN(Spc/BTop(d)) \\
Bord_d^t \arrow[ur] & & &
\end{tikzcd}$$

PULL-BACK ON MAPPING CATEGORIES $d\neq 4$

$\rightsquigarrow$ MAPPING CATEGORIES FROM P TO $\emptyset$ YIELDS A PULL-BACK OF CATS

$\rightsquigarrow$ TAKE CORE TO GET A PULL-BACK OF SPACES

Thm c) $S_\partial^{Disc,0}(M) \xrightarrow{\simeq} S_\partial^{Diff,t}(M)$ $d\neq 4$

Rem: CAN'T BE TRUE FOR $d=4$

3) NON-TRIVIALITY

Thm D': $S_\partial^{Disc}(M)$ IS NON-TRIVIAL IF M IS (1-CONN) SPIN, $d\geq 8$, EVEN, AFTER LOOPING ARBITRARILY MANY TIMES.

QUICK REDUCTION: $S_\partial^{\mathrm{Disc}}(D^d) \xrightarrow[(B)]{\simeq} S_\partial^{\mathrm{Disc}}(M)$

BECAUSE $D^d \hookrightarrow M$ IS AN EQUIV. ON TANGENTIAL 2-TYPES.

REM: THIS IS EQUIVALENT TO NON-CONVERGENCE OF EMBEDDING CALCULUS IN CODIM 0.

$$\frac{T_\infty \mathrm{Emb}_\partial^{o}(M,M)^\times}{\mathrm{Diff}_\partial(M)} \simeq \frac{T_\infty \mathrm{Emb}_\partial^{t}(M,M)}{\mathrm{Homeo}_\partial(M)}$$

$$\cap$$

$$S_\partial^{\mathrm{Disc}}(M)$$

3.1 A DELOOPING TRICK

$$S_\partial^{\mathrm{Disc}}(D^d) = \mathrm{fib}\left[\underset{\simeq\ \ast}{B\mathrm{Homeo}_\partial(D^d)} \longrightarrow BT_\infty \mathrm{Emb}_\partial^t(D^d,D^d)^\times\right]$$

$$\simeq T_\infty \mathrm{Emb}_\partial^t(D^d,D^d)^\times$$

"ALEXANDER TRICK FOR CONF. CAT"

TO COMPUTE IT, RELATE TO PARTICLE VERSION

$$\begin{array}{ccc} BT_\infty \mathrm{Emb}^t(D^d,D^d)^\times & \xrightarrow{(*)} & BT_\infty \mathrm{Emb}^p(D^d,D^d)^\times \\ \downarrow & & \downarrow \\ \Omega^d B\mathrm{Top}(d) & \longrightarrow & \Omega^d B\mathrm{Aut}(E_d) \end{array}$$

LEMMA: $BT_\infty \mathrm{Emb}_\partial^\theta(D^d, D^d)^\times \simeq *$

PROOF SKETCH: FOR $\theta : B \longrightarrow BAut(E_d)$

$$BT_\infty \mathrm{Emb}_\partial^\theta(D^d, D^d) \simeq BAut_{\mathrm{Mod}\,E_{S^{d-1}\times I}(\mathrm{RMod}(E_d^\theta))}(E_{D^d}, E_{D^d})$$

$$\underset{\substack{\text{[LURIE}\\ \text{"ON CENTRALIZERS"]}}}{\simeq} \Omega_0^d(\mathrm{fib}(\theta))$$

$\rightarrow$ TAKE $\theta = id$ SO $\mathrm{fib}(\theta) \simeq *$.

COR: $S_\partial^{\mathrm{DISC}}(D^d) \simeq \Omega^{d+1} \dfrac{Aut(E_d)}{Top(d)}$

3.2 QUESTION OF DWYER-HESS

Q: IS $BTop(d) \longrightarrow BAut(E_d)$ EQUIV ?

A: YES IF $d \leq 2$ [HOREL]

NO IF $d \geq 3$ [KK].

THM: IF $d \geq 8$, $\Omega^n BTop(d) \rightarrow \Omega^n BAut(E_d)$

IS NOT AN EQUIV $\forall n \geq 0$

PROOF BY CONTRADICTION:

$$\begin{tikzcd}
BSTop(d-2) \arrow[r] \arrow[d] & BSAut(E_{d-2}) \arrow[r] \arrow[d] & BSAut(E_{d-2}^{\mathbb{Q}}) \arrow[d, "\simeq 0"] \\
BSTop(d) \arrow[r, "\simeq"', "\text{hypothesis}"'] & BSAut(E_d) \arrow[r, "r"'] & BSAut(E_d^{\mathbb{Q}})
\end{tikzcd}$$

<u>CASE 1</u>: $\pi_* BAut(E_d)$ <u>COUNTABLE</u>

$\Longrightarrow$ r RATIONALIZATION.

$$\begin{tikzcd}
BSTop(d-2) \arrow[r, "\simeq_{\mathbb{Q}} 0"] \arrow[rrd, bend right, "\text{surj on } \pi_*(-)\otimes\mathbb{Q}"'] & BSTop(d) \arrow[r] & BSTop^{\mathbb{Q}} \arrow[d, phantom, "\text{is}"] \\
 & & \prod_{i>0} K(\mathbb{Q}, 4i)
\end{tikzcd}$$

$\Longrightarrow$ CONTRADICTION!

<u>CASE 2</u>: <u>NOT</u> COUNTABLE.

$\Leftarrow$ CONTRADICTS $\pi_* BSTop(d)$ COUNTABLE SINCE $STop(d)$ IS LOCALLY CONTRACTIBLE SECOND COUNTABLE.