MANUEL KRANNICH : LECTURE I

PONTRYAGIN–WEISS CLASSES

BASED ON JT WORK
WITH SANDER KUPERS

CONVENTIONS: • $H^* = H^*(\, , \mathbb{Q})$

• ALL MFDS ARE ORIENTED.

LECTURE 1: INTRO & REDUCTION TO THE BDL MANIPULATION THM

1.1 PONTRYAGIN CLASSES

Fix $d \geq 0$ $\quad p_i \in H^{4i}(BO(d)) \quad i \geq 1$

MAIN PROPERTIES:

• STABILITY: $p_i(\pi) = p_i(\pi \oplus \mathbb{R}) \quad \pi : E \to X$

$\Leftrightarrow \iota^* p_i = p_i$ FOR $\iota : BO(d) \to BO(d+1)$
CLASSIFIES $\pi \mapsto \pi \oplus \mathbb{R}$

$\Leftrightarrow p_i$'s FOR DIFFERENT d's ASSEMBLE TO $p_i \in H^{4i}(BO)$ WHERE
$BO = \mathrm{colim}\,(BO(d) \xrightarrow{\iota} BO(d+1) \xrightarrow{\iota} \cdots)$

• VANISHING: $\forall \pi : E \to X$ d-dim v.b.
$p_i(\pi) = 0 \quad \forall \, i > d/2$

$$\Leftrightarrow H^{4i}(BO) \longrightarrow H^{4i}(BO(d)) \quad i > d/2.$$
$$p_i \longmapsto 0$$

1.2 EUCLIDEAN BDL

AN $\mathbb{R}^d$-BDL $\pi : E \to X$ IS A FIBRE BDL WITH FIBER HOMEOMORPHIC TO $\mathbb{R}^d$ (NO LINEAR STRUCTURE).

$$\rightsquigarrow \quad BO(d) \longrightarrow BTop(d).$$

EX: EVERY TOPOLOGICAL MFD M^d HAS A TOPOLOGICAL TANGENT BDL T^tM, WHICH IS AN $\mathbb{R}^d$-BDL IN THE ABOVE SENSE.

EXISTENCE OF SMOOTH STRUCTURE $\Longleftrightarrow$ WHETHER THIS TOPOLOGICAL BDL IS LIFTABLE TO AN $O(d)$-BDL / [illegible] AN ACTUAL VECTOR BDL.

$$\begin{tikzcd}
H^*(BO(d)) \arrow[r] \arrow[d, shift left] & H^*(BO(d+1)) \arrow[r] \arrow[d, shift left] & \cdots \arrow[r] & H^*(BO) \arrow[d, shift left, "\cong \mathbb{Q}"] & \ni p_i \\
H^*(BTop(d)) \arrow[r] \arrow[u, shift left] & H^*(BTop(d+1)) \arrow[r] \arrow[u, shift left] & \cdots \arrow[r] & H^*(BTop) \arrow[u, shift left] & \\
p_i \arrow[u, phantom, sloped, "\in"] & p_i \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"] & & & \exists p_i \arrow[lll, mapsto] \arrow[ul, phantom, sloped, "\in"]
\end{tikzcd}$$

THM (NOVIKOV, SULLIVAN, KERVAIRE–MILNOR, KIRBY–SIEBENMANN)

$$BO \xrightarrow[\cong \mathbb{Q}]{} BTop$$

$\Longrightarrow \exists\, p_i \in BTop(d)$ COMING FROM THE THM + ABOVE DIAGRAM.

PROPERTIES?

- STABILITY ✓ (BY DEFINITION)

- VANISHING ?

CONJ (SULLIVAN '70)

$p_i = 0$ IN $H^{4i}(BTop(d))$ FOR $i > \frac{d}{2}$

THM (WEISS '15)

(i) $\exists$ "MANY" PAIRS (i,d) WITH $i > d/2$ AND ST $\exists$ $\mathbb{R}^d$-BDL $E \xrightarrow{\pi} X$ WITH $p_i(\pi) \neq 0$

(NOT COMING FROM A VECTOR BDL!)

(ii) IN (i), X CAN BE CHOSEN TO BE A SPHERE.

THM [GALATIUS-RANDAL-WILLIAMS '22] PART (i) HOLDS FOR ALL PAIRS (i,d) WITH $d \geq 6$.

($\Leftarrow$ $p_i \neq 0$ IN $H^{4i}(BTop(6))$ $\forall\, i \geq 4$).

THM [K-KUPERS '24) PART (ii) ALSO HOLDS $\forall\, (i,d)$ WITH $d \geq 6$.

($\Leftrightarrow$ $p_i \neq 0$ IN $\mathrm{Hom}(\pi_{4i}(BTop(6)), \mathbb{Q})$ (IMAGE OF HUREWICZ HOMOMORPH.)

$\Leftrightarrow$ $BTop(6) \longrightarrow BTop$ HAS A RATIONAL SPLITTING AFTER LOOPING ONCE.

$\Leftrightarrow$ CERTAIN CHARACT. CLASSES

$$B\mathrm{Diff}_\partial(D^d) \longrightarrow \prod_{i>d/2}^{\infty} K(\mathbb{Q}[4i-d-1])$$

↑ EILENBERG-MACLANE SPACE

IS SURJECTIVE ON $\pi_*(-)_{\mathbb{Q}}$

FOR $d \geq 6$.

SOURCES OF $\mathbb{R}^d$-BUNDLES

① T^tM^d FOR TOPOLOGICAL d-MFD

↑ TOPOLOGICAL $\mathbb{R}^d$-BDL

NO GOOD: $p_i(T^tM^d) \in \underset{\to 0}{H^{4i}(M)}$ FOR $4i>d$

SO IN PARTICULAR FOR $i > d/2$.

② TOPOLOGICAL TANGENT BUNDLES OF "FAMILIES OF TOP MFDS"

BETTER.

1.3 M-BDLS:

M^d CLOSED TOP d-MFD, AN M-BDL $E \xrightarrow{\pi} X$ IS A FIBER BUNDLE WITH FIBRES $\underset{\text{HOMEO}}{\cong} M$.

VERTICAL tg-BDL $T^v\pi$ EXISTS:

$$\begin{tikzcd}
M^d \arrow[d] \arrow[drr, "T^tM"] & & \\
E \arrow[rr, "T^v\pi"'] \arrow[d, "\pi"'] & & B\mathrm{Top}(d) \\
X & &
\end{tikzcd}$$

NOTE: $p_i(T^*\pi) \in H^{4i}(E)$

CAN BE CHOSEN MUCH HIGHER DIMENSIONAL.

FROM $c \in H^k(BTop(d))$, GET A CHARCT. CLASS FOR M-BDLS $\pi: E \to X$ BY

$$\int_M c(T^v\pi) \in H^{k-d}(X)$$

$$\| \qquad \kappa_c(\pi)$$

IF X IS A MFD, THIS INTEGRATION IS PD TO THE MAP IN H_*

UNIVERSALLY, GIVES $\kappa_c \in H^{k-d}(BHomeo(M))$

NOTE: $\kappa_c \neq 0 \Rightarrow c \neq 0$.

⇝ SEARCH FOR M-BDLS $E \xrightarrow{\pi} X$ WITH $\kappa_{p_i}(\pi) \neq 0$! NEED SUCH THAT DO NOT ADMIT A SMOOTH STRUCTURE.

PLAN: START WITH A SMOOTH BDL AND MODIFY IT...

1.4 L-CLASSES AND SIGNATURES

HIRZEBRUCH L-CLASSES

$L_i \in H^{4i}(BO) \cong H^{4i}(BTop)$

$L_1 = \frac{1}{3} p$

$L_2 = \frac{7}{45} p_2 - \frac{1}{45} p_1^2 \in H^8(BTop)$

$\vdots$

$L_i = A.p_i + (\text{DECOMPOSABLES})$ WITH $A \neq 0$

FAMILY SIGNATURE THM: $E \xrightarrow{\pi} X$ M-BDL.

$$\sigma_i(\pi) = \kappa_{L_i}(\pi) \in H^{4i-d}(X)$$ WHERE

$\sigma_i = \{$ UNIQUE CHARACT. CLASS OF M-BDLS S.T. $\forall \pi : E \longrightarrow \underset{\text{STABLE PARALLELIZABLE}}{X^{4i-d}}$, $\sigma_i(\pi) = \underset{\text{SIGNATURE OF INT. FORM ON } E}{\mathrm{sig}(E)}$ $\}$

1.5 THE BDL MANIPULATION THM

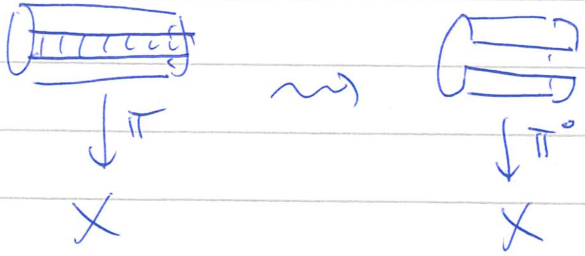
- M^d CLOSED d-MFD, $d \geq 6$, 2-CONNECTED, STABLY PARALLELIZABLE. $D^d \subseteq M$ FIXED.
- π M-BDL WITH A TRIVIAL D^d-SUBBUNDLE.

WRITE

$$\begin{tikzcd} E^0 := E \setminus e(\mathring{D}^d \times X) \arrow[d, "\pi^0"] \\ X \end{tikzcd}$$

FIBRE BDL WITH FIBER $M^0 = M \setminus D^n$.

$$\begin{tikzcd} D^d \arrow[r, hook] \arrow[d] & M \arrow[d] \\ D^d \times X \arrow[r, hook, "e"] \arrow[dr] & E \arrow[d, "\pi"] \\ & X \end{tikzcd}$$



BTM (K-KUPERS) IF $T^t_{\pi^\circ} \cong \mathbb{R}^2 \oplus \xi$ FOR SOME $\mathbb{R}^{d-2}$-BDL ξ, THEN $\exists$ M-BDL

$$\begin{tikzcd} \tilde{\pi} : \tilde{E} \arrow[r] & X \\ D^d \times X \arrow[u, "\tilde{e}"] \arrow[ur, "pr_2"'] & \end{tikzcd}$$

s.t. (a) $K_{L_i}(\pi) = K_{L_i}(\tilde{\pi})$ $\forall i$

(b) $K_c(\tilde{\pi}) = 0$ $\forall c \in H^*(BTop(d))$ DECOMPOSABLE

(a') $\tilde{\pi}$ AND π HAVE THE SAME UNDERLYING FIBRATION.

(b') $T^t_{\tilde{\pi}} : \tilde{E}^\circ \longrightarrow BTop(d)$ RATIONALLY TRIVIAL. RELATIVE TO ⓢ

Fix $* \in \partial M^\circ$

$\rightarrow$ SECTION

$$X \underset{s = * \times id}{\overset{pr_2}{\rightleftarrows}} D^d \times X$$

((a') $\Rightarrow$ (a), (b') $\Rightarrow$ (b))

BY THE FAMILY SIGNATURE THEOREM.

SUPPOSE THAT WHENEVER

As in BMT ✓

(*) $\forall i \geq 0$, $d \geq 6$, $\exists$ M-BDL $\pi : E \to B$ WITH $K_{L_i}(\pi) \neq 0$.

PROOF OF THE THM ASSUMING BMT:

WRITE $L_i = A \cdot p_i$ + (DECOMPOSABLES)

TAKE $i \geq 1$, $d \geq 6$. BY (*), PICK M-BDL $\pi : E \to X$ WITH $K_{L_i}(\pi) \neq 0$.

$$0 \neq K_{L_i}(\pi) \overset{(a)}{=} K_{L_i}(\tilde{\pi}) = A . K_{p_i}(\tilde{\pi}) + \underset{\underset{0}{\overset{(b)}{\|}}}{\underline{K_{(DECOMP)}(\tilde{\pi})}}$$

$\Rightarrow K_{p_i}(\tilde{\pi}) \neq 0 \Rightarrow p_i \neq 0$ in $H^{4i}(BTop(d))$.

LEFT: $p_i \in H^{4i}(BTop(d))$ EVALUATES NON-TRIVIALLY AGAINST HUREWICZ IMAGE

$$\begin{tikzcd}
S^{d-1} \times X \arrow[r] \arrow[d, hook] & \tilde{E}^0 \arrow[r] \arrow[d, "\cap" description] \arrow[dr, phantom, "(b')"] & * \arrow[r] \arrow[d] & * \arrow[d] \\
D^d \times X \arrow[r] \arrow[d] & \tilde{E} \arrow[r, "t\tilde{\pi}"] \arrow[rr, bend right, "p_i", "\not\simeq *"'] & BTop(d)_{\mathbb{Q}} \arrow[r] & K(\mathbb{Q}, 4i) \\
\Sigma^d(X_+) \arrow[urrr, bend right, "\not\simeq *"'] & & &
\end{tikzcd}$$

(REALISING THE NON-TRIVIAL TOP RATIONAL H_*-CLASS FROM TWO LINES BY $\pi_*(\Sigma X)_{\mathbb{Q}} \xrightarrow{\cong} H_*(\Sigma X)$)

NOTE: $K_{L_i}(\pi) \neq 0$ COULD EVEN BE REALISED BY A SMOOTH BDL, WITH SMALL p_i's CONTRIBUTING TO THIS NON-VANISHING.

IN FACT, (*) IS IMPLIED BY WORK OF GALATIUS-RANDAL-WILLIAMS.

MANUEL KRANNICH LECTURE 2

ON THE PROOF OF THE BUNDLE MANIPULATION THEOREM (BMT)

RECALL: (K-K) $Top(6) \longrightarrow Top$ ADMITS A RATIONAL SECTION.

REM: PROOF IS AN EXTENSION OF WEISS' ORIGINAL STATEMENT

LAST TIME: REDUCED THE PROOF* TO BMT

M CLOSED 2-CONN. STABLY //, TOPOLOGICAL d-MFD $d \geq 6$

$$\begin{tikzcd} E \arrow[dr, "\pi"'] & & D^d \times X \arrow[ll, hook'] \arrow[dl] \\ M\text{-BDL} & X & \end{tikzcd} \qquad * \in \partial D^d \rightsquigarrow \quad \begin{tikzcd} \overset{\circ}{E} \arrow[d, "\pi_0"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$

$\overset{\circ}{E} : E \setminus e(\overset{\circ}{D}{}^d \times X)$

$T^t\pi^\circ \cong \mathbb{R}^2 \oplus \xi$

$$\Rightarrow \exists \quad \begin{tikzcd} \tilde{E} \arrow[dr, "\tilde{\pi}"'] & & D^d \times X \arrow[ll, hook', "\tilde{e}"'] \arrow[dl] \\ M\text{-BDL} & X & \end{tikzcd}$$

a) $\tilde{\pi} \simeq \pi$ AS M-FIBRATIONS

b) $T^t\tilde{\pi}^\circ : \tilde{E}^\circ \longrightarrow BTop(d)$ $\mathbb{Q}$-TRIV

(RELATIVE TO $s(X) \subseteq \tilde{E}^\circ$)

TODAY: WILL [illegible] REPHRASE THIS IN TERMS OF CLASSIFYING SPACES $\longrightarrow$ WILL ALLOW TO USE SANDERS' LECTURES.

$D^d \subseteq M, \; M^\circ = M \setminus \mathring{D}^d$

$$\begin{tikzcd}
B\mathrm{Homeo}(M) & B\mathrm{Homeo}_{D^d}(M) \arrow[l, "\text{FORGET}"'] & B\mathrm{Homeo}_\partial(M^\circ) \arrow[l, "\simeq"', "\substack{\text{extend}\\ \text{BY id}}"]
\end{tikzcd}$$

CLASSIFIES: M-BDLS

$$\begin{tikzcd}
M \arrow[d] \\
E \arrow[d, "\pi"'] \\
X
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
M \arrow[d] & \\
E \arrow[d, "\pi"'] & D^d \times X \arrow[l, hook', "c"'] \arrow[dl] \\
X &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
M^\circ \arrow[d] & \partial M^\circ \arrow[l, hook'] \arrow[d] \\
E \arrow[d] & \partial M^\circ \times X \arrow[l, hook'] \arrow[dl] \\
X &
\end{tikzcd}$$

WE CONSTRUCT A CERTAIN SELF-MAP $\Psi : B\mathrm{Homeo}_\partial(M^\circ) \circlearrowleft$

CLASSIFYING $\pi \longmapsto \tilde{\pi}$

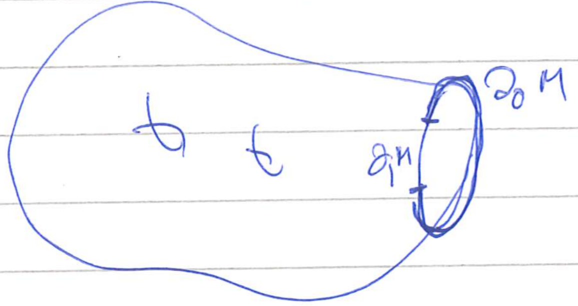
$$\begin{tikzcd}
M \arrow[d] & \\
E \arrow[r] \arrow[d] & B\mathrm{Top}(d) \\
X &
\end{tikzcd}
\quad \text{CLASSIFIES} \qquad
\begin{tikzcd}
M \arrow[d] \\
E \arrow[d] \\
X
\end{tikzcd} \text{fib}
\qquad
\begin{tikzcd}
M^\circ \arrow[d] & \partial M^\circ \arrow[l, hook'] \arrow[d] \\
E \arrow[d] & \partial M^\circ \times X \arrow[l, hook'] \arrow[dl] \arrow[u, phantom, "\simeq"] \\
X &
\end{tikzcd}$$

$$\begin{tikzcd}
B\mathrm{Homeo}(M) \arrow[r] & \mathrm{Map}(M, B\mathrm{Top}(d))/\!\!/h\mathrm{Aut}(M) \arrow[rr] & & Bh\mathrm{Aut}(M) & \\
B\mathrm{Homeo}_\partial(M^\circ) \arrow[u] \arrow[r] \arrow[ddd, "\simeq"] & \mathrm{Map}_\partial(M^\circ, B\mathrm{Top}(d))/\!\!/h\mathrm{Aut}(M) \arrow[u] \arrow[rr] \arrow[dr] & & Bh\mathrm{Aut}_\partial(M^\circ) \arrow[u] & \\
& \subseteq \mathrm{Mod}_{E_{\partial_0}M \times I}(\mathrm{RMod}^{red}(E^t_d))^{\simeq} \arrow[ddd] \arrow[rrr, "\substack{\ulcorner\text{PULL-BACK}\\ \text{(SMOOTHING THEORY)}}"] & \mathrm{Map}_{\partial_0}(M^\circ, B\mathrm{Top}(d))/\!\!/h\mathrm{Aut}_\partial(M) \arrow[ur] & & \\
& & & & \subseteq (\mathrm{Spc}_{/B\mathrm{Top}(d)})^{\partial M/\,\simeq} \arrow[dd] \\
BT_\infty\mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ) \arrow[rr, Rightarrow] \arrow[dd, "\substack{\ulcorner\text{UP TO }\pi_0,\pi_1\\ \text{PULL-BACK}}"] & & \mathrm{Map}_{\partial_0}(M^\circ, B\mathrm{Top}(d))/\!\!/h\mathrm{Aut}_{\partial_0}(M) \arrow[dd, Rightarrow, "⊛"] & & \\
& \subseteq \mathrm{Mod}_{E_{\partial_0}M \times I}(\mathrm{RMod}^{red}(E^p_d)) \arrow[rrr] & & & \subseteq (\mathrm{Spc}_{/B\mathrm{Aut}(E_d)})^{\partial M/\,\simeq} \\
BT_\infty\mathrm{Emb}^p_{\partial_0}(M^\circ, M^\circ) \arrow[rr, Rightarrow] & & \mathrm{Map}_{\partial_0}(M, B\mathrm{Aut}(E_d))/\!\!/h\mathrm{Aut}_\partial(M) & &
\end{tikzcd}$$

"PARTICLE EMBEDDING CALCULUS"

CONCLUSION (a) IN BMT $\Longleftrightarrow$ Φ COVERING $\mathrm{id}_{BhAut(M)}$ — IN FACT IT WILL COVER $\mathrm{id}_{BhAut_\partial(M^\circ)}$.

CONSIDER M° AS A TRIAD $(M^\circ, \partial_0 M, \partial_1 M)$. $S^{d-1} = \underset{\partial_0 M}{\overset{\shortparallel}{D^{d-1}_+}} \cup \underset{\partial_1 M}{\overset{\shortparallel}{D^{d-1}_-}}$



CONCLUSIONS (a) + (b) (NON-RATIONALLY) $\Longleftrightarrow$ Φ COVERING $\mathrm{const}_{/\!/hAut_\partial(M)} : \mathrm{Map}_{\partial_0}(M, BTop(d))_{/\!/hAut_\partial(M)}$ $\circlearrowleft$

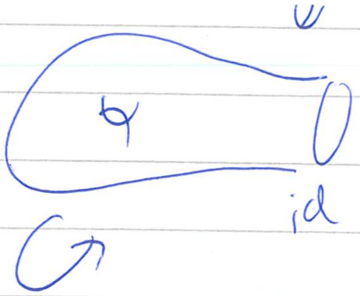
2.2 EMBEDDING CALCULUS ENTERING

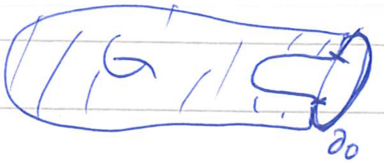
NEXT: EXPRESS $BHomeo_\partial(M^\circ)$ IN TERMS OF EMBEDDING CALCULUS.

NAIVELY: $Homeo_\partial(M^\circ) = Emb^t_\partial(M^\circ, M^\circ) \xrightarrow{(*)} T_\infty Emb^t(M^\circ, M^\circ)$

NO GOOD SINCE (*) IS NOT AN EQUIVALENCE. (NOTE: $\partial_1 M = \emptyset$ HERE, AND $\emptyset \subseteq M$ IS NOT A TANGENTIAL 2-TYPE EQUIV — SEE CONDITION IN SANDER'S LECTURE.)

$$\begin{tikzcd} \text{BETTER: } \mathrm{Homeo}_\partial(M^\circ) \arrow[r, "①"] & \mathrm{Emb}_{\partial_0}(M^\circ, M^\circ) \arrow[d, "②"] \\ & T_\infty \mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ) \end{tikzcd}$$





Claim ① & ② ARE EQUIVALENCES

PF: ① ALEXANDER TRICK: $\mathrm{Homeo}_\partial(D^d) \simeq *$
+ TOPOLOGICAL POINCARÉ CONJECTURE

② CONVERGENCE OF EMBEDDING CALCULUS APPLIES SINCE $\partial_0 M = D^{d-1} \subseteq M$ IS A TANGENTIAL 2-TYPE EQUIVALENCE (EXERCISE).

[NOTE: THE ALEXANDER TRICK FAILS IN EMBEDDING CALCULUS LAND!]

THE MAP ⊛ IN THE DIAGRAM IS INDUCED BY POST COMPOSITION WITH THE MAP $BTop(d) \longrightarrow BAut(E_d)$. SUPPOSE ———— IS NULL-HOMOTOPIC.

USING THE +/- PULL-BACK SQUARE IT SITS IN, CAN CONSTRUCT ψ ON $BT_\infty \mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ)$ BY USING $\mathrm{id}_{BT_\infty \mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ)}$ AND CONST${}_{/\!/ hAut_{\partial_0}(M)}$ ON

$$\mathrm{Map}_{\partial_0}(M^0, B\mathrm{Top}(d)) /\!\!/_h \mathrm{Aut}_{\partial_0}(M) \, .$$

<u>TWO PROBLEMS</u>: ① $B\mathrm{Top}(d) \longrightarrow B\mathrm{Aut}(E_d)$ is NOT NULL-HOMOTOPIC! (EXERCISE WITH EULER CLASS.)

② Ψ CONSTRUCTED THIS WAY WOULD ONLY COVER $\mathrm{const} /\!\!/ \mathrm{id}_{h\mathrm{Aut}_\partial(M)}$ AND NOT $\mathrm{const} /\!\!/ \mathrm{id}\, h\mathrm{Aut}_\partial(M)$

(+ THE π_0, π_1 ISSUE WITH THE PULL-BACK THAT WE WILL IGNORE FOR CLARITY.)

2.3 REGARDING PROBLEM ①

THE CONDITION $T^0 \cong \mathbb{R}^2 \oplus \xi$ IN THE BMT SAYS THAT $\pi : E \to X$ is CLASSIFIED BY A MAP TO THE PULL-BACK

$$\begin{tikzcd}
B\mathrm{Homeo}_\partial^\theta(M^0) \arrow[r] \arrow[d] & \mathrm{Map}_{\partial_0}(M^0, B\mathrm{Top}(d-2)) /\!\!/_h \mathrm{Aut}_\partial(M) \arrow[d] \\
B\mathrm{Homeo}_\partial(M^0) \arrow[r] & \mathrm{Map}_{\partial_0}(M^0, B\mathrm{Top}(d)) /\!\!/_h \mathrm{Aut}_\partial(M)
\end{tikzcd}$$

⇝ CAN ADD SOME "-2"'s IN THE DIAGRAM AND "θ"'s.

NOW NEED

$$\begin{tikzcd}
B\mathrm{Top}(d-2) \arrow[rr] \arrow[dr] & & B\mathrm{Aut}(E_d) \\
& B\mathrm{Top}(d) \arrow[ur] &
\end{tikzcd}$$

TRIVIAL.

THM (K-KUPERS) THE COMPOSITION

$$\begin{tikzcd}
BSTop(d-2) \arrow[r] \arrow[ddr, bend left, "?*"'] & BSTop(d) \arrow[d] \\
 & BSAut(E_d) \arrow[d] \\
 & BSAut(E_d^{\mathbb{Q}})
\end{tikzcd}$$

ORIENTATION PRESERVING

RATIONALIZED

PROOF: NEXT LECTURE.

2.4 REGARDING PROBLEM ②

$$\begin{tikzcd}
\text{WANT:} & B\mathrm{Homeo}_\partial(M^\circ) \arrow[r] \arrow[dl] & & B\mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ) \arrow[d, "\simeq"] \\
B\mathrm{hAut}_\partial(M) \arrow[dr] & & & BT_\infty\mathrm{Emb}^t_{\partial_0}(M^\circ, M^\circ)^\times \arrow[d] \\
 & B\mathrm{hAut}_{\partial_0}(M) & & BT_\infty\mathrm{Emb}^p_{\partial_0}(M^\circ, M^\circ)^\times \arrow[ll] \arrow[ulll, dashed] \arrow[dlll, no head] \\
T_\infty\mathrm{Emb}^p_{\partial_0}(M^\circ, M^\circ) \arrow[rr] & & T_1\mathrm{Emb}^p_{\partial_0}(M, M) = \mathrm{Map}_{\partial_0}(M, M) \arrow[dr] & \\
\cup & & & \mathrm{Map}_{\partial_0}(M, M) \\
T_\infty\mathrm{Emb}^p_{\partial_0}(M^\circ, M^\circ)^\times \arrow[drrr] & & & \cup \\
 & & & \mathrm{hAut}_{\partial_0}(M)
\end{tikzcd}$$

HTPY INV. COMPONENTS

BAN(EN

MANUEL KRANNICH LECTURE 3

PARTICLE ACTIONS ON BOUNDARIES AND THE DOUBLE STABILISATION OF $E_d^{\otimes}$

● TWO MISSING INGREDIENTS.

(I) M^0 cpct, 2-conn, smoothable, $\partial_1 M^0 = S^{d-1}$, $(M^0, \partial_0 M, \partial_1 M)$. ∃ "DOTTED LIFT"

$$\begin{tikzcd}
\mathrm{Homeo}_\partial(M^0) \arrow[r] \arrow[dd] & \mathrm{Emb}^t_{\partial_0}(M^0;M^0)^\times \arrow[r] & T_\infty\mathrm{Emb}^t_{\partial_0}(M^0;M^0)^\times \arrow[dr] & \\
& & & T_\infty\mathrm{Emb}^p_{\partial}(M^0;M^0)^\times \arrow[dlll, dashed] \arrow[dd] \\
\mathrm{hAut}_\partial(M^0) \arrow[drrr] & & & \\
& & & \mathrm{hAut}_{\partial_0}(M^0)
\end{tikzcd}$$

AS E_1-MAPS.

MORE GENERAL RESULT: $(W^d, \partial_0 W, \partial_1 W)$ COMPACT d-dim MFD TRIAD, $d \geq 6$, W SMOOTHABLE.

$$\begin{tikzcd}
\mathrm{Homeo}_{\partial_0}(W) \arrow[r] \arrow[d] & \mathrm{Emb}^t_{\partial_0}(W,W)^\times \arrow[r] & T_\infty\mathrm{Emb}^t_{\partial_0}(W,W)^\times \arrow[r] & T_\infty\mathrm{Emb}^p_{\partial_0}(W)^\times \arrow[d] \arrow[dlll, dashed, "\exists ?"'] \\
\mathrm{hAut}_{\partial_0}(W, \partial_1 W) \arrow[rrr] \arrow[d, equal] & & & \mathrm{hAut}_{\partial_0}(W) \\
\mathrm{End}^{(1)}_{Sp_{\partial_{01}W \xrightarrow{1} \partial_0 W}}(\partial_1 W \to W)^\times & & &
\end{tikzcd}$$

$\uparrow$ SETWISE PRESERVED

$\partial_{01} W = \partial_0 W \cap \partial_1 W$

EXERCISE: DEDUCE THE PREVIOUS DOTTED LIFT FROM THIS ONE USING THE ALEXANDER TRICK.

THM (WEISS, K-KUPERS) IF $\partial_1 W \subseteq W$ IS A TANGENTIAL 2-TYPE EQUIV, THEN $\exists$ DOTTED FILLER.

WILL DISCUSS THE CASE $\partial_0 W = \emptyset$, $\partial_1 W = \partial W$.

i.e. PARTICLE EMBEDDING CALCULUS REMEMBERS ENOUGH OF THE BOUNDARY...

START WITH:

$$\begin{tikzcd} \mathrm{Homeo}(W) \arrow[r] \arrow[d] & \mathrm{Emb}(W,W)^\times \arrow[d] \arrow[dl, dashed] \\ h\mathrm{Aut}(W,\partial W) \arrow[r] & h\mathrm{Aut}(W) \end{tikzcd}$$

UNDER THE ASSUMPTIONS OF THE THM,

LEMMA: $\exists$ PREFERRED EQUIVALENCE

$$\begin{pmatrix}\partial W\\ \cap\\ W\end{pmatrix} \simeq \left(\begin{tikzcd} \mathrm{fib}_{\mathrm{id}}\, T_\infty\mathrm{Emb}^{\times,p}(W \sqcup \mathbb{R}^d, W)/\!\!/T_\infty\mathrm{Emb}^p(\mathbb{R}^d,\mathbb{R}^d)^\times \arrow[r] \arrow[d] & T_\infty\mathrm{Emb}^{\times,p}(W,W) \\ T_\infty\mathrm{Emb}^{\times p}(\mathbb{R}^d,W)/\!\!/T_\infty\mathrm{Emb}^p(\mathbb{R}^d,\mathbb{R}^d)^\times & \end{tikzcd}\right)$$

EXERCISE: COLLAR NBHD THM & PARAMETRIZED ISOTOPY EXTENSION

+ SMOOTHING THEORY, NEUTRALITY OF CUBICAL THM.

· CONVERGENCE OF EMBEDDING CALC.

(II) THE COMPOSITION

$$\begin{tikzcd} BS\mathrm{Top}(d-2) \arrow[r] & BS\mathrm{Top}(d) \arrow[r] & BS\mathrm{Aut}(E_d) \arrow[d] \\ & & BS\mathrm{Aut}(E_d^{\mathbb{Q}}) \end{tikzcd}$$

IS NULL-HOMOTOPIC.

$$E_d(k) = \{ \underline{k} \times \mathring{D}^d \overset{e}{\hookrightarrow} \mathring{D}^d \mid e(i,-) = \lambda_i \cdot (-) + t_i \}$$
$$\lambda_i > 0, \ t_i \in \mathring{D}^d$$

OPERAD STRUCTURE BY COMPOSITION.

<u>NOTE</u>: $E_d(k) \xrightarrow[\simeq]{\mathrm{res}_0} \mathrm{Conf}_k(\mathring{D}^d) \simeq \mathrm{Conf}_k(\mathbb{R}^d)$

<u>Eg</u>: $\mathrm{Conf}_2(\mathbb{R}^d) \cong \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \simeq S^{d-1}$

↳ RESTRICT TO ARITY 2 GIVES

$$\begin{tikzcd} & & \mathrm{Aut}_{\mathrm{Opd}}(E_d) \arrow[r] \arrow[d] & h\mathrm{Aut}(E_d(2)) \simeq h\mathrm{Aut}(S^{d-1}) \arrow[r, phantom, "="] & G(d) \arrow[d, "\text{ACT ON } H_{d-1}(S^{d-1})"] \\ & \mathrm{Top}(d) \arrow[ur] & \mathrm{Aut}_{\mathrm{Opd}}(E_d^{\mathbb{Q}}) \arrow[r] & h\mathrm{Aut}(S^{d-1}_{\mathbb{Q}}) \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}^\times \arrow[d, phantom, sloped, "\subset"] \\ O(d) \arrow[ur] & & & G^{\mathbb{Q}}(d) \arrow[r] & \mathbb{Q}^\times \end{tikzcd}$$

<u>NOTE</u>: "S" IN FRONT OF AUTOMORPHISM SPACES
= KERNEL OF MAP TO $\mathbb{Q}^\times$

COMPATIBLE EXTERNAL PRODUCT MAPS

- $\mathrm{Top}(d) \times \mathrm{Top}(d') \xrightarrow{(-)\times(-)} \mathrm{Top}(d+d')$
- $\mathrm{Aut}(E_d^{\otimes}) \times \mathrm{Aut}(E_{d'}^{\otimes}) \xrightarrow{\otimes_{BV}} \mathrm{Aut}(E_d^{\otimes} \otimes E_{d'}^{\otimes})$

IS [DUNN-LURIE]

$E_{d+d'}^{\otimes}$

[BOAVIDA de BRITO - HOREL]

II.2 CYCLOTOMIC ACTIONS

$$\begin{tikzcd} \mathrm{SAut}(E_{d-2}^{\otimes}) \times \mathrm{SAut}(E_2^{\otimes}) \arrow[r, "\otimes_{BV}"] & \mathrm{SAut}(E_d^{\otimes}) \\ \mathrm{Aut}(E_2^{\otimes}) \arrow[u, phantom, "\circlearrowleft"] & \mathrm{Aut}(E_2^{\otimes}) \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

CONJUGATION ACTION

RESTRICTING TO $\mathrm{SAut}(E_{d-2}^{\otimes})$ ~~$\times\{id\}$~~ GIVES

$$\begin{tikzcd} B\mathrm{SAut}(E_{d-2}^{\otimes}) \arrow[r] \arrow[d] & B\mathrm{SAut}(E_d^{\otimes}) \\ B\mathrm{SAut}(E_d^{\otimes})^{h\mathrm{Aut}(E_2^{\otimes})} \arrow[ur] & \end{tikzcd}$$

THM [KK] $B\mathrm{SAut}(E_{d-2}^{\otimes}) \xrightarrow[-\otimes id_{E_2^{\otimes}}]{\simeq *} B\mathrm{SAut}(E_d^{\otimes})$

WE USE $B\mathrm{SAut}(E_d^{\otimes})^{h\mathrm{Aut}(E_2^{\otimes})} \simeq \lim_{k} \left(B\mathrm{SAut}(E_{d,\leq k}^{\otimes})^{h\mathrm{Aut}(E_2^{\otimes})}\right)$

KEY PROPOSITION: $\forall i \geq 0, d \geq 2, k \geq 1$

$\pi_0 \mathrm{Aut}(E_2^{\mathbb{Q}}) \curvearrowright \pi_i(\mathrm{BSAut}(E_{d, \leq k}^{\mathbb{Q}}))$

IS gr-CYCLOTOMIC OF NON-TRIVIAL WEIGHT.

DEF: M GROUP WITH ACTION OF

$GT_{\mathbb{Q}} := \pi_0 \mathrm{Aut}(E_2^{\mathbb{Q}})$.

- M IS CYCLOTOMIC OF NON-TRIVIAL WEIGHT IF ~~[illegible]~~ M $\mathbb{Q}$-VECTOR SPACE AND THE ACTION IS GIVEN BY $(GT^{\mathbb{Q}} \xrightarrow{\chi} \mathbb{Q}^\times)$, $g \cdot m = \chi(g)^i \cdot m$ FOR SOME $i \neq 0$.

- M IS gr-CYCLOTOMIC ON N-TRIVIAL WEIGHT IF $\exists$ SUBNORMAL SERIES

$$0 \trianglelefteq M_0 \trianglelefteq M_1 \trianglelefteq \dots \trianglelefteq M_k \trianglelefteq \dots M$$

S.T. M_k / M_{k-1} IS CYCLOTOMIC OF NON-TRIVIAL WEIGHT.

(WHERE THE WEIGHTS CAN VARY FROM ONE QUOTIENT TO THE NEXT).

LEMMA (DRINFELD) $\chi : GT_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{\times}$ IS SURJECTIVE

(WHICH IS THE REASON IT'S GOOD TO RATIONALIZE
$\longrightarrow$ ∃ MANY MORE AUTOMORPHISMS)

EXERCISE: SHOW THAT BEING gr-CYCL. OF NON-TRIVIAL W. IS PRESERVED BY TAKING $GT_{\mathbb{Q}}$-INVARIANT $\left\{\begin{array}{l}\text{SUBGROUPS}\\ \text{QUOTIENTS}\\ \text{EXTENSIONS}\end{array}\right.$

PROOF OF THE NULL-HOMOTOPY KEY PROP:

PICK $\Lambda \in GT_{\mathbb{Q}}$ WITH $\chi(\Lambda) \neq \pm 1$.

ENOUGH: $\mathrm{SAut}(E^{\mathbb{Q}}_{d, \leq 2})^{\langle \Lambda \rangle} \simeq *$

HOMOTOPY FIXED pt SS: $H^{-p}(\Lambda ; \pi_q(\mathrm{SAut}(E^{\mathbb{Q}}_{d, \leq 2})))$

$$\Downarrow$$

$$\pi_{p+q}(\mathrm{SAut}(E_d^{\leq 2})^{h\Lambda})$$

ENOUGH: $H^*(\Lambda ; M) = 0$ FOR M (gr)-CYCLOTOMIC OF NON-TRIVIAL WEIGHT -

"CENTER-KILLS TRICK": $H^*(\Lambda ; M)$ IS $(\chi(\Lambda)^i - 1)$-TORSION $\Longrightarrow$ TRIVIAL.

III.3 PROOF OF KEY PROPOSITION (A SKETCH)

HAVE FIBER SEQUENCE (GÖPPL-WEISS)

$$\left\{\begin{tikzcd} & \partial F_k(\mathbb{R}^d)_{\mathbb{Q}} \arrow[dl] \arrow[dr] & \\ F_k(\mathbb{R}^d)_{\mathbb{Q}} \arrow[rr, dashed, "\Sigma_k\text{-EQUIV}"] \arrow[dr] & & F_k(\mathbb{R}^d)_{\mathbb{Q}} \arrow[dl] \\ & \lim\limits_{S \subsetneq \underline{k}} F_S(\mathbb{R}^d)_{\mathbb{Q}} & \end{tikzcd}\right\} \longrightarrow \mathrm{SAut}(E_{d \leq k}^{\mathbb{Q}}) \longrightarrow \mathrm{SAut}(E_{d \leq k-1}^{\mathbb{Q}})$$

CERTAIN COMPACTIFICATION OF THE ORDERED CONF SPACE (ONLY DEFINED FOR $\mathbb{R}^d$)

$(-)_{\mathbb{Q}}$ OF THE BDRY INCLUSION OF A COMPACT $((k-1)\cdot d-1)$-MFD

AFTER SEVERAL SS, INDUCTION ON k & SOME TRICKS (e.g. USING POINCARÉ DUALITY FOR $F_k(\mathbb{R}^d)$) IT EVENTUALLY REDUCES TO

- $H_*(E_d^{\mathbb{Q}})$ IS, AS AN OPERAD IN $\mathbb{Q}$-VECTOR SPACE, GENERATED IN ARITY 2, WHERE $H_*(E_d(2)_{\mathbb{Q}}) \cong \mathbb{Q}[d-1]$ WHERE Λ ACTS BY $\chi(\Lambda)^i$
- $\pi_{*+1}(F_k(\mathbb{R}^d)_{\mathbb{Q}})$ IS A LIE ALG GENERATED BY $\pi_{d-1}(F_k(\mathbb{R}^d)_{\mathbb{Q}}) \cong H_{d-1}(F_k(\mathbb{R}^d)_{\mathbb{Q}})$ WHERE IT ACTS BY $\chi(\Lambda)^i$.