Copenhagen Masterclass on Higher Representation Theory: Notes and Exercises

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Contents

1. Recollections on ∞–categories

We begin these notes by gathering some basics from the theory of higher categories. As far as possible, our presentation will be done in a model–independent way. The purpose of this section is merely to provide a convenient reference point for the reader not so familiar with ∞–categories. As such, it will mostly be just a list of standard results and informal definitions that we need in the main body of this document, with precise references where possible, without it being woven into a cogent story. Our recommendation is for the reader to begin reading other parts and come back to this section as needed in order to find references and precise statements for ∞–categorical manoeuvres that are mentioned only in passing in the later sections.

Finally, a word of warning: since these results are so embedded in the canon, it is a bit difficult to make the correct attribution as to where the result first appeared. As such, many of these references will be pointing to *a* place where it appeared without the implicit claim about origins. Of course, the most biblically comprehensive sources for all things ∞–categorical are Lurie's pair of tomes [Lur09; Lur17], but some good (and shorter) one–stop locations for many of the basic results are the excellent set of lecture notes by Fabian Hebestreit, as expanded by Ferdinand Wagner [HW21], as well as the survey by David Gepner [Gep19].

The section is shaped with the following list of subsections:

- Basic setup
- Presentability
- Stability
- General multiplicative matters
- Duality and dualisability

Basic setup

Definition 1.1. A *Bousfield localisation* is an adjunction $L \dashv R$ whose right adjoint R is fully faithful. This is equivalent to the assertion that the counit $LR \Rightarrow \mathrm{id}_{\mathcal{D}}$ is an equivalence. Dually, a *Bousfield colocalisation* is an adjunction whose left adjoint is fully faithful.

Notation 1.2. In this document, we always write left adjoints on top of its right adjoint. In other words, when we write $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : R$ or

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

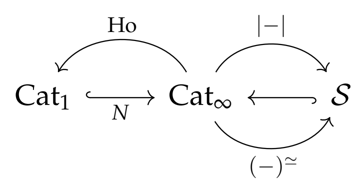
we mean that L is the left adjoint to R.

Observation 1.3. It is sometimes useful to observe that a Bousfield localisation is an equivalence if and only if the left adjoint L is conservative. Proving this is a simple and instructive exercise.

Fact 1.4. Let $f\colon I \to J$ be a functor and $\mathcal{C}$ an ∞–category with all small (co)limits. The left adjoint $f_!$ (resp. right adjoint f_*) of the restriction functor $f^*\colon \mathrm{Fun}(J,\mathcal{C}) \to \mathrm{Fun}(I,\mathcal{C})$ are called the *left Kan extension (resp. right Kan extension) along* f. If f is fully faithful, then $f_!$ and f_* are fully faithful too.

Notation 1.5. We write $\mathcal{S}$ for the ∞–category of spaces/∞–groupoids/anima/homotopy types. It is the ∞–category freely generated by the point under small (∞–)colimits. We write Cat_∞ for the ∞–category of small ∞–categories. We also write Cat_1 for the ∞–category of 1–categories. It turns out that Cat_1 is itself a 2–category.

Fact 1.6 (Cosmic adjunctions). By virtue the so–called *Grothendieck homotopy hypothesis*, any valid theory of higher categories should in particular admit the result that spaces are equivalent to ∞–groupoids (ie. ∞–categories where all morphisms are equivalences); moreover, 1–categories (ie. ordinary, classical, pre–grad–school categories) should be an instance of an ∞–category. From this, we have the adjunctions



where Ho is taking the homotopy 1–category of an ∞–category (ie. taking the connected components of mapping spaces in an ∞–category); N is viewing a 1–category as an ∞–category, where the notation N is supposed to remind the quasi–categorically–minded reader of taking the nerve simplicial set of a 1–category; $|-|$ is the functor which inverts all morphisms in an ∞–category; the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ is viewing spaces as ∞–groupoids; and finally $(-)^{\simeq}$ is taking the so–called *core groupoid* of an ∞–category, ie. remembering only the morphisms which are equivalences.

Importantly from these adjunctions, we see that the inclusion $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ preserves limits, and the inclusion $\mathcal{S} \subseteq \mathrm{Cat}_\infty$ preserves both limits and colimits. In other words, taking limits of 1–categories in Cat_∞ still yields a 1–category; and taking (co)limits of spaces in Cat_∞ still yields a space.

Fact 1.7. In $\mathcal{S}$, filtered colimits commute with finite limits. Moreover, geometric realisations (ie. colimits indexed by the 1–category $\mathbf{\Delta}^{\mathrm{op}}$), commute with finite products.

Construction 1.8 (Mapping spaces)**.** For objects x, y in an ∞–category $\mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x,y) \in \mathcal{S}$ is defined as the pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow[(x,y)]{} & \mathcal{C}\times\mathcal{C} \end{array}$$

Fact 1.9. For $X \in \mathcal{S}$, there is a formula for the mapping space of $\mathrm{Fun}(X,\mathcal{C})$ given by the following: for $\varphi,\psi \in \mathrm{Fun}(X,\mathcal{C})$, we have

$$\mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(\varphi,\psi) \simeq \lim_{x\in X} \mathrm{Map}_{\mathcal{C}}(\varphi(x),\psi(x))$$

This will be proved in Exercise 4.11 Item 3.

Construction 1.10 (Straightening–unstraightening)**.** For a functor $F\colon I \to \mathrm{Cat}_\infty$, there is a construction producing an ∞–category called the *unstraightening* $\mathrm{Un}(F)$ equipped with a map to I satisfying the property of being a *cocartesian fibration*. This is a very important notion in the general theory, but we will not be needing too much of the full details in this masterclass, and so we refrain from explaining it here for the sake of brevity. As with many things, this is a key construction due to Grothendieck. The point of this construction is that there is then an equivalence of ∞–categories

$$\mathrm{coCart}(I) \simeq \mathrm{Fun}(I,\mathrm{Cat}_\infty)$$

where the left hand side denotes the ∞–category of cocartesian fibrations over I. This is a difficult theorem first proved by Lurie in the ∞–categorical setting in [Lur09] with subsequent easier proofs by many other people. Suffice to say, when the functor F is constant with value $\mathcal{C} \in \mathrm{Cat}_\infty$, $\mathrm{Un}(F) \to I$ is given simply by the projection $I \times \mathcal{C} \to I$.

Among other things, we can use this construction to compute limits in Cat_∞. Writing $\Gamma(-)$ for the ∞—category of sections (and $\Gamma_{\mathrm{cocart}}(-)$ for the sections which are cocartesian, whose explanation we again omit), we have:

Theorem 1.11 (Lurie, [HW21, Prop I.36])**.** *Given a functor* $F\colon I \to \mathrm{Cat}_\infty$*, we have the following formulae for (co)limits:*

$$\operatorname*{colim}_I F \simeq \mathrm{Un}(F)[\{\textit{cocart edges}\}^{-1}] \quad \textit{and} \quad \lim_I F \simeq \Gamma_{\mathrm{cocart}}(\mathrm{Un}(F) \to I)$$

In particular, if $F\colon I \to \mathcal{S} \subseteq \mathrm{Cat}_\infty$*, then we have*

$$\operatorname*{colim}_I F \simeq |\mathrm{Un}(F)| \quad \textit{and} \quad \lim_I F \simeq \Gamma(\mathrm{Un}(F) \to I)$$

Definition 1.12. An object $X \in \mathcal{C}$ is said to be *compact* if $\mathrm{Map}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathcal{S}$ preserves filtered colimits. More generally, for a regular cardinal κ, X is called κ-compact if $\mathrm{Map}_{\mathcal{C}}(X,-)$ preserves κ-filtered colimits.

Definition 1.13. A set of objects $S \subseteq \mathcal{C}$ is said to be *jointly conservative* if any morphism in $\mathcal{C}$ that gets sent to an equivalence under the functor

$$\prod_{X \in S} \mathrm{Map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \longrightarrow \prod_{S} \mathcal{S}$$

is already an equivalence in $\mathcal{C}$.

The following is an important standard result due originally in 1–categories to [MP87, Lem. 1.7.ii]. The ∞–categorical version is well–known and is recorded for example in [CDH+].

Proposition 1.14. *Let $\mathcal{C}$ be cocomplete and let $S \subseteq \mathcal{C}$ be a jointly conservative set of compact objects. Then $\mathcal{C}$ is generated under small colimits by the full subcategory spanned by S.*

This is also true if S is a jointly conservative set of κ-compact objects.

Presentability

A presentable ∞-category is, roughly, a cocomplete ∞-category (thus, usually a large ∞-category) which is controlled by "small" objects, and thus describable with a set's worth of data. In more detail:

Definition 1.15. An ∞-category $\mathcal{C}$ is presentable if it is cocomplete, and there is a cardinal κ and a small set of κ-compact objects $S \subset \mathcal{C}$ that generate $\mathcal{C}$ under colimits.

This is, in principle, a very strong restriction, but in practice, most "natural" cocomplete ∞-categories you'll encounter are presentable[1], so that it almost becomes a mild assumption.

This type of size control affords many pleasant properties, among which the existence of objects satisfying various universal properties. This is neatly encoded in the adjoint functor theorem:

Theorem 1.16 (Adjoint Functor Theorem (AFT), [Lur09, Cor. 5.5.2.9])**.** *Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{D}$ be a cocomplete ∞-category.*

1. *A functor $f : \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.*
2. *Suppose $\mathcal{D}$ is also presentable. In this case, a functor $g : \mathcal{D} \to \mathcal{C}$ is a right adjoint if and only if it preserves limits and is accessible.*

"Recall" that a functor is called accessible if it preserves κ-filtered colimits for some cardinal κ. As for presentability, this is in principle rather restrictive, but in practice, most functors you've encountered are accessible (it is in fact quite hard to come up with a non-accessible functor).

[1]With the caveat that the opposite of a presentable ∞-category is almost never presentable.

Remark 1.17. Item 2 in the AFT, specialized to $\mathcal{C} = \mathcal{S}$ is a corepresentability criterion. Indeed, for $\mathcal{D}$ cocomplete, a functor $g : \mathcal{D} \to \mathcal{S}$ is corepresentable if and only if it admits a left adjoint. Hence, by the AFT, when $\mathcal{D}$ is presentable, this is the case if and only if it is accessible and preserves limits.

Item 1, specialized to $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$ is a representability criterion[2], in the same way. Thus, for presentable $\mathcal{C}$, a functor $f : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is representable if and only if it preserves limits.

Stability

The condition of stability on a category has been well–recognised for a long time to be a crucial idea in carrying out homological methods in generalised settings. Traditionally, its manifestations include the notion of chain complexes, triangulated categories, and spectra. One of the many key advantages of ∞–category theory over 1–categories is that this condition is most naturally a higher categorical notion, owing to the fact that the shift functors Ω and Σ (or $[1]$ and $[-1]$ for chain complexes) necessitate the notion of mapping *spaces* as opposed to just mapping sets.

In this subsection, we introduce the basics of the theory of stable ∞–categories and the various key constructions we need for the main body of these notes.

Notation 1.18. Let $\mathcal{C}$ be a pointed ∞–category, ie. it has a zero object. Let $X \in \mathcal{C}$. When they exist, we write ΩX (resp. ΣX) for the pullback (resp. pushout)

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \big\downarrow & \lrcorner & \big\downarrow \\ 0 & \longrightarrow & X \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \big\downarrow & \ulcorner & \big\downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Definition 1.19 ([Lur17, Def. 1.1.1.9, Prop. 1.4.2.27])**.** Let $\mathcal{C}$ be a pointed ∞–category. We say that it is *stable* if the following equivalent conditions are satisfied:

1. pullbacks and pushouts exist, and a commuting square is a pullback if and only if it is a pushout;
2. finite colimits exist, and the functor $\Sigma \colon \mathcal{C} \to \mathcal{C}$ is an equivalence;
3. finite limits exist, and the functor $\Omega \colon \mathcal{C} \to \mathcal{C}$ is an equivalence.

Remark 1.20. A stable ∞-category is in particular additive.

Terminology 1.21. One can check that a functor $\varphi \colon \mathcal{C} \to \mathcal{D}$ between stable ∞–categories preserve finite colimits if and only if it preserves finite limits. We will say that it is an *exact functor* if it satisfies either one (and so both) of the equivalent conditions.

[2] Here, it's important that this part of the statement does not require $\mathcal{D}$ to be presentable, as $\mathcal{S}^{\mathrm{op}}$ is not presentable.

Construction 1.22 (Stabilisations and mapping spectra, [Lur17, Cor. 1.4.2.23])**.** Write $\mathrm{Cat}_\infty^{\mathrm{lex}}$ for the ∞–category of small ∞–categories that have finite limits and morphisms the left exact functors (ie. those that preserve finite limits), and write $\mathrm{Cat}_\infty^{\mathrm{ex}}$ for the ∞–category of small stable ∞–categories and morphisms the exact functors. By the preceding paragraphs, we see that there is a fully faithful embedding $\mathrm{Cat}_\infty^{\mathrm{ex}} \subseteq \mathrm{Cat}_\infty^{\mathrm{lex}}$. The result now is that there is a right adjoint, denoted as $\mathrm{Sp}(-)$, to this inclusion. Concretely, this is given as follows: writing $\mathcal{D}_*$ for the ∞–category of pointed objects in a $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{lex}}$, the ∞–category $\mathrm{Sp}(\mathcal{D})$ is defined as $\lim(\cdots \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_*)$.

In fact, this is even an $(\infty, 2)$–adjunction in that if we have $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$ and $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{lex}}$, then the adjunction counit – which is usually denoted $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ – induces an equivalence of ∞–categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

One can check from the concrete model for $\mathrm{Sp}(-)$ that this universal property is true even when $\mathcal{D}$ is large. From this, we may obtain that the mapping spaces in stable ∞–categories canonically lift to mapping spectra. To wit, the mapping space functor can also be written as

$$\mathrm{Map}_{\mathcal{C}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}$$

and it is easy to see that it preserves limits. Hence by the universal property above, this functor lifts to yield

$$\begin{array}{ccc} & & \mathrm{Sp}(\mathcal{S}) \simeq \mathrm{Sp} \\ & \overset{\mathrm{map}_{\mathcal{C}}}{\nearrow} & \downarrow \Omega^\infty \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow[\mathrm{Map}_{\mathcal{C}}]{} & \mathcal{S} \end{array}$$

In other words, in a stable ∞–category $\mathcal{C}$, we may deloop its mapping space to get a mapping spectrum, ie. $\mathrm{Map}_{\mathcal{C}}(-,-) \simeq \Omega^\infty \mathrm{map}_{\mathcal{C}}(-,-)$.

Definition 1.23. A stable subcategory of a stable ∞–category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ which is stable under finite limits and colimits.

It is called *thick* if it is furthermore closed under retracts in $\mathcal{C}$ (and a "thick subcategory" is implicitly assumed to be a stable subcategory). If $\mathcal{C}$ is idempotent-complete, this latter condition is equivalent to $\mathcal{D}$ being idempotent-complete.

One of the reasons to consider thick subcategories is that they are precisely the kernels of exact functors. One direction (that kernels of exact functors are thick subcategories) is an easy observation, and the other direction comes from Verdier quotients, which we describe in the following omnibus theorem (good sources for which include [NS18, Thm. I.3.3] and [CDH+20, § A.2]):

Theorem 1.24 (Omnibus Verdier quotients)**.** *Given a thick subcategory $\mathcal{D}$ of a stable ∞–category $\mathcal{C}$, there is a stable ∞–category $\mathcal{C}/\mathcal{D}$, called the Verdier quotient of $\mathcal{C}$ by $\mathcal{D}$, with a projection functor $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ and the following properties:*

1. *$\mathcal{C}/\mathcal{D}$ is the cofiber of the inclusion $\mathcal{D} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ and as such induces, for every stable ∞–category $\mathcal{E}$, a fully faithful functor $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \xrightarrow{p^*} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ with essential image those exact functors $\mathcal{C} \to \mathcal{E}$ that vanish on $\mathcal{D}$.*

2. *The kernel of p is exactly $\mathcal{D}$.*

3. *$p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ witnesses the latter as the localization of $\mathcal{C}$ at "mod-$\mathcal{D}$ equivalences", i.e. maps $f : x \to y$ whose cofiber (or equivalently, fiber) is in $\mathcal{D}$. In particular p is essentially surjective.*

4. *For any pair of objects $x, y \in \mathcal{C}$, the canonical map $\mathrm{colim}_{f \in \mathcal{D}_{/y}} \mathrm{Map}_{\mathcal{C}}(x, \mathrm{cofib}(f)) \to \mathrm{Map}_{\mathcal{C}/\mathcal{D}}(p(x), p(y))$ is an equivalence, and $\mathcal{D}_{/y}$ is a filtered ∞-category. Dually, the canonical map $\mathrm{colim}_{g \in \mathcal{D}_{x/}} \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(g), y) \to \mathrm{Map}_{\mathcal{C}/\mathcal{D}}(p(x), p(y))$ is an equivalence.*

Finally, if $\mathcal{D}$ is only a stable subcategory, then the theorem remains true except for Item 2 which is replaced by "the kernel of p is exactly the closure of $\mathcal{D}$ under retracts in $\mathcal{C}$".

Remark 1.25. It is an instructive exercise to get used to ∞-categories to work out exactly what "the canonical map" is in Item 4.

It is also an instructive exercise to deduce that the formula for mapping spaces in Item 4 also holds for mapping spectra.

In the "big world", that is, when $\mathcal{C}$ has all small colimits, there is a better suited notion of subcategory, namely:

Definition 1.26. Let $\mathcal{C}$ be a cocomplete stable ∞–category. A stable subcategory $\mathcal{D}$ of $\mathcal{C}$ is called *localizing* if it is closed under all colimits.

In this situation, we have:

Proposition 1.27. *Suppose $\mathcal{C}$ is a cocomplete stable ∞–category and $\mathcal{D}$ is a localizing subcategory of $\mathcal{C}$. In this case, $\mathcal{C}/\mathcal{D}$ is also cocomplete, and $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ preserves colimits.*

If $\mathcal{C}, \mathcal{D}$ are furthermore presentable, then $\mathcal{C}/\mathcal{D}$ is an accessible localization of $\mathcal{C}$, i.e. it is also presentable and $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ admits a fully faithful right adjoint with essential image the c's in $\mathcal{C}$ such that for all $d \in \mathcal{D}$, $\mathrm{Map}_{\mathcal{C}}(d, c) = 0$, or equivalently, for all $d \in \mathcal{D}$, $\mathrm{map}_{\mathcal{C}}(d, c) = 0$.

Lastly, here is a bridge between the "small" and the "big" worlds of stable ∞–categories:

Theorem 1.28 ([Lur09, Prop. 5.5.7.8])**.** *The Ind–completion and compact objects functor participate in the following equivalence of ∞–categories*

$$\mathrm{Ind}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_{L,\mathrm{st},\omega} : (-)^\omega$$

where $\mathrm{Cat}_\infty^{\mathrm{perf}}$ *denotes the* ∞*–category of small idempotent–complete stable* ∞*–categories and* $\mathrm{Pr}_{L,\mathrm{st},\omega}$ *denote the* ∞*–category of* ω*–compactly generated presentable stable* ∞*–categories and morphisms given by functors which preserve colimits and compact objects.*

General multiplicative matters

Symmetric monoidal structures in the ∞–categorical setting are much more intricate than their 1–categorical counterparts, the reason being that we have to specify a lot more coherence structures. While this can be done very neatly by a key insight of Graeme Segal in [Seg74], we will nevertheless forego precise discussions of these matters in the interest of brevity. As such, we will content ourselves with informal "definitions" in this subsection, and refer the reader to [Lur17, §2.1] for more details on the basic precise notions.

"Definition" 1.29. A *symmetric monoidal* ∞*–category* $\mathcal{C}^\otimes$ is roughly speaking an ∞–category equipped with symmetric monoidal structures which include a tensor map

$$-\otimes - : \mathcal{C}\times\mathcal{C}\longrightarrow\mathcal{C}$$

which is coherently associative and commutative, as well as a unit object $\mathbb{1}\in\mathcal{C}$ such that $\mathbb{1}\otimes(-)\simeq\mathrm{id}_{\mathcal{C}}$. Given this, we will often also denote $\mathcal{C}^\otimes$ with $(\mathcal{C},\otimes,\mathbb{1})$.

A symmetric monoidal functor $f^\otimes\colon\mathcal{C}^\otimes\to\mathcal{D}^\otimes$ between symmetric monoidal ∞–categories is an underlying functor $f\colon\mathcal{C}\to\mathcal{D}$ together with structures witnessing the compatibility of f with $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$. For example, it includes the datum of an equivalence

$$\mathbb{1}_{\mathcal{D}}\xrightarrow{\simeq} f(\mathbb{1}_{\mathcal{C}})$$

and it will also include the data of equivalences

$$f(X)\otimes_{\mathcal{D}} f(Y)\xrightarrow{\simeq} f(X\otimes_{\mathcal{C}} Y)$$

for every $X,Y\in\mathcal{C}$ etc. If one knows what $\mathbf{E}_\infty$–monoids $\mathrm{CMon}(\mathcal{D})$ in an ∞–category $\mathcal{D}$ having finite products, then the ∞–category of symmetric monoidal ∞–categories can be defined simply as $\mathrm{CMon}(\mathrm{Cat}_\infty)$.

A *lax symmetric monoidal functor* $f^\otimes\colon\mathcal{C}^\otimes\to\mathcal{D}^\otimes$ has the same structures as a symmetric monoidal functor, except that the maps

$$\mathbb{1}_{\mathcal{D}}\to f(\mathbb{1}_{\mathcal{C}}) \qquad f(X)\otimes_{\mathcal{D}} f(Y)\to f(X\otimes_{\mathcal{C}} Y)$$

are no longer required to be equivalences.

The following is a bread–and-butter result, and a convenient source for a very general discussion of it is [HHL+21].

Proposition 1.30. *Let* $L^\otimes\colon\mathcal{C}^\otimes\to\mathcal{D}^\otimes$ *be a symmetric monoidal functor whose underlying functor* $L\colon\mathcal{C}\to\mathcal{D}$ *has a right adjoint* $R\colon\mathcal{D}\to\mathcal{C}$*. In this case, the right adjoint can canonically be refined with the structure of a lax symmetric monoidal functor* $R^\otimes\colon\mathcal{D}^\otimes\to\mathcal{C}^\otimes$*.*

"Definition" 1.31. In a symmetric monoidal ∞–category $\mathcal{C}^{\otimes}$, one may speak of an $\mathbf{E}_\infty$–algebra object (we will often just call these commutative algebra objects). Writing $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ for the ∞–category of commutative algebra objects, roughly speaking with some justifiable abuse of notations, an object $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$ is the datum of an object $A \in \mathcal{C}$ equipped with "multiplication maps"

$$\mu \colon A \otimes_{\mathcal{C}} A \longrightarrow A$$

which is coherently associative and commutative, as well as a "unit map" $1 \colon \mathbb{1} \to A$ together with the datum of a homotopy to the identity id_A for the composite

$$A \simeq A \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes 1} A \otimes A \xrightarrow{\mu} A$$

Fact 1.32. Lax symmetric monoidal functors preserve commutative algebra objects, ie. any lax symmetric monoidal functor $f^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ induces a functor

$$f \colon \mathrm{CAlg}(\mathcal{C}^{\otimes}) \longrightarrow \mathrm{CAlg}(\mathcal{D}^{\otimes})$$

Morally, this is because if we started with an $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$, then the lax maps provide us with enough structure to define a multiplication map

$$f(A) \otimes_{\mathcal{D}} f(A) \xrightarrow{\mathrm{lax}} f(A \otimes_{\mathcal{C}} A) \xrightarrow{f\mu} f(A)$$

and so on.

Fact 1.33 (Presentably symmetric monoidal ∞–categories). There is a symmetric monoidal structure on Pr_L, which is usually called the Lurie tensor product, whose tensor unit is the ∞–category of spaces $\mathcal{S}$. This tensor product is defined by the following universal property: writing $\mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for the full subcategory of bicocontinuous functors (ie. those which preserve small colimits in each variable), the Lurie tensor product $\mathcal{C} \otimes \mathcal{D}$ is equipped with a bicocontinuous functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which induces an equivalence

$$\mathrm{Fun}^{L}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

Consequently, a commutative algebra object $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}_L^{\otimes})$ is a presentable ∞–category equipped with a tensor product

$$- \otimes - \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mu} \mathcal{C}$$

which is bicocontinuous. To distinguish this important extra criterion, we will call an object in $\mathrm{CAlg}(\mathrm{Pr}_L^{\otimes})$ *presentably symmetric monoidal ∞–categories*. Crucially, by the adjoint functor theorem, presentably symmetric monoidal ∞–categories are always closed symmetric monoidal in that for any $X \in \mathcal{C}$, $- \otimes X$ has a right adjoint that we denote by $\mathrm{Hom}_{\mathcal{C}}(X, -)$ called the *internal hom object*.

Moreover, Lurie proved that $\mathrm{Sp} \in \mathrm{Pr}_L$ is an idempotent object, meaning $\mathrm{Sp} \otimes \mathrm{Sp} \otimes - \simeq \mathrm{Sp} \otimes -$. This endows Sp with a presentably symmetric monoidal functor whose unit map is written as

$$\mathbb{S}[-] = \Sigma^\infty_+ \colon \mathcal{S} \longrightarrow \mathrm{Sp}$$

recovering the classical suspension spectrum functor whose right adjoint is Ω^∞. Furthermore, writing $\mathrm{Pr}_{L,\mathrm{st}} \subseteq \mathrm{Pr}_L$ for the full subcategory of stable presentable ∞–categories, one obtains that this inclusion has a left adjoint given by applying $\mathrm{Sp} \otimes -$. This endows $\mathrm{Pr}_{L,\mathrm{st}}$ with a symmetric monoidal structure whose tensor unit is Sp and such that both adjoints in

$$\mathrm{Pr}_L \underset{\longleftarrow}{\overset{\mathrm{Sp}\otimes -}{\longrightarrow}} \mathrm{Pr}_{L,\mathrm{st}}$$

refine to the structure of symmetric monoidal functors.

While we are mainly interested in $\mathbf{E}_\infty$–structures, we will often also need to talk about $\mathbf{E}_1$–structures. These are the higher algebraic analogue of associative algebras, relevant examples of which include group rings $R[G]$ (that we will see later) as well as the example in the famous result of Schwede–Shipley which says:

Theorem 1.34 (Schwede–Shipley, [Lur17, Thm. 7.1.2.1], [HW21, Thm. II.58])**.** *Let $\mathcal{C}$ be a stable presentable ∞–category. If it has a compact generator $X \in \mathcal{C}$, then* $\mathrm{map}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Sp}$ *lifts to an equivalence*

$$\mathrm{map}_{\mathcal{C}}(X, -)\colon \mathcal{C} \xrightarrow{\simeq} \mathrm{RMod}_{\mathrm{Sp}}(\mathrm{End}(X))$$

where $\mathrm{End}(X)$ *is the* $\mathbf{E}_1$*–ring spectrum with underlying object* $\mathrm{map}_{\mathcal{C}}(X, X)$ *and multiplication given by composition.*

Duality and dualisability

In the groundbreaking paper [DP84], among other things, Dold and Puppe axiomatised the notion of duality, providing a far–reaching generalisation of dual vector spaces. We summarise this notion now in the modern setting.

Definition 1.35 ([Lur17, Def. 4.6.1.1])**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞–category. A *duality datum* in $\mathcal{C}$ is a tuple $(X, X^\vee, c, e)$ where $X, X^\vee \in \mathcal{C}$ and c, e are morphisms

$$c\colon \mathbb{1} \to X \otimes X^\vee \qquad e\colon X^\vee \otimes X \to \mathbb{1}$$

such that the composites

$$X \xrightarrow{c\otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}\otimes e} X$$

$$X^\vee \xrightarrow{\mathrm{id}\otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e\otimes \mathrm{id}} X^\vee$$

are homotopic to the identity morphisms. An object $X \in \mathcal{C}$ which participate in such a datum is said to be *dualisable*.

Observation 1.36. Since duality data are tuples satisfying homotopy conditions, they are really notions that are controlled in the homotopy category with the induced 1–symmetric monoidal structure $(\mathrm{Ho}(\mathcal{C}), \otimes, \mathbb{1})$. That is, a tuple $(X, X^{\vee}, c, e)$ is a duality datum in $(\mathcal{C}, \otimes, \mathbb{1})$ if and only if it is one in $(\mathrm{Ho}(\mathcal{C}), \otimes, \mathbb{1})$.

There is an alternative characterisation of duality data that is often used interchangeably when talking about dualisable objects. The proof is a straightforward unwinding of definitions which we will omit in this note.

Proposition 1.37 ([Lur17, Lem. 4.6.1.6]). *If an object $X \in \mathcal{C}$ is dualisable with dual $X^{\vee}$, then the duality datum provides a natural equivalence of functors*

$$\mathrm{Map}_{\mathcal{C}}(-, X \otimes -) \simeq \mathrm{Map}_{\mathcal{C}}(- \otimes X^{\vee}, -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}$$

Observation 1.38. In particular, when $(\mathcal{C}, \otimes, \mathbb{1})$ is presentably symmetric monoidal stable, we have an equivalence of functors $\mathrm{Hom}_{\mathcal{C}}(X, -) \simeq X^{\vee} \otimes -$. This in particular implies the special property that $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves small colimits, which is very much not true in general.

Finally, we record the following "descent" property of the notion of dualisability.

Proposition 1.39 (Descent for dualisability, [Lur17, Prop. 4.6.1.11]). *Let $\{\mathcal{C}_i\}_{i \in I}$ be a diagram of symmetric monoidal ∞–categories with limit $\mathcal{C}$. In this case, an object in $\mathcal{C}$ is dualisable if and only if its image in each $\mathcal{C}_a$ is dualisable.*

2. Recollections on classical representation theory

This section will be a brisk recollection of some of classical representation theory of finite groups. The main emphasis is to touch on the basics of modular representation theory including Serre's *cde* formalism, Brauer characters, and Block theory. A great reference for most of what we discuss in this section is Serre's book [Ser+77]. A reference that (among other things) relates these techniques to Mackey theory is Swan's book [Swa06].

Terminology 2.1. Let R be a commutative ring. By a G–representation over R, we mean an object in $\mathrm{Fun}(BG, \mathrm{Mod}_R^{\mathrm{fg,proj}})$. That is, in these notes, we will only ever be concerned with finite–dimensional representations and so we will always omit this clunky adjective.

This section is shaped with the following list of subsections:

- K–theoretic gadgets in representation theory
- The non–modular case: standard character theory
- The modular case setup: Serre's *cde* formalism
- Modular case tool I: Brauer characters
- Modular case tool II: block theory

K–theoretic gadgets in representation theory

Construction 2.2. Let $\mathcal{A}$ be a small exact category. We define its *K–theory group* $K_0(\mathcal{A})$ to be the abelian group generated by the isomorphism classes of objects subject to the relation that $[M] = [N] + [Q]$ whenever we have a short exact sequence $0 \to N \to M \to Q \to 0$. If $\mathcal{A}$ has a symmetric monoidal structure where $\otimes$ commutes with $\oplus$ in each variable, then $K_0(\mathcal{A})$ admits a canonical ring structure.

We now introduce two important players of representation theory:

Definition 2.3. Let G be a finite group and A a commutative ring. We define

$$P(A[G]) := K_0\Big(\operatorname{Fun}(BG, \operatorname{Mod}_A)^{\mathrm{fg,proj}}\Big) \quad \text{and} \quad R_A(G) := K_0\Big(\operatorname{Fun}(BG, \operatorname{Mod}_A^{\mathrm{fg,proj}})\Big)$$

Since $\operatorname{Fun}(BG, \operatorname{Mod}_A^{\mathrm{fg,proj}})$ has a natural symmetric monoidal structure given by $\otimes_A$, $R_A(G)$ has a natural commutative ring structure.

Remark 2.4. Note that $\operatorname{Fun}(BG, \operatorname{Mod}_A)^{\mathrm{fg,proj}} \simeq \operatorname{Mod}_{A[G]}^{\mathrm{fg,proj}}$.

One of the goals of the representation theory for finite groups G is to understand $P(k[G])$ and $R_k(G)$ as much as possible. Here is a way to relate them:

Construction 2.5 (Cartan morphisms)**.** There is an inclusion of exact categories $\operatorname{Fun}(BG, \operatorname{Mod}_k)^{\mathrm{fg,proj}} \subseteq \operatorname{Fun}(BG, \operatorname{Mod}_k^{\mathrm{fg,proj}})$. This induces a group homomorphism

$$P(k[G]) \xrightarrow{c} R_k(G)$$

which is usually called the *Cartan morphism*.

In the case where $|G| \in k^\times$, we can get a relatively good understanding of these groups. The key notion here is that of semi-simplicity:

Definition 2.6. Let k be a field and V be a G–representation over k. We say that it is *simple* if it has no non–trivial G–subrepresentations (ue. it has no nontrivial $k[G]$–submodules); it is said to be *semi–simple* if it is a finite direct sum of simple representations.

Proposition 2.7 (Maschke's theorem)**.** *Let R be a commutative ring in which $|G|$ is invertible, and let M, N be G-representations in R-modules, with an R-linear, equivariant map $f : M \to N$. If M has an R-linear section (resp. retraction), then it has one which is also G-equivariant.*

In particular, if $R = k$ is a field in which $|G|$ is invertible, then any $k[G]$-linear injection (resp. surjection) admits a retraction (resp. section).

Proof. The key point here will be our ability to *average* over G[3]. We deal with the case of a section, but the case of a retraction is completely symmetric.

[3]This averaging trick also works in the setting of compact topological groups over $\mathbb{R}$, using Haar measures; but this is out of the scope of these notes.

Fix an R-linear section $s : N \to M$, i.e. $f \circ s = \mathrm{id}_N$. The map s might not be G-equivariant, equivalently, we might not have $s = gsg^{-1}$. We fix this by defining $\tilde{s} := \frac{1}{|G|} \sum_{g \in G} gsg^{-1}$. It is easy to verify the following facts: first, $\tilde{s}$ is R-linear and G-equivariant, and second, $f \circ \tilde{s} = \mathrm{id}_N$ (this latter part is where we use the fact that we divided by $|G|$, and that f is itself G-equivariant). $\square$

Corollary 2.8. *Let k be a field in which $|G|$ is invertible. The category of G-representations in k-vector spaces is semi-simple: any $k[G]$-module is a direct sum of irreducible representations, i.e. simple modules (which are necessarily projective).*

The non–modular case: standard character theory

Classifying representations in characteristic prime to $|G|$ thus boils down to classifying irreducible representations, and this is done via *character theory*. This is a key concept which makes sense without $|G| \in k^\times$ and will be relevant also when $|G|$ is not invertible, so we mention it briefly now.

Construction 2.9 (Characters)**.** Let M be a finite–dimensional G–representation over k. We define its character as the map $\chi_M : G \to k$ given by $g \mapsto \mathrm{tr}(g \mid M)$, where the latter is the trace of the k-linear endomorphism of M given by the action of g. Because traces are cyclically invariant, this factors canonically through the set of conjugacy classes of G, $G/\mathrm{conj} \to k$. Writing $C(G/\mathrm{conj}, k)$ for the abelian group of set maps $G/\mathrm{conj} \to k$, the above construction yields the homomorphism

$$\chi\colon R_k(G) \longrightarrow C(G/\mathrm{conj}, k)$$

called the *character map*.

Theorem 2.10. *Let k be a field in which $|G|$ is invertible. The character map $\chi\colon R_k(G) \to C(G/\mathrm{conj}, k)$ is injective. If furthermore k is algebraically closed, this induces an isomorphism $R_k(G) \otimes_{\mathbb{Z}} k \cong C(G/\mathrm{conj}, k)$.*

Corollary 2.11. *Let k be an algebraically closed field in which $|G|$ is invertible. In this case, the number of irreducible representations of G over k is equal to the number of conjugacy classes of elements of G.*[4]

The modular case setup: Serre's *cde* formalism

In characteristic dividing $|G|$, the preceding results break down, as we can no longer average over G and thus lose semi-simplicity. Consequently, the Cartan morphism $P(k[G]) \to R_k(G)$ is almost never an isomorphism.

We start by discussing an extreme example of this failure, namely the case of p-groups. In this case, we can completely describe $P(k[G])$ and $R_k(G)$:

[4]However, there is no explicit bijection in general.

Proposition 2.12. *Let G be a p-group and k a field of characteristic p.*

- *The only simple $k[G]$-module is k with the trivial G-action.*
- *Any projective $k[G]$-module is free.*

In particular, the Cartan morphism is identified with the map $\mathbb{Z} \xrightarrow{|G|} \mathbb{Z}$, which is not an isomorphism.

Example 2.13. Let G be a finite group, and let $f : G \to k$ a morphism to the additive group of k. Let V_f be the vector space k^2 with the following G-action: $g \cdot (x, y) = (f(g)y + x, y)$. This is an extension of k by itself, and the assignment $f \mapsto V_f$ induces an isomorphism between $\hom(G, k)$ and the set of isomorphism classes of extensions of k by itself.

More qualitatively, one can ask how complicated the representation theory of G is, in computational terms. For instance, a theorem of Higman says that if p is odd and the p–Sylow subgroups of G are not cyclic, then $k[G]$ is of wild representation type[5].

This is really a key difference with characteristic 0: representations are way too complicated, and the simple modules and projective modules only represent a tiny piece of it. Understanding them is, however, a first step to understanding the whole thing, and we can actually analyze them. Some tools are available for this thanks in large parts to the work of Brauer. The general setting for all these machinery is:

Setting 2.14 (Coefficients in modular representation theory)**.** Fix a field k of characteristic p, a complete discrete valuation ring R (with maximal ideal $\mathfrak{m}$) with residue field k and quotient field K of characteristic 0. A typical example would be to start with k, take $R = W(k)$, the Witt vectors of k, and K the fraction field of R. The goal is to use the span

$$K \longleftarrow R \longrightarrow k$$

as a bridge to transport knowledge from the left (where non–modular methods are available) to the right.

The reason this kind of setting can be helpful is that it will help us relate characteristic p to characteristic 0 is *idempotent lifting*:

Lemma 2.15 (Idempotent–lifting)**.** *Let A be a ring and I a 2-sided ideal for which A is I-complete. Any idempotent e in A/I has a lift to A.*

Applying this to $M_n(A)$ with the ideal $M_n(I)$ also gives lifts for projective modules, and one can prove that they are in fact unique up to isomorphism, thus getting an isomorphism $K_0(A) \to K_0(A/I)$.

[5] This term essentially means that for any finite dimensional k-algebra B, the category of finite dimensional representations of B embeds into the category of finite dimensional representations of $k[G]$. This has interpretations in terms of computability theory and essentially means: computationally intractable.

Proof. A being I-complete means that $A \cong \lim_n A/I^n$, so if we can lift e at any stage A/I^n, we can lift it to A. Furthermore, each $A/I^{n+1} \to A/I^n$ is a quotient by a square zero ideal, so we may in fact assume $I^2 = 0$.

In this case, given an arbitrary lift r of e, we find that $\tilde{e} := 3r^2 - 2r^3$ is still a lift of e, and it is in fact idempotent.[6] □

We will now describe Serre's cde triangle, which is going to look as follows:

$$\begin{array}{ccc} P(k[G]) & \xrightarrow{c} & R_k(G) \\ & \searrow^{e} & \uparrow d \\ & & R_K(G) \end{array} \tag{1}$$

Here, c is the Cartan morphism which we have already described. The philosophical point of this is that this factors the map c in the modular world via the non–modular object $R_K(G)$, and the hope is then that we can use this to transport insights from the easier non–modular setting to the modular one. We now define d, e. The simplest is e:

Construction 2.16 (The idempotent–lifting map e)**.** By idempotent-lifting Lemma 2.15, and the fact that R is $\mathfrak{m}$-complete, basechange along $R \to k$ induces an isomorphism $P(R[G]) \xrightarrow{\cong} P(k[G])$.

So we may define $e : P(k[G]) \to R_K(G)$ as the composite $P(k[G]) \xleftarrow{\cong} P(R[G]) \to P(K[G]) \cong R_K(G)$. Concretely, given a projective $k[G]$-module P, e lifts it to the unique (up to non-unique isomorphism) projective $R[G]$-module $\tilde{P}$ lifting it, and then takes $K \otimes_R \tilde{P}$. Presumably the letter e was used because it is the standard notation for idempotents.

Construction 2.17 (The decomposition map d)**.** Now for d, first recall we make the following observation: if P is a K-module with a G-action and $L \subset P$ is an R-lattice therein, i.e. a sub-R-module with $K \otimes_R L \cong P$ along the canonical map, then we can "average" L over G to get a lattice which is stable under G, namely, one can take $L' := \sum_{g \in G} gL$. This is stable under G , and it is still a lattice, thus it is an $R[G]$-module, which is R-projective. The claim is then that the class of $L' \otimes_R k \cong L'/\mathfrak{m}L'$ in $R_k(G)$ is independent of the choice of L, L', and this thus gives a definition of $d : R_K(G) \to R_k(G)$, the "decomposition" morphism.

Here is a result summarising the basic points about the cde triangle.

Proposition 2.18 (Omnibus cde)**.** *Let (R, k, K) be as in Setting 2.14.*

1. *The cde triangle diagram (1) commutes;*

[6]This magical "$3r^2 - 2r^3$" does not come out of nowhere, see https://sites.google.com/view/maxime-ramzi-en/notes/idempotent-lifting for a slightly more enlightening proof, and a derivation of this formula.

2. *The cde triangle is compatible with inductions and restrictions along subgroup inclusions. In the language of §9, this is a commuting diagram of Mackey functors;*

3. *The map $d\colon R_K(G) \to R_k(G)$ is a ring homomorphism.*

Proof. We will only prove point (3). To do this, simply note that if $L_i \subset P_i$ are R-lattices, stable under G for $i \in \{0,1\}$, then $L_0 \otimes_R L_1 \subset P_0 \otimes_K P_1$ is an R-lattice, stable under G, and $k \otimes_R -$ is symmetric monoidal. □

We now sketch a situation that illustrates how the cde triangle can successfully import characteristic 0 knowledge into the characteristic p setting.

Corollary 2.19. *Let G be a finite group, k a field of characteristic p, and p^n the highest power of p dividing $|G|$. The cokernel of the Cartan map Construction 2.5 is p^n-torsion.*

Sketch of proof. The proof consists of three main steps.

1. For simplicity, we assume K is large enough for the theorem of Brauer to apply. One can deduce the general case from this one.

2. We may reduce the question to the case when G is a so–called *elementary group*. We defer the details of this step to Exercise 2.25 Item 4. The key point is that we will transport a Dress induction theorem (cf. Theorem 9.6) on $R_K(G)$ onto $R_k(G)$ via the d morphism in *cde*.

3. In the case of elementary groups, we may work out explicitly that the Cartan map looks like $\mathbb{Z}^k \xrightarrow{p^n} \mathbb{Z}^k$, whose cokernel is then p^n–torsion as required. This is analogous Proposition 2.12.

This completes the proof sketch. □

The point of this corollary was not to state a particular result but to showcase several proof-techniques in modular representation theory. In step (1), we see the standard manouevre of separating "absolute" questions (over sufficiently large fields) and rationality questions (given the answer to a question over a large field K', how does one deduce things about the same question over smaller fields?). Next, step (2) shows us how to use the *cde* triangle - in this case, the d morphism - to push characteristic 0 results to characteristic p. Here we have also used the so–called *Dress induction theorem* in Mackey theory (more on this in §9) to reduce the problem to simpler groups for which explicit computations are possible.

Modular case tool I: Brauer characters

Another way of connecting characteristic 0 and characteristic p, still in the same Setting 2.14 (with K sufficiently large, say), is via so-called *Brauer* characters. From the perspective of the cde triangle, one can describe them as follows: ordinary characters provide a morphism $R_K(G) \to C(G/\mathrm{conj}, K)$, and if we restrict our class functions to the p-regular conjugacy classes, i.e. the conjugacy classes of elements that have order prime to p, the corresponding morphism $R_K(G) \to C(G_{reg}/\mathrm{conj}, K)$ *factors* through $d : R_K(G) \to R_k(G)$. A construction is given as follows:

Construction 2.20 (Brauer characters)**.** By Hensel's lemma, prime-to-p roots of unity in k admit lifts to R. Thus if k has all prime-to-p roots of unity, so does K, and furthermore, there is an isomorphism (given by reduction mod $\mathfrak{m}$) between the corresponding groups $\mu_K^{p'}$ and $\mu_k^{p'}$. We let $\lambda \mapsto \tilde{\lambda}$ denote the inverse of this isomorphism.

Given a finite dimensional $k[G]$-module V and $g \in G_{reg}$, as g has order prime to p, we find that the action of g on V is diagonalizable, with eigenvalues λ_i (counted with multiplicity). We let $\chi(g) := \sum_i \tilde{\lambda}_i \in K$.

The basic properties of these characters are easy to check: it behaves similarly to ordinary characters, e.g. it is additive along short exact sequences, $\chi(1_G) = \dim(V)$, it is conjugation invariant, and finally the corresponding map $R_k(G) \to C(G_{reg}/\mathrm{conj}, K)$, restricted to $R_K(G)$, recovers (the restriction to G_{reg} of) ordinary characters.

Modular case tool II: block theory

The final tool we introduce in these recollections is the theory of blocks and defect groups. The idea is to split the representation theory of G into different pieces, called blocks.

Warning 2.21. The word "block" is a bit overloaded, in the sense that it can correspond to different *types* of objects. The point is that there are canonical ways of matching these different types of objects, so there is in general no possible confusion.

For $\Lambda \in \{R, k, K\}$, the algebra $\Lambda[G]$ admits a decomposition $\bigoplus_i B_i$ into a finite direct sum of indecompsable two-sided ideals. These are generated by a central, primitive idempotent $e_i \in \Lambda[G]$.

Given a primitive central idempotent e in an algebra A, three things correspond to it: firstly, the idempotent e itself, secondly, the summand eA it generates, and thirdly, on a more representation-theoretic note, the full sub-category of A-modules M such that $eM = M$. Any of these three things is called a *block* of the algebra A.

We start by noting that, essentially by idempotent-lifting (cf. Lemma 2.15),for a triple such as (R, k, K) from Setting 2.14, there are really only two notions of block:

Proposition 2.22. *Reduction mod $\mathfrak{m}$ induces an isomorphism between central idempotents in $R[G]$ and central idempotents in $k[G]$; in particular it induces a bijection their respective sets of primitive central idempotents.*

We can also identify the blocks of $K[G]$:

Proposition 2.23. *The blocks of $K[G]$ bijects with the set of irreducible representations of G over K. If K is sufficiently large, this is equal to the number of conjugacy classes of G.*

We further note that the block decomposition $\Lambda[G] \cong \bigoplus_i B_i$ induces a decomposition of the whole module category, namely $\mathrm{Mod}_{\Lambda[G]} \simeq \prod_i \mathrm{Mod}_{B_i}$. A $\Lambda[G]$-module M "belongs" to a block B_i, corresponding to the idempotent e_i, if in the above product decomposition, it belongs to the factor Mod_{B_i}, or, equivalently, if $e_i M = M$, or equivalently if $e_j M = 0$ for all the other primitive central idempotents e_j.

This is also the maximal decomposition of $\mathrm{Mod}_{\Lambda[G]}$ as a product of categories, so the blocks of $\Lambda[G]$ are a way to simplify the study of $\mathrm{Mod}_{\Lambda[G]}$ as much as possible.

Example 2.24. As the trivial $\Lambda[G]$-module Λ is indecomposable, it belongs to a single block. This is called the *principal block*, often denoted B_0.

Exercise 2.25.

1. The goal of this exercise is to prove Proposition 2.12. Throughout, k is a field of characteristic p.
   a) Prove that a p-group always has a nontrivial center. **Hint:** use the orbit-stabilizer theorem for the conjugation action of the group on itself.
   b) Using induction, deduce that any G-representation over k has a nonzero invariant vector, that is, an element v with $gv = v$ for all $g \in G$. **Hint:** We are in characteristic p, so $g^{p^k} - 1 = (g-1)^{p^k}$. Apply this to a nontrivial element in the center to induct on the size of the group G.
   c) Deduce that any G-representation V has a filtration $0 = V_0 \subset V_1 \subset ... \subset V_n = V$ where each V_{i+1}/V_i has a trivial G-action.
   d) Using the previous item over $k[G]$, deduce that $k[G]$ is a local ring and hence that any projective module is free. Conclude with a proof of Proposition 2.12.
   e) If you know about projective covers, state and prove an analogous result for groups of the form $P \times S$ where P is a p-group and S a p'-group.
2. Let $k \subset k'$ be a field extension. Prove that the induced morphism $R_k(G) \to R_{k'}(G)$ is injective, for any finite group G. As a hint, you'll want to prove that given two nonisomorphic irreducible representations V_0, V_1 over k, $\hom_{k'[G]}(k' \otimes_k V_0, k' \otimes_k V_1)$ is trivial.
3. Prove that the decomposition morphism $d : R_K(G) \to R_k(G)$ is well-defined. As a hint, you'll want to notice that given two R-lattices L_0, L_1 in a K-vector space V, there is some integer k such that $\mathfrak{m}^k L_0 \subset L_1$. If you know about K-theory, try to prove that there is even a morphism of spectra $K(\mathrm{Fun}(BG, \mathrm{Perf}(K))) \to K(\mathrm{Fun}(BG, \mathrm{Perf}(k)))$ which induces d on π_0. Prove also that the cde triangle commutes.
4. We work out step (2) in Corollary 2.19. Using the compatibility of the ring map d with induction and restriction Proposition 2.18 (2), prove that if G is a group

and $\mathcal{F}$ a family of subgroups of G such that $\bigoplus_{H\in\mathcal{F}} R_K(H) \xrightarrow{(\mathrm{Ind}_H^G)_{H\in\mathcal{F}}} R_K(G)$ is surjective, then so is the corresponding map from R_k. Compare with Theorem 9.6.

Now step (2) is then a direct consequence of a theorem of Brauer which says that if K is "large enough" (contains enough roots of unity), then the map

$$\oplus_H \mathrm{Ind}_H^G\colon \bigoplus_{H\leq G,\ H \text{ elementary}} R_K(H) \longrightarrow R_K(G)$$

is surjective. Here, an elementary subgroup of G is a subgroup which, for some prime q, is isomorphic to $C \times Q$ where Q is a q-group and C is a cyclic group of order prime to q[7].

3. Basic chromatic homotopy theory

Chromatic homotopy theory begins with the observation that certain cohomology theories naturally give rise to so-called *formal groups* (cf. §6 for more details). This connection between stable homotopy theory and the theory of formal groups allows us to better understand the large-scale structure of the ∞-category of spectra. Among other things, the notion of *height* of a formal group has an incarnation in stable homotopy theory through the *chromatic filtration* and the corresponding notions of height.

Our goal in this section is not to give a complete overview of this theory, rather to introduce the players and ideas that will be relevant for the masterclass. Lurie's lecture notes [Lur10] are a good introduction to go into more depth than what we have presented here. There are also two chromatic books by Doug Ravenel that cover the foundations of chromatic homotopy theory [Rav23], [Rav92].

We begin by an important theorem, one that more or less justifies the idea that chromatic homotopy theory is a way to organize the "large-scale structure" of Sp. To state it, we need a black–boxed "definition":

"Definition" 3.1. For every prime p (usually implicit in the notation), and every $0 \leq n \leq \infty$ (the "height"), there is a spectrum $K(n)$, called Morava K-theory at height n. For $0 < n < \infty$, its homotopy groups are $\mathbb{F}_p[v_n^{\pm 1}]$ where $|v_n| = 2(p^n - 1)$.

For $n = 0$, $K(0) = \mathbb{Q}$, and for $n = \infty$, $K(\infty) = \mathbb{F}_p$.

Warning 3.2. When $0 < n < \infty$, this spectrum is *not* an Eilenberg-MacLane spectrum, and in particular it is *not* $\bigoplus_{k\in\mathbb{Z}} \Sigma^{2(p^n-1)k}\mathbb{F}_p$. In particular, it does *not* come from classical discrete algebra.

Example 3.3. At height 1, Morava K-theory is related to complex topological K-theory, namely $K(1) = \mathrm{KU}/p$. In particular, v_1 can be thought of as the Bott periodicity element.[8]

[7] Not to be confused with "elementary abelian" groups, which are not the same as "elementary groups that are abelian"

[8] The analogous fact at higher heights involves so-called Morava E-theory, or Lubin-Tate theory.

With these preliminaries, we can now state the thick-subcategory theorem:

Theorem 3.4 (The thick subcategory theorem, [HS98]). *Fix a prime p. The following is an exhaustive and non-redundant list of the thick subcategories of $\mathrm{Sp}_{(p)}^{\omega}$: For any $0 \leq n \leq \infty$, $C_{\geq n} := \{X \in \mathrm{Sp}_{(p)}^{\omega} \mid K(n-1) \otimes X = 0\}$, as well as $C_{\geq 0} := \mathrm{Sp}_{(p)}^{\omega}$.*

Furthermore, for all $0 \leq n \leq \infty$, we have $C_{\geq n+1} \subset C_{\geq n}$, and finally $C_{\geq \infty} = 0$.

Remark 3.5. Among other things, this theorem states that for a finite p-local spectrum X, if $K(n+1) \otimes X = 0$, then $K(n) \otimes X = 0$. This is *not* true in general for non-finite spectra. For example, $K(n+1) \otimes K(n) = 0$, but $K(n) \otimes K(n)$ is not zero, as $K(n)$ admits a homotopy ring structure.

In fact, for commutative ring spectra, the implication is reversed! See [Hah16, Theorem 1.1] or [BSY22, Theorem 1.5].

This suggests the following important definition:

Definition 3.6. A finite p-local spectrum X is said to be of type n if $K(n-1) \otimes X = 0$ and $K(n) \otimes X \neq 0$.

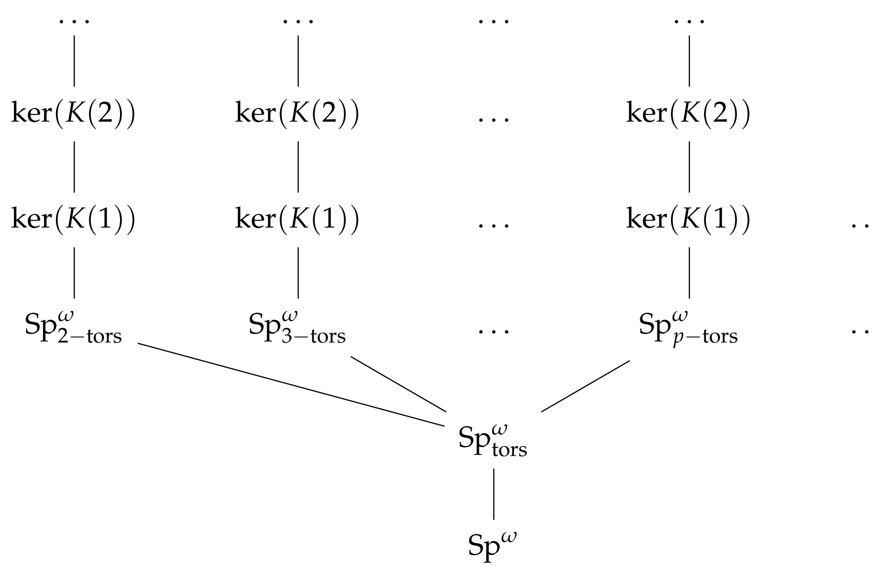
Remark 3.7. By the thick subcategory theorem, for any type n spectrum X, the thick subcategory generatd by X is exactly $C_{\geq n}$.

The fact that this list of thick subcategories is non-redundant is a consequence of the theory of v_n-self maps, which we will mention a bit later; while the fact that it is exhaustive is a consequence of the nilpotence theorem, another foundational result in chromatic homotopy theory, showcasing the importance of Morava K-theories. It has several equivalent formulations, here we give the one that is the most suitable to prove the thick subcategory theorem (which we will not do in this document):

Theorem 3.8 (The nilpotence theorem [HS98]). *Let $f : X \to Y$ be a map of finite p-local spectra such that for all $0 \leq n \leq \infty$, $K(n)_* f : K(n)_* X \to K(n)_* Y$ is zero.*

There is an integer k such that $f^{\otimes k} : X^{\otimes k} \to Y^{\otimes k}$ is nullhomotopic.

The thick subcategory theorem suggests the following pictorial description of Sp^ω:



(where, of course, the $K(n)$'s lying over 2 are not at all the same as the ones lying over 3, or over a different prime p)

This is one incarnation of the chromatic filtration. In this picture, the $\leq n$-part can be seen as an open in the "spectrum" of Sp (precisely, the Balmer spectrum of Sp^ω). It turns out that there is a chromatic analogue of restriction to this open piece, namely L_n-localization.

Definition 3.9. A spectrum X is L_n-acyclic if for all $k \leq n$, $K(k) \otimes X = 0$.

A spectrum Y is L_n-local if for any L_n-acyclic spectrum X, $\mathrm{Map}(X, Y) = 0$.

One can reinterpret L_n-acyclicity as acyclicity with respect to Morava E-theory. For this, we again need a fake definition:

"Definition" 3.10. For every prime p (implicit in the notation), every perfect field k of characteristic p and every height $0 < n < \infty$, there is a $K(n)$-local spectrum E_n, called Morava E-theory at height n. For $0 < n < \infty$, its homotopy groups are $W(k)[\![u_1, ..., u_{n-1}]\!][u^{\pm 1}]$ with $|u_i| = 0, |u| = 2$, where $W(k)$ is the ring of Witt vectors of k.

Warning 3.11. As in the case of Morava K-theory, the above is very far from a definition. The spectrum E_n is again very far from an Eilenberg-MacLane spectrum, and it is not captured by its homotopy groups alone.

Warning 3.12. In principle, E_n depends not only on k and n, but also on a formal group $\mathbf{G}$ on k, of height n, so one should really write $E(k, \mathbf{G})$. When k is algebraically closed, any two formal groups on k of the same height are isomorphic; and in practice, many of the properties of $E(k, \mathbf{G})$ really only depend on n (and p, of course), so this abuse of notation is not so problematic.

Remark 3.13. The spectra $E(k, \mathbf{G})$ depend functorially on the pair $(k, \mathbf{G})$, and are in fact commutative ring spectra, functorially (and in a more or less unique way). They are extremely important in chromatic homotopy theory. For instance they have properties analogous to those algebraically closed fields (see [Rog08],[BSY22]).

Example 3.14. At height 1, and for $k = \mathbb{F}_p$, E_1 is simply KU_p, p-completed topological K-theory.

Remark 3.15. In the above example, we see that KU is an "integral" version of all the E_1's at different primes. At higher heights, it is not clear whether there are analogues, that is, spectra that can be completed at any prime to give some variant of E_n (even if not exactly the spectrum E_n). At height 2 for instance, topological modular forms can be seen as an attempt to get such an integral version of E-theory.

We can now state without the proof the following

Proposition 3.16. *A spectrum X is L_n-acyclic if and only if it is E_n-acyclic.*

Remark 3.17. For example in this proposition, we see that E_n-acyclicity only depends on the prime p and the height n.

We can now introduce most of the "chromatic localizations", and make basic observations about them.

Example 3.18. With $E = E_n$, Morava E-theory, $L_{E_n} = L_n$ because a spectrum is E_n-acyclic if and only if it is L_n-acyclic.

A deep theorem about L_n is that it is a smashing localization:

Theorem 3.19 (The smash product theorem, [Rav92])**.** *Localization at E_n is smashing, namely the canonical map $L_n\mathbb{S} \otimes X \to L_n X$ is an equivalence for any spectrum X.*

Remark 3.20. Note that $L_n\mathbb{S}$ is *not* equivalent to E_n.

This theorem shows that localizing at E_n really behaves like restriction to an open, restriction to "heights $\leq n$". Once we are in height $\leq n$, we can try to complete *at* the height. One way to do that is to $K(n)$-localize:

Example 3.21. Localization at $K(n)$, $L_{K(n)}$ is *not* a smashing localization.

We will later introduce another localization, namely $T(n)$-localization, which behaves somewhat like $L_{K(n)}$, i.e. like a completion at the height n.

Before going there, we introduce a variant of L_n.

Example 3.22. We let L_n^f denote the localization at the class of acyclics $\mathrm{Ind}(C_{\geq n+1})$. Note that, while the kernel of L_n-localization consists exactly of the E_n-acyclics, the kernel of L_n^f-localization consists exactly of the spectra that are filtered colimits of *compact* E_n-acyclics.

This justifies the name/notation: the f stands for "finite".

It is not hard to prove that L_n^f-localization is also smashing.

We now aim to introduce $T(n)$-localizations, the last chromatic localization we will mention. For this, we need the notion of a v_n-self map, a central object in chromatic homotopy theory. Historically, they were discovered as non-trivial "periodic" families of elements in the stable homotopy groups of spheres. From the perspective of this short introduction, they can be seen as ways to build type $(n+1)$-spectra from type n spectra, thus ensuring that each of the inclusions $C_{\geq n+1} \subset C_{\geq n}$ described above is proper.

Definition 3.23. A v_n-self map[9] on a finite p-local spectrum F is a a self-map, possibly of non-zero degree, $v : \Sigma^k F \to F$ such that $K(m)_* v$ is nilpotent for $m \neq n$, and an isomorphism for $m = n$.

Observation 3.24. If F is of type $> n$, then in particular $K(n)_* F = 0$ so the zero map is a v_n-self map.

If F is of type $< n$, then F does not admit a v_n-self map.

This observation says that v_n-self map are really only interesting for type n spectra.

Proposition 3.25. *Suppose F is a type n-spectrum, and v is a v_n-self map. The cofiber of v, F/v is a type $n+1$-spectrum.*

One can prove many properties of the v_n-self maps, such as their asymptotic uniqueness, or more generally their asymptotic compatibility with any map. The striking theorem here is the following:

Theorem 3.26 (The periodicity theorem [HS98])**.** *Any type n spectrum admits a v_n-self map.*

Further, for any type n spectrum F with a v_n-self-map v, one can define the telescope of (F, v):

Definition 3.27. If $v : \Sigma^k F \to F$ is a self-map, one may define its telescope as the sequential colimit of the diagram

$$F \to \Sigma^{-k} F \to \Sigma^{-2k} F \to \Sigma^{-3k} F \to \dots$$

The various basic properties of v_n-self maps guarantee that the telescope of (F, v) is independent of the choice of the chosen v_n-self map v, and in fact:

Lemma 3.28. *If F, F' are type n-spectra, then* $\mathrm{Tel}(F)$ *is in the thick subcategory generated by* $\mathrm{Tel}(F')$ *(and conversely, by symmetry).*

In particular, they have the same Bousfield class, i.e. for any spectrum X, $\mathrm{Tel}(F) \otimes X = 0 \iff \mathrm{Tel}(F') \otimes X = 0$.

Because of this, one can define:

[9] At the implicit prime p

Definition 3.29. A spectrum X is $T(n)$-local if and only if for some (and hence all) type n spectrum F, X is $\mathrm{Tel}(F)$-local.

The corresponding localization is called $T(n)$-localization, denoted $L_{T(n)}$.

Remark 3.30. Note that $T(n)$ itself is not a well-defined spectrum, only the corresponding Bousfield class is defined. Nonetheless, it is not uncommon to let $T(n)$ be $\mathrm{Tel}(F)$ for some type n spectrum F. The corresponding localization does not depend on this choice, but some arguments are simpler or possible through specific choices, e.g. if F is chosen to be a ring spectrum (which is always possible, as one can prove that $\mathrm{End}(F) \simeq F \otimes F^\vee$ is always type n, whenever F is).

Example 3.31. For a type n spectrum F, $L_{T(n)}F \simeq \mathrm{Tel}(F)$. For a type m spectrum X, where $m > n$, $L_{T(n)}X = 0$.

Like $K(n)$-localization, $T(n)$-localization is like a "completion at the height n" - as such, it is also not smashing. They share many formal properties, partly because:

Observation 3.32. Any $K(n)$-local spectrum is $T(n)$-local.

But in fact, the similarities go deeper and do not all follow from this fact. In fact, the *telescope conjecture* (now believed to be false by most experts) is equivalent to the statement that $T(n)$-local also implies $K(n)$-local.

Example 3.33. At height $n = 1$, the telescope conjecture holds, namely $K(1)$-local is equivalent to $T(1)$-local.

Example 3.34. The version of the telescope conjecture that states that the map $L_{T(n)}X \to L_{K(n)}X$ is an equivalence is known to hold in some cases, e.g. if X is an MU-module.

Example 3.35. For a commutative ring spectrum R, $L_{T(n)}R = 0$ if and only if $L_{K(n)}R = 0$. Note that this does not imply that $L_{T(n)}R \simeq L_{K(n)}R$.

Remark 3.36. Another formulation of the telescope conjecture is that L_n^f is equivalent to L_n, i.e. that any X with $E_n \otimes X = 0$ is a filtered colimit of compact spectra with the same property.

One of the key features that distinguish $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$ from Sp is their relationship with the Tate construction:

Theorem 3.37 (Telescopic Tate vanishing [Kuh04])**.** *Let G be a finite group. In $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$, the Tate construction with respect to G is always* 0*, i.e. there is a natural equivalence $M_{hG} \simeq M^{hG}$ for any $T(n)$-local (resp. $K(n)$-local) spectrum M with G-action. Note that here, the homotopy orbits are taken in $\mathrm{Sp}_{T(n)}$ (resp. $\mathrm{Sp}_{K(n)}$).*

Remark 3.38. In fact, this is true more generally for π-finite spaces, cf. [HL13], [CSY22].

We mention one final key tool about $T(n)$-localization, namely, the existence of the Bousfield-Kuhn functor. This is a functor, usually denoted $\Phi : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$ for $n > 0$, with the following remarkable property:

Theorem 3.39 (Bousfield-Kuhn)**.** *The composite $\Phi \circ \Omega^\infty : \mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ is equivalent to $T(n)$-localization.*

This is remarkable because this means that the $T(n)$-localization of a spectrum X only depends on its underlying *space*, with no information abouts its deloopings, or its highly structured commutative monoid structure. This rather striking fact is by now a key fact in chromatic homotopy theory, and can be used to rederive, among other things, Kuhn's telescopic Tate vanishing, cf. [CM17].

Remark 3.40*.* An immediate corollary is that the functor of $K(n)$-localization also factors through $\mathcal{S}_*$.

Exercise 3.41.

1. Let $C \subset \mathrm{Sp}^\omega$ be a stable subcategory, and let L denote the localization at the class of acyclics given by $\mathrm{Ind}(C)$. Prove that L is smashing. Formulate and prove a more general statement, replacing Sp with a presentably symmetric monoidal stable ∞-category which is compactly generated, and where compacts agree with dualizables.

2. Prove Observation 3.24, i.e. that type $< n$-spectra cannot admit v_n-self maps. If $v : \Sigma^k X \to X$ is such a v_n-self map, one can try to consider the cofiber X/v.

3. Prove Proposition 3.25, i.e. that if F is a type n spectrum, and v a v_n-self map, then F/v is a type $n+1$-spectrum.

4. Prove that if F is a type n finite spectrum, then so is $\mathrm{End}(F) \simeq F \otimes F^\vee$.

5. Prove that if F is a type n spectrum, $L_n^f F \simeq \mathrm{Tel}(F)$. Prove that L_n^f-acyclics are also $T(n)$-acyclic, and that $\mathrm{Tel}(F)$ is $T(n)$-local. Deduce the first part of Example 3.31, namely that $L_{T(n)} F \simeq L_n^f F \simeq \mathrm{Tel}(F)$.

6. Prove the second part of Example 3.31, that is, if X is a finite p-local spectrum of type $> n$, then $L_{T(n)} X = 0$.

7. Prove Observation 3.32, namely that a $K(n)$-local spectrum is $T(n)$-local.

8. For $n > 0$, prove that any bounded above spectrum is $T(n)$-acyclic. In particular, all Eilenberg-MacLane spectra are $T(n)$-acyclic. Think about how this relates to the Bousfield-Kuhn functor.

Part I.
Week 1

4. Borel equivariant theory

Definition 4.1. Let G be a group and $\mathcal{C}$ an ∞–category. The *associated Borel equivariant ∞–category* is defined to be $\mathrm{Fun}(BG, \mathcal{C}) = \mathcal{C}^{BG}$.

Remark 4.2. Concretely, an object in $\mathrm{Fun}(BG, \mathcal{C})$ consists of the following data:

- An object $X \in \mathcal{C}$,
- the datum of an equivalence $X \xrightarrow[\simeq]{g} X$ for every $g \in G$,
- the datum of an equivalence $h \circ g \simeq hg$ of maps $X \to X$ for every pair $(g, h) \in G \times G$,
- and higher and higher coherences...

Warning 4.3. When G is a finite group, the ∞–category BG is very much *not* finite! The proof of this will be deferred to the exercise. In particular, this means that $(-)^{hG}$ does not commute with all colimits and that $(-)_{hG}$ does not commute with all limits in general. This is a very important philosophical point underpinning why studying Borel equivariant situations is a hard and interesting problem in the homotopical setting.

Remark 4.4. We collect here the basic adjunctions enjoyed by these Borel equivariant ∞–categories. Let $p \colon BG \to *$ be the unique map. And for a subgroup $H \leq G$, write $BH \xrightarrow{i} BG$ for the induced map on classifying spaces (which is also a covering space with fibre G/H). Then for $\mathcal{C}$ an ∞–category which has the appropriate (co)limits, taking the appropriate Kan extensions, we get the two sets of adjunctions

$$\mathcal{C} \underset{(-)^{hG} := p_*}{\overset{(-)_{hG} := p_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{triv}_G := p^*}}}}} \mathrm{Fun}(BG, \mathcal{C}) \underset{\mathrm{Coind}_H^G := i_*}{\overset{\mathrm{Ind}_H^G := i_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{Res}_H^G := i^*}}}}} \mathrm{Fun}(BH, \mathcal{C}) \qquad (2)$$

The functor $(-)_{hG}$ is called the *G–homotopy orbits*, $(-)^{hG}$ the *G–homotopy fixed points*, Ind_H^G the *induction*, and Coind_H^G the *coinduction*. When $\mathcal{C}$ is semiadditive, the Kan extension formula for the map $BH \to BG$ shows that $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$. Furthermore, under the pointwise symmetric monoidal structures on the functor categories that appear in (2), the middle functors are symmetric monoidal (in particular $\mathrm{triv}_G \mathbb{1} \in \mathrm{Fun}(BG, \mathcal{C})$ is always the tensor unit), and so their respective right adjoints $(-)^{hG}$ and Coind_H^G are lax symmetric monoidal.

Proposition 4.5. *Let G be a group and $\mathcal{C}$ an ∞–category. For any $A, B \in \mathcal{C}^{BG}$, we have an equivalence*

$$\mathrm{Map}_{\mathcal{C}^{BG}}(A, B) \simeq \mathrm{Map}_{\mathcal{C}}(A, B)^{hG}$$

Remark 4.6. Combining the last few points, we see that when G is a finite group, $\mathrm{triv}_G X \in \mathrm{Fun}(BG, \mathcal{C})$ need no longer be compact even if $X \in \mathcal{C}$ were. For instance, the sphere spectrum with trivial G–action $\mathrm{triv}_G \mathbb{S} \in \mathrm{Fun}(BG, \mathrm{Sp})$ is no longer compact in the Borel equivariant ∞–category.

Lemma 4.7. *Let $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ considered as an object with trivial G–action. Then we have the equivalences*

$$\operatorname*{colim}_{BG} \mathcal{C} \simeq \mathrm{Fun}(BG, \mathrm{Ind}\mathcal{C})^\omega \qquad \lim_{BG} \mathcal{C} \simeq \mathrm{Fun}(BG, \mathcal{C})$$

Proof. The second equivalence is covered in Exercise 4.11 Item 2. For the first, recall that we have an equivalence $\mathrm{Ind}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_{\mathrm{st},L,\omega} : (-)^\omega$ from Theorem 1.28. So we could use this equivalence to compute in presentable ∞–categories and consider $\mathrm{Ind}\mathcal{C}$ with the trivial G–action. The value of this manoeuvre is that we have a further equivalence $\mathrm{Pr}_{\mathrm{st},L,\omega} \simeq \mathrm{Pr}^{\mathrm{op}}_{\mathrm{st},R,\mathrm{filt}}$ where the latter is the ∞–category of stable presentable ∞–categories where morphisms are right adjoint functors preserve ω–filtered colimits. Since this was a contravariant equivalence, we need to compute the limit of the diagram indexed over BG in $\mathrm{Pr}_{\mathrm{st},R,\mathrm{filt}}$: here we have used that BG was an ∞–groupoid and so passing to right adjoints is just taking the inverse. But now by the second equivalence, we see that the limit is $\mathrm{Fun}(BG, \mathrm{Ind}\mathcal{C})$. And so passing back to $\mathrm{Cat}_\infty^{\mathrm{perf}}$, we obtain $\mathrm{colim}_{BG}\, \mathcal{C} \simeq \mathrm{Fun}(BG, \mathrm{Ind}\mathcal{C})^\omega$ as wanted. □

We end this section with a few words about the various relationship between different kinds of compact objects in the setting of the so–called *parametrised homotopy theory*, ie. the theory of $\mathrm{Fun}(Z, \mathcal{C})$ where $\mathcal{C}$ is a presentable ∞–category and $Z \in \mathcal{S}$. The two cases that we will consider are $Z = BG$ where G is a finite group and $Z = X$ where X is a compact space. We include this discussion about compact spaces only as a cautionary tale to be careful about compactness in the parametrised setting.

The first case is left as an exercise to the reader:

Proposition 4.8. *Let G be a finite group and $\mathcal{C}$ a presentable ∞–category. The inclusion* $\mathrm{Fun}(BG, \mathcal{C})^\omega \subseteq \mathrm{Fun}(BG, \mathcal{C})$ *factors through* $\mathrm{Fun}(BG, \mathcal{C}^\omega)$.

Remark 4.9. For $R \in \mathrm{CAlg}(\mathrm{Sp})$, it is a standard fact that an object in Mod_R is compact if and only if it is dualisable, so that we have an equivalence $\mathrm{Mod}_R^\omega \simeq \mathrm{Mod}_R^{\mathrm{dbl}}$. Thus, even though we only have an inclusion $\mathrm{Fun}(BG, \mathrm{Mod}_R^\omega) \subseteq \mathrm{Fun}(BG, \mathrm{Mod}_R)^\omega$ which is in general not an equivalence, we do have an equivalence

$$\mathrm{Fun}(BG, \mathrm{Mod}_R^\omega) \simeq \mathrm{Fun}(BG, \mathrm{Mod}_R^{\mathrm{dbl}}) \simeq \mathrm{Fun}(BG, \mathrm{Mod}_R)^{\mathrm{dbl}}$$

by virtue of Proposition 1.39.

Now for the other case, we have a reverse inclusion, and as this is tangential to the masterclass, we will be brief with the proof.

Proposition 4.10. *Let $X \in \mathcal{S}$ be compact and $\mathcal{C}$ a presentable ∞–category. The inclusion* $\mathrm{Fun}(X, \mathcal{C}^{\omega}) \subseteq \mathrm{Fun}(X, \mathcal{C})$ *factors through* $\mathrm{Fun}(X, \mathcal{C})^{\omega}$.

Proof. Since $\pi_0 X$ is finite, without loss of generality we can suppose $\pi_0 X = \{x\}$. Since compact objects are retracts of finite colimits, using functoriality of limits, we may reduce to the case when $X \in \mathcal{S}$ is finite, ie. it is built as a colimit of a finite diagram. In this case, let $\zeta \in \mathrm{Fun}(X, \mathcal{C}^{\omega})$ and $\partial \colon J \to \mathrm{Fun}(X, \mathcal{C})$ be a filtered diagram. We need to show that the map of spaces

$$\operatorname*{colim}_J \mathrm{Map}_{\mathcal{C}^X}(\zeta, \partial_j) \longrightarrow \mathrm{Map}_{\mathcal{C}^X}(\zeta, \operatorname*{colim}_J \partial_j)$$

is an equivalence. Now by the formula for mapping spaces in functor categories Fact 1.9, this is equivalent to

$$\operatorname*{colim}_J \lim_X \mathrm{Map}_{\mathcal{C}}(\zeta(x), \partial_j(x)) \longrightarrow \lim_X \mathrm{Map}_{\mathcal{C}}(\zeta(x), \operatorname*{colim}_J \partial_j(x))$$

Now finite limits commute with filtered colimits in spaces, so the source is equivalent to $\lim_X \operatorname{colim}_J \mathrm{Map}_{\mathcal{C}}(\zeta(x), \partial_j(x)) \simeq \lim_X \mathrm{Map}_{\mathcal{C}}(\zeta(x), \operatorname{colim}_J \partial_j(x))$ where the equivalent was by our hypothesis that $\zeta(x) \in \mathcal{C}^{\omega}$. This completes the proof. □

Exercise 4.11.

1. Prove Warning 4.3 that BG does not have a finite cell structure when G is a finite group. **Hint (which we learnt from Thomas Goodwillie):** *consider the covering space over BG associated to a nontrivial subgroup C_p and the $\mathbb{F}_p$–cohomology of BC_p.*

2. Prove that $(\mathrm{triv}_G \mathcal{C})^{hG} \simeq \mathrm{Fun}(BG, \mathcal{C})$ and $(\mathrm{triv}_G X)^{hG} \simeq \mathrm{Map}(BG, X)$. **Hint:** *one way is to use the formula for limits in* Cat_∞ *from §1.*

3. Prove Proposition 4.5. **Hint:** *By virtue of Item 2, you can prove more generally that mapping spaces in limits of ∞-categories are computed as limits of mapping spaces - for this, given a diagram $\mathcal{C}_\bullet : I \to \mathrm{Cat}_\infty$ of ∞-categories, and two points $x, y : * \to \lim_I \mathcal{C}_i$, consider the I-indexed diagram*

$$* \to \mathcal{C}_i \times \mathcal{C}_i \leftarrow \mathrm{Fun}(\Delta^1, \mathcal{C}_i)$$

and use the fact that limits commute with pullbacks to express $\mathrm{Map}_{\lim_I \mathcal{C}_i}(x, y)$ in terms of mapping spaces in the $\mathcal{C}_i$'s.[10]

4. Prove Proposition 4.8, i.e. that there is a natural inclusion $\mathrm{Fun}(BG, \mathcal{C})^{\omega} \subseteq \mathrm{Fun}(BG, \mathcal{C}^{\omega})$. **Hint:** *letting $i \colon * \to BG$ be the unique map, compute the right adjoint i_*. You might want to use the fact that a left adjoint preserves compact objects if and only if the right adjoint preserves filtered colimits.*

[10]Why does this work for $\mathrm{Fun}(BG, \mathcal{C})$, but not for $\mathrm{Fun}(J, \mathcal{C})$ for ∞-categories that aren't ∞-groupoids ?

5. Tate constructions and stable module categories

Construction 5.1 (Norm maps in classical algebra)**.** Let G be a finite group acting on an abelian group M. Write

$$M_G := M/(m - gm\colon m \in M, g \in G) \qquad M^G := \{m \in M\colon gm = m \forall g \in G\} \subseteq M$$

The *norm map* is defined as

$$\mathrm{Nm}\colon M_G \longrightarrow M^G \quad :: \quad [m] \mapsto \sum_{g\in G} gm$$

Crucially, when $|G|$ acts invertibly on M, Nm is an isomorphism with inverse

$$\mathrm{Nm}^{-1}\colon M^G \longrightarrow M_G \quad :: \quad m \mapsto \frac{1}{|G|}m$$

Construction 5.2 (The Tate construction)**.** An analogous notion of a norm map also exists at the higher algebraic level, but since a proper construction of it is quite fiddly, we content ourselves with merely stating here that, when $\mathcal{C}$ is semiadditive, there is an analogous transformation of functors $\mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$

$$\mathrm{Nm}\colon (-)_{hG} \Longrightarrow (-)^{hG}$$

For more details, the reader is referred to [Lur17, §6.1.6]. In suitable cases, this "higher" norm map recovers the classical one above. For example, using the spectral sequences

$$H_s(G, \pi_t X) \Rightarrow \pi_{t+s} X_{hG} \qquad H^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{hG}$$

when X is connective and $|G|$ acts invertibly on $\pi_i X$ for $i \geq 1$, this higher norm map will become the classical one upon applying π_0.

When $\mathcal{C}$ is moreover stable, taking the cofibre cofib(Nm) of the norm map is an extremely useful construction: this is what is called the *Tate construction* $(-)^{tG}$. The following string of lemmas is one of the reasons why this is such a fruitful consideration. We will indicate the contents of some of their proofs in the exercises.

Lemma 5.3 ([NS18, §I.3], see also [GM95, Prop. 3.5])**.** *Suppose $\mathcal{C}$ is presentably symmetric monoidal stable. Then the functor $(-)^{tG}\colon \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$ canonically refines to a lax symmetric monoidal functor. In particular, for any $\mathbf{E}_\infty$–ring object R in $\mathrm{Fun}(BG, \mathcal{C})$, R^{tG} refines to an $\mathbf{E}_\infty$–ring object in $\mathcal{C}$; for any R–module M, M^{tG} then refines to an R^{tG}–module.*

Lemma 5.4 (Tate vanishing on induced objects)**.** *The Tate construction $(-)^{tG}$ vanishes on induced objects, ie. those of the form $\mathrm{Ind}_e^G X$.*

Lemma 5.5 (Tate vanishing away from the group order)**.** *Suppose $X \in \mathrm{Fun}(BG, \mathcal{C})$ is an object on which $|G|$ acts invertibly. Then $X^{tG} \simeq 0$.*

Lemma 5.6 (Torsionness of Tate)**.** *Let $M \in \mathrm{Fun}(BG, D(\mathbb{Z}))$. For any $n \in \mathbb{Z}$, $\pi_n M^{tG}$ is a $|G|$–torsion abelian group.*

We now introduce an important construction, the *stable module category*, as a categorification of the Tate construction. Let $\mathcal{C}$ a presentable stable ∞–category. Recall from Proposition 4.8 that we have the inclusion $\mathrm{Fun}(BG, \mathcal{C})^\omega \subseteq \mathrm{Fun}(BG, \mathcal{C}^\omega)$. We can use this to make the following

Definition 5.7. Let G be a finite group and $\mathcal{C}$ be a presentable stable ∞–category. We define the *small* and *large stable module ∞–categories*, respectively, to be the Verdier quotients (cf. Theorem 1.24)

$$\mathrm{stmod}_G(\mathcal{C}) := \mathrm{Fun}(BG, \mathcal{C}^\omega) / \mathrm{Fun}(BG, \mathcal{C})^\omega$$

$$\mathrm{StMod}_G(\mathcal{C}) := \mathrm{Ind}(\mathrm{stmod}_G(\mathcal{C})) \simeq \mathrm{Ind}\big(\mathrm{Fun}(BG, \mathcal{C}^\omega)\big) / \mathrm{Fun}(BG, \mathcal{C})$$

Notation 5.8. For a connective $\mathbf{E}_\infty$–ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})^{\geq 0}$, we write $\mathrm{stmod}_G(R) := \mathrm{stmod}_G(\mathrm{Mod}_R)$ for the G–stable module category for R. It is not hard to see that $\mathrm{Fun}(BG, \mathrm{Mod}_R)^\omega$ is a $\otimes$–ideal so that $\mathrm{StMod}_G(R)$ inherits a natural symmetric monoidal structure from $\mathrm{Fun}(BG, \mathrm{Mod}_R)$ with the tensor unit given by $\mathrm{triv}_G R$.

The following is the fundamental result justifying our assertion earlier that stable module categories categorify the Tate construction. The proof will be sketchy and we invite the reader to fill in the details in the exercise at the end.

Proposition 5.9. *For $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $X, Y \in \mathrm{stmod}_G(R)$, we have an equivalence*

$$\mathrm{map}_{\mathrm{stmod}_G(R)}(X, Y) \simeq \mathrm{map}_R(X, Y)^{tG}$$

In particular, the endomorphism spectrum of the tensor unit $\mathrm{triv}_G R$ is given by $(\mathrm{triv}_G R)^{tG}$.

Proof. Since $\mathrm{Fun}(BG, \mathrm{Mod}_R)^\omega \subseteq \mathrm{Fun}(BG, \mathrm{Mod}_R^\omega)$ is the thick subcategory generated by $R[G]$, we will use the justifiable notation $\langle R[G] \rangle$ to denote $\mathrm{Fun}(BG, \mathrm{Mod}_R)^\omega$ for short. By Proposition 4.5 and by the colimit description of mapping spectra in Verdier quotients, we get that $\mathrm{map}_{\mathrm{stmod}_G(R)}(X, Y)$ is computed as the cofibre of the map

$$\operatorname*{colim}_{A \in \langle R[G] \rangle_{/Y}} \mathrm{map}_R(X, A)^{hG} \to \mathrm{map}_R(X, Y)^{hG}$$

Now look at the square

$$\begin{array}{ccc} \mathrm{colim}_{A \in \langle R[G] \rangle_{/Y}} \mathrm{map}_R(X, A)_{hG} & \longrightarrow & \mathrm{map}_R(X, Y)_{hG} \\ \downarrow & & \downarrow \\ \mathrm{colim}_{A \in \langle R[G] \rangle_{/Y}} \mathrm{map}_R(X, A)^{hG} & \longrightarrow & \mathrm{map}_R(X, Y)^{hG} \end{array} \tag{3}$$

Since X was compact, and so dualisable, we get $\mathrm{map}_R(X, A) \in \langle R[G] \rangle$ and so the left vertical map is an equivalence. We claim that the top map is also an equivalence, and this reduces to showing that $\operatorname{colim}_{A\in\langle R[G]\rangle_{/Y}} A \to Y$ is an equivalence, or equivalently, that $\operatorname{colim}_{A\in\langle R[G]\rangle_{/Y}} Y_A \simeq 0$ where we write $Y_A := \operatorname{cofib}(A \to Y)$. Given this claim (which we will work out in the exercise), the cofibre of the bottom map can be computed as the cofibre of the right vertical, which is $\operatorname{map}_R(X,Y)^{tG}$ as required. □

In fact, interestingly for p–groups, we have the following theorem which we will neither prove nor use in the rest of this document.

Theorem 5.10 ([Mat15, Thm. 2.9]). *If G is a p–group, then there is a symmetric monoidal equivalence* $\operatorname{StMod}_G(R) \simeq \operatorname{Mod}_{R^{tG}}$.

Exercise 5.11.

1. Show that $(\operatorname{Coind}_e^G -)^{hG} \simeq \mathrm{id}$ and $(\operatorname{Ind}_e^G -)_{hG} \simeq \mathrm{id}$. This should indicate to some extent why Lemma 5.4 is true.

2. Prove Lemma 5.5 about Tate vanishing away from the group order in the special case when $\mathcal{C} = \mathrm{Sp}$. **Hint:** *show this first for $X = \operatorname{triv}_G(\mathbb{S}[\frac{1}{|G|}])$ using the π_*–long exact sequence associated to the defining sequence for $(-)^{tG}$.*

3. Prove Lemma 5.6 about Tate torsionness. **Hint:** *prove this first for $\pi_0(\mathbb{Z}^{tG})$ using the π_*–long exact sequence associated to the defining sequence for $(-)^{tG}$.*

4. Fill in the details for the proof of the mapping spectrum formula for stable module categories Proposition 5.9. More specifically,

   a) Show from the mapping spectrum formula for Verdier quotients that
   $$\operatorname{map}_{\operatorname{stmod}_G(R)}(X,Y) \simeq \operatorname{cofib}\Big(\operatorname*{colim}_{A\in\langle R[G]\rangle_{/Y}} \operatorname{map}_R(X,A)^{hG} \to \operatorname{map}_R(X,Y)^{hG}\Big)$$

   b) Show using dualisability of X that $\operatorname{map}_R(X,A) \in \langle R[G]\rangle$. Deduce from this that the left vertical map in (3) is an equivalence.

   c) Prove the claim that $\operatorname{colim}_{A\in\langle R[G]\rangle_{/Y}} Y_A \simeq 0$ by showing that $\pi_n \operatorname{colim}_{A\in\langle R[G]\rangle_{/Y}} Y_A = 0$ for all n. **Hint:** *show first that the indexing category $\langle R[G]\rangle_{/Y}$ is filtered. Now show that any element $\alpha \in \pi_n \operatorname{colim}_{A\in\langle R[G]\rangle_{/Y}} Y_A$ must be zero by showing that $\operatorname{cofib}(\alpha\colon \Sigma^n R \to Y_A)$ must also be of the form $\operatorname{cofib}(B \to Y)$ for some $B \in \langle R[G]\rangle$.*

6. Basics of formal group laws (by Lennart Meier)

One can tensor line bundles. This induces a multiplication map $m\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. Commutativity and associativity of the tensor product show that this makes $\mathbb{CP}^\infty$ into a homotopy commutative H-space.

Let E be a complex-oriented cohomology theory. One can take as the definition that an isomorphism $E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$ with $|x| = 2$ exists and is chosen. Thus, m induces a map

$$m^*\colon E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!] \to E^*[\![x_1, x_2]\!] \cong E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

This map is continuous for the usual topologies defined on rings of power series, i.e. for every k, l the preimage $(m^*)^{-1}(x_1^k, x_2^l)$ contains (x^n) for some n. This follows from the fact that the image of $m|_{\mathbb{CP}^{k-1}\times\mathbb{CP}^{l-1}}$ lies in some $\mathbb{CP}^{n-1}$ because $\mathbb{CP}^{k-1} \times \mathbb{CP}^{l-1}$ is compact.

As furthermore, m^* is an E^*-algebra morphism (as it is induced by a map of spaces), it follows that m^* is equivalent data to the power series $F = m^*(x)$. We record how the axioms for a homotopy commutative H-space translate into properties of F.

We know that the composition $\mathbb{CP}^\infty \xrightarrow{\mathrm{id}\times\mathrm{pt}} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{m} \mathbb{CP}^\infty$ is homotopic to the identity (right unitality). As the map $E^*[\![x]\!] \to E^*$ induced by $\mathrm{pt} \to \mathbb{CP}^\infty$ sets $x = 0$, we see that this translates into $F(x_1, 0) = x_1$. Likewise, left unitality translates into $F(0, x_2) = x_2$. These two conditions are equivalent to

$$F(x_1, x_2) = x_1 + x_2 + \text{ higher terms.} \tag{4}$$

The twist map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ just permutes x_1 and x_2. Thus, the homotopy commutativity of $\mathbb{CP}^\infty$ translates into

$$F(x_1, x_2) = F(x_2, x_1). \tag{5}$$

The homotopy associativity of m translates into

$$F(x_1, F(x_2, x_3)) = F(F(x_1, x_2), x_3). \tag{6}$$

Definition 6.1. Let R be a commutative ring. A power series $F \in R[\![x_1, x_2]\!]$ satisfying (4), (5) and (6) is called a formal group law over R.

Example 6.2. If E is a complex oriented ring spectrum, we obtain a formal group law over $E_* = E^{-*}$. For E being ordinary homology, this formal group law is $x + y$. For $E = KU$ being complex K-theory, we obtain $x + y \pm uxy$ for u the Bott element in $\pi_2 KU$ and the sign depending on the conventions.

For a commutative ring R, denote by CAlg the category of commutative rings.[11]

Exercise 6.3. Denote by $\widehat{\mathcal{A}}^1$ the functor CAlg $\to$ Set, sending each S to its set of nilpotent elements, and by $\mathbb{G}_m\colon$ CAlg $\to$ Set, sending each S to the units in S.

1. Show that $\mathbb{G}_m$ is representable. Moreover, show that it lifts to a functor to abelian groups.

[11]In homotopy theory, one often uses CAlg for commutative ring spectra instead and uses $\mathrm{CAlg}^\heartsuit$ for commutative rings. We drop the heart here for simplicity.

2. Denote by $\widehat{\mathbb{G}_m} \subset \mathbb{G}_m$ the subfunctor on those units x such that $1 - x$ is nilpotent. Show that this is isomorphic to $\widehat{\mathcal{A}}^1$.

3. Show that a lift of $\widehat{\mathcal{A}}^1$ to a functor to abelian groups is equivalent to a formal group law over $\mathbb{Z}$. Compute this in the case of $\widehat{\mathbb{G}_m}$. How does this compare to the formal group law for KU?

Exercise 6.4. 1. Let $\mathcal{C}$ be any category with all finite limits. Define what an abelian group object in $\mathcal{C}$ is. If A is such an abelian group object and G a finitely generated abelian group, define $\mathrm{Hom}(G, A)$.

2. Let G be a compact abelian Lie group and let $\widehat{G} \cong \mathrm{Hom}(G, U(1))$ its Pontryagin dual. Show $\mathrm{Spec}\, R(G) \cong \mathrm{Hom}(\widehat{G}, \mathbb{G}_m)$. (If you're not familiar with schemes, define $\mathrm{Spec}\, R(G)$ to be the functor represented by the representation ring $R(G)$ on CAlg.)

Exercise 6.5. Every elliptic curve C over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$ (i.e. a discrete subgroup isomorphic to $\mathcal{Z}^2$).

1. Visualize the n-torsion in C.

2. Consider the subspace X of $C \times C$ of points adding to zero. Contemplate that $X/\Sigma_2 \cong \mathbb{P}^1_{\mathbb{C}}$.

7. Classical Smith theory

In this section we sketch a proof of Smith's famous theorem using ideas from a so–called localisation theorem expanding on [MCC+96]. For the remainder of the section, we assume G is a p-group and X a finite dimensional G-CW-complex (considered as a genuine G-space). All cohomology groups considered are with mod p-coefficient

Theorem 7.1 (P. A. Smith)**.** *Let X be a genuine G-space, if X is a mod-p cohomology sphere of dimension n, then X^G is also a mod-p cohomology sphere of smaller dimension m. If p is odd, then $n - m$ is even and X^G is non-empty if n is even.*

Recall that p-groups have the special property of being solvable, ie. we have a normal series

$$e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_i \triangleleft \ldots \triangleleft G_n = G$$

where $\mid G_i \mid$ is p^i, so that $G_i/G_{i-1} \cong C_p$. The proof for this is not too difficult and uses the same orbit–counting method from Exercise 2.25 Item 1 (a).

Observe that we have $X^G = (X^{G_i})^{G/G_i}$, and hence we can prove the above theorem by induction on the order of the group. We are therefore reduced to the case where $G = C_p$. So from now on, we fix $G = C_p$.

Recall that the group cohomology of $G = C_p$ with mod p coefficients, as an algebra, is isomorphic to $\mathbb{F}_p[t] \otimes \Lambda(x), \mid t \mid = 2, \mid x \mid = 1$ when p is odd and $\mathbb{F}_p[t], \mid t \mid = 1$ when p is even.

Further recall that the Borel equivariant cohomology of a space with a G–action, (i.e, an object in $\mathrm{Fun}(BG, \mathcal{S})$) is defined as $H^*_G(X) := H^*(X_{hG}) = \pi_{-*} \operatorname{map}(X_{hG}, \mathbb{F}_p)$, which naturally admits the structure of a $H^*(BG)$-algebra via the fibration $X \to X_{hG} \to BG$

A genuine G-space can be considered as a space with G-action, and hence we can also define its Borel equivariant cohomology the same way.

Proposition 7.2 (Localisation Theorem). *For a finite dimensional G-CW complex X we have that the canonical map*

$$H^*_G(X)[t^{-1}] \to H^*_G(X^G)[t^{-1}]$$

is an isomorphism, where t is the degree 2 generator of $H^*(BC_p)$ when p is odd and the degree 1 generator when p is even.

Proof. Since our group G is C_p, we have the following cofibre sequence of objects in $\mathrm{Fun}(BG, \mathrm{Sp})$

$$\mathbb{S}[X^G] \longrightarrow \mathbb{S}[X] \longrightarrow \bigoplus_{a \in A} \mathrm{Ind}_e^G \mathbb{S}^{k_a}$$

where A is a set and $0 \leq k_a \leq n$ for all $a \in A$, by finite dimensionality. Applying $(-)_{hG}$ then yields the cofibre sequence

$$\mathbb{S}[X^G]_{hG} \longrightarrow \mathbb{S}[X]_{hG} \longrightarrow \bigoplus_{a \in A} \mathbb{S}^{k_a} \tag{7}$$

over $\mathbb{S}[BC_p]$ in Sp. Here, we have also used Exercise 4.11 Item 1.

Upon application of $\operatorname{map}(-, \mathbb{F}_p)[t^{-1}]$, we get the fibre sequence

$$\Big(\prod_{a \in A} \operatorname{map}(S^{k_a}, \mathbb{F}_p)\Big)[t^{-1}] \longrightarrow \operatorname{map}((X^G)_{hG}, \mathbb{F}_p)[t^{-1}] \longrightarrow \operatorname{map}(X_{hG}, \mathbb{F}_p)[t^{-1}] \tag{8}$$

Now, $\prod_{a \in A} \operatorname{map}(S^{k_a}, \mathbb{F}_p)$ is concentrated in finitely many degrees, and so inverting any element in nonzero degree will yield the zero object. Therefore the fibre term in (8) is zero, whence the equivalence $\operatorname{map}((X^G)_{hG}, \mathbb{F}_p)[t^{-1}] \simeq \operatorname{map}(X_{hG}, \mathbb{F}_p)[t^{-1}]$ as desired. ☐

In fact we have something more from the proof: applying H^* to the cofibre sequence (7), we have:

Corollary 7.3. *$H^i_G(X) \to H^i_G(X^G)$ is an isomorphism for $i > n = dim(X)$.*

Let's now give the proof of Theorem 7.1:

Proof. The proof will distinguish two cases: the case where X has fixed points, and when it does not. In the case where X has at least one fixed point, we will give a direct poof of all the claims, while in the case where X has no fixed points, we will prove that n must be odd. As in this situation, $X^G = \emptyset$ is the (-1)-sphere, it follows that $n - m$ is even too.

The key tool in the proof is the (cohomological) Serre spectral sequence associated to the fibration $X \to X_{hG} \to BG$:

$$E_2^{p,q} = H^p(BG, H^q(X)) \Longrightarrow H_G^{p+q}(X)$$

As our group $G = C_p$ and we are working with mod p coefficients, we note that the action of G on $H^q(X)$ is trivial : indeed, X is a mod p cohomology sphere, so $H^n(X) \cong \mathbb{F}_p$ and there is no nontrivial action of C_p on this group. This implies that the local coefficients here are trivial. Hence $E_2^{p,q} \cong H^p(BG) \otimes H^q(X)$.

We now assume X has fixed points. In this case, it also has homotopy fixed points, and so the fibration $X_{hG} \to BG$ admits a section.

This implies (together with the fact that X is a mod p cohomology sphere) that the spectral sequence collapses at the E^2-page: there is a map from this spectral sequence to the spectral sequence of the fibration $* \to BG \to BG$ which induces an isomorphism on the $q = 0$ line, which forbids any nonzero differential.

As a consequence of the collapse together with the hypothesis that X is a homology sphere, we have that the reduced cohomology $\tilde{H}_G^*(X)$ is a free $H^*(BG)$-module of rank 1 with generator in degree n.

Now by Corollary 7.3 we have an isomorphism in high degrees$\tilde{H}_G^*(X) \simeq \tilde{H}_G^*(X^G) \simeq \tilde{H}^*(X^G) \otimes H^*(BG)$. Hence by counting dimension we have that X^G is a mod p cohomology sphere of dimension $m \leq n$. This proves the very first part of the statement.

We now assume that p is odd. The $H^*(BG)$-module structure is what implies that $n = m \bmod 2$: indeed, the map $X^G \to X$ induces a map $H_G^*(X) \to H_G^*(X^G)$ of free $H^*(BG)$-modules of rank 1 which is an isomorphism in high enough degrees. Because of the exterior algebra generator in degree 1 of the group cohomology $H^*(BC_p)$, for any such map $M_* \to N_*$, the degree of the free generators in M_*, N_* must differ by an even amount.

Finally, we prove that if p is odd and n even,then X^G is non-empty. For this, note that if there is no fixed point, the spectral sequence converges to the cohomology of a finite dimensional space. We will see that this is impossible, by inspecting the multiplicative structure of our spectral sequence. Recall that $E_2^{p,q} \cong H^p(BG) \otimes H^q(X)$, and let η denote the generator of $H^n(X)$.

As the abutment is finite dimensional $d = d_{n+1} : H^*(BG) \otimes H^n(X) \to H^{*+n+1}(BG)$ must an isomorphism for $* >> 0$. For degree reasons, $d(\alpha) = 0$ for all $\alpha \in H^*(BG)$, so that $d(\alpha\eta) = (-1)^{|\alpha|}\alpha d(\eta)$, by the Leibniz rule. But $d(\eta) \in H^{n+1}(BG)$ is in odd degree (n is even), and hence it must be of the form $\lambda x t^k$ for some $\lambda \in \mathbb{F}_p$, where x, t are the generators of $H^*(BG)$. We thus see immediately that $d(xt^r\eta) = 0$ for arbitrarily large r, which is a contradiction with the fact that d is an isomorphism for sufficiently large $*$. □

8. Sheaf theory

In this section, we explain the basic theory of ∞–categorical sheaves on topological spaces. We will be mainly following [Lur17, §5.5.5], [Vol17] and unpublished notes of Oscar Bendix Harr.

Let X be a toplogical space, and $\mathcal{C}$ be an ∞-category which is bicomplete (that is it has all small limits and colimits). We can define the ∞-category of $\mathcal{C}$-valued sheaves on X, $\mathrm{Sh}_{\mathcal{C}}(X)$. For this, let $\mathcal{U}(X)$ be the poset of open subsets of X under inclusion.

Definition 8.1. $\mathrm{Sh}_{\mathcal{C}}(X)$ is the full subcategory of the presheaf ∞-category $\mathrm{Psh}_{\mathcal{C}}(X) = \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C})$ spanned by the presheaves $\mathcal{F}$ such that for all open covers $\bigsqcup U_i \to U$ the following descent/gluing condition holds: the map

$$\mathcal{F}(U) \to \lim_{V} \mathcal{F}(V)$$

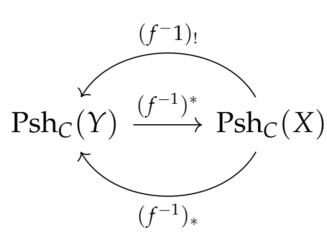
is an equivalence, where V runs over all open subsets which are contained in one of the U_i's.

Proposition 8.2. *If $\mathcal{C}$ is stable then $\mathrm{Sh}_{\mathcal{C}}(X)$ is stable*

For sheafification, the situation is a bit more subtle than in 1-categories. In the presentable case, there is no problem, but in general, it does not seem to be known whether sheafification exists.

Proposition 8.3. *If X is locally compact Hausdorff or if $\mathcal{C}$ is presentable, then the inclusion $\mathrm{Sh}_{\mathcal{C}}(X) \subset \mathrm{Psh}_{\mathcal{C}}(X)$ admits a left adjoint. Moreover, if filtered colimits in $\mathcal{C}$ are left exact, then the sheafification functor is also left exact.*

Given a continuous map $f : Y \to X$ we have an induced functor $f^{-1} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{U}(Y)^{\mathrm{op}}$ which, in turn, by precomposition induces a functor $(f^{-1})^* : \mathrm{Psh}_{\mathcal{C}}(Y) \to \mathrm{Psh}_{\mathcal{C}}(X)$. This functor has a left and a right adjoint given respectively by left and right Kan extensions



We define $f_* : \mathrm{Sh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(X)$ to be the restriction of $(f^{-1})^*$: this restriction will be checked to preserve sheaves in the exercise, so this makes sense. We will refer to this as the pushforward along f, which fits into an adjuction (as long as sheafifcation for X exists)

$$\mathrm{Sh}_{\mathcal{C}}(Y) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Sh}_{\mathcal{C}}(X) \tag{9}$$

where f^* is defined as

$$\mathrm{Sh}_{\mathcal{C}}(X) \subset \mathrm{Psh}_{\mathcal{C}}(X) \xrightarrow{(f^{-1})_!} \mathrm{Psh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(Y)$$

where the last arrow is the sheafification functor.

Definition 8.4 (Stalk)**.** Let $x \in X$ be a point and $i_x : * \to X$ be its inclusion. For $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(X)$, the stalk at x is defined as $\mathcal{F}_x := i_x^* \mathcal{F}$.

The following fact will be checked in the exercise at the end.

Fact 8.5 (Stalk formula). Prove that $\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$, where the colimit is defined over the directed system of open sets containing $x \in X$

Another subtlety of ∞-categories is that in general, stalks do *not* detect equivalences between sheaves - this is related to the question of "hypercompleteness" of ∞-topoi. In sufficiently finite-dimensional situations, though, this problem disappears, cf. for example:

Theorem 8.6 ([Lur09, Thms. 7.2.1.17 and 7.2.3.6])**.** *Let X be a manifold. Suppose $\mathcal{C}$ is compactly generated then $\varphi : \mathcal{F} \to \mathcal{G}$ is equivalence if and only if $\varphi_x = i_x^*(\varphi)$ is an equivalence for all x*

We now turn to the matter of cosheaves, which will lead to the phenomenon of Verdier duality and exceptional functors.

Definition 8.7 (co-Sheaves)**.** The category of $\mathcal{C}$-valued cosheaves is defined as $\mathrm{coSh}_{\mathcal{C}}(X) := \mathrm{Sh}_{\mathcal{C}^{\mathrm{op}}}(X)^{\mathrm{op}}$

Hence, given a map $f : Y \to X$ between locally compact Hausdorff spaces, dualizing the adjunction $f^* \dashv f_*$ yields an adjunction

$$\mathrm{coSh}_{\mathcal{C}}(Y) \underset{f^+}{\overset{f_+}{\rightleftarrows}} \mathrm{coSh}_{\mathcal{C}}(X) \tag{10}$$

From now on we assume all our sheaves are valued in stable ∞–categories.

Construction 8.8 (Compactly supported sections)**.** For every presheaf $\mathcal{F} \in \mathrm{Psh}_{\mathcal{C}}(X)$ we associate the co-presheaf of "compactly supported sections" $\mathcal{F}_c \in \mathrm{Psh}_{\mathcal{C}^{op}}(X)^{\mathrm{op}}$

$$\mathcal{F}_c \colon U \mapsto \underset{K \subset U}{\mathrm{colim}} \Big(\mathrm{fib} \left(\mathcal{F}(X) \to \mathcal{F}(X \setminus K) \right) \Big)$$

where K ranges over compact subsets of U (since X is Hausdorff, $X \setminus K$ is open, hence the above expression makes sense). See also [Lur17, Def. 5.5.5.9] for this construction. Importantly, this takes sheaves to cosheaves (cf. [Lur17, Cor. 5.5.5.12]).

Theorem 8.9 (Verdier Duality)**.** *Let X be a locally compact Hausdorff space. The compactly supported sections above gives a canonical equivalence D_X*

$$D_X : \mathrm{Sh}_{\mathcal{C}}(X) \xrightarrow{\simeq} \mathrm{coSh}_{\mathcal{C}}(X)$$

Proof. Refer to section 5.5.5 of [Lur17] $\square$

Construction 8.10 (Exceptional functors)**.** Verdier duality is a valuable notion partly because it yields certain exceptional functorialities. To wit, given a map $f : Y \to X$ of locally compact Hausdorff spaces, Verdier duality gives us two new, so–called "exceptional" functors defined as

$$\begin{array}{ccc} \mathrm{coSh}_{\mathcal{C}}(X) & \xrightarrow{f^+} & \mathrm{coSh}_{\mathcal{C}}(Y) \\ {\scriptstyle D_X}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle D_Y} \\ \mathrm{Sh}_{\mathcal{C}}(X) & \xrightarrow{f^!} & \mathrm{Sh}_{\mathcal{C}}(Y) \end{array} \qquad \begin{array}{ccc} \mathrm{coSh}_{\mathcal{C}}(X) & \xleftarrow{f_+} & \mathrm{coSh}_{\mathcal{C}}(Y) \\ {\scriptstyle D_X}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle D_Y} \\ \mathrm{Sh}_{\mathcal{C}}(X) & \xleftarrow{f_!} & \mathrm{Sh}_{\mathcal{C}}(Y) \end{array}$$

Thus, collecting the adjunctions (9) and (10), we see that f gives rise to the following adjunctions

$$\mathrm{Sh}_{\mathcal{C}}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}_{\mathcal{C}}(X) \qquad \mathrm{Sh}_{\mathcal{C}}(Y) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathrm{Sh}_{\mathcal{C}}(X)$$

We now work towards a relationship between the $*$ and the ! constructions. We learnt of the proof of the following from Oscar Harr.

Theorem 8.11. *Let $f : Y \to X$ be a map of locally compact Hausdorff spaces. There is a canonical natural transformation $\mathrm{Nm}_f : f_! \to f_*$ which is an equivalence if f is proper.*

Proof. By definition of $f_!$, constructing Nm_f is the same as constructing a natural transformation $f_+ D_Y \to D_X f_*$.

For $Z \subset Y$ closed we let $\mathcal{F}_Z$ denote the fiber of the map $\mathcal{F}(Y) \to \mathcal{F}(Y \setminus Z)$. Unpacking the definitions (which is a good exercise) yields for any open $V \subset X$ that

$$f_+ D_Y \mathcal{F}(V) = \underset{\substack{K \subseteq f^{-1}V \\ \text{cpt}}}{\mathrm{colim}}\, \mathcal{F}_K \qquad D_X f_* \mathcal{F}(V) = \underset{\substack{L \subseteq V \\ \text{cpt}}}{\mathrm{colim}}\, \mathcal{F}_{f^{-1}L}$$

where K ranges over compact subsets of $f^{-1}(V)$ and similarly for L in V. Now note that we have natural maps $\mathcal{F}_K \to \mathcal{F}_{f^{-1}(f(K))}$ given as maps of fibers

$$\begin{array}{ccccc} \mathcal{F}_K & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y \setminus K) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{f^{-1}f(K)} & \longrightarrow & \mathcal{F}(Y) & \longrightarrow & \mathcal{F}(Y \setminus f^{-1}f(K)) \end{array}$$

inducing a natural map $(f_+ D_Y \mathcal{F})(V) \to (D_X f_* \mathcal{F})(V)$ which is also natural in $\mathcal{F}$, and thus we have the desired natural transformation $f_+ D_Y \to D_X f_*$.

Moreover when f is proper, $f^{-1}(L)$ is compact, for any compact $L \subset X$ and the inclusion $K \subset f^{-1}(f(K))$ guarantees that the functor f^{-1} is cofinal, viewed as a functor from compact subsets of V to compact subsets of $f^{-1}(V)$. It follows that the above natural transformation is an equivalence. □

By very similar arguments we also have a comparison map between the upper $*$ and ! functors, which we briefly state as follows:

Proposition 8.12. *Let $j : V \hookrightarrow X$ be an open embedding. Then there is a canonical natural equivalence $j^* \simeq j^!$*

Now let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category, where the tensor product preserves colimits in each variable. The presheaf category $\mathrm{Psh}_{\mathcal{C}}(X)$ can be given the structure of a symmetric monoidal ∞-category with the pointwise tensor product. This in turn gives a symmetric monoidal structure on $\mathrm{Sh}_{\mathcal{C}}(X)$ where the tensor product of sheaves is given as the sheafification of their tensor product as presheaves.

Proposition 8.13. *Assume $\mathcal{C}^\otimes$ is presentably symmetric monoidal. The symmetric monoidal ∞-category $\mathrm{Sh}_{\mathcal{C}}(X)^\otimes$ is also presentably symmetric monoidal. In particular, it admits internal-hom objects denoted by $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$.*

Given a map $f\colon Y \to X$ between locally compact Hausdorff spaces, we get an enriched adjunction - informally, this means that for $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$ and $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$, we have natural equivalences of the form:

$$f_* \mathrm{Hom}_X(f^* \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_Y(\mathcal{F}, f_* \mathcal{G})$$

This is obtained by a straightforward unwinding of adjunctions, using also the that f^* is symmetric monoidal.

What is less trivial is the following, whose proof we omit in these notes. The interested reader may refer to [Vol17] for more details.

Proposition 8.14. *(Projection formula) Let $f\colon Y \to X$ be a map of locally compact Hausdorff spaces and let $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$ and $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$. We then have*

$$f_!(\mathcal{F} \otimes f^* \mathcal{G}) \simeq f_! \mathcal{F} \otimes \mathcal{G}$$

By a straightforward unravelling of adjunctions again, this gives us another form of enriched adjunction

$$f_* \mathrm{Hom}(\mathcal{F}, f^! \mathcal{G}) \simeq \mathrm{Hom}(f_! \mathcal{F}, \mathcal{G})$$

Exercise 8.15.

1. Show that $(f^{-1})^* : \mathrm{Psh}_{\mathcal{C}}(Y) \to \mathrm{Psh}_{\mathcal{C}}(X)$ takes sheaves to sheaves. **Hint:** *use that (f^{-1}) takes cover to a cover*

2. Show the formula for stalks Fact 8.5. **Hint:** *use the pointwise formula for left Kan extensions. Note that no sheafification is needed since $Y = *$.*

3. Show the claim that $f_+ D_Y \mathcal{F}(V) = \operatorname{colim}_{\substack{K \subseteq f^{-1}V \\ \text{cpt}}} \mathcal{F}_K$ and $D_X f_* \mathcal{F}(V) = \operatorname{colim}_{\substack{L \subseteq V \\ \text{cpt}}} \mathcal{F}_{f^{-1}L}$ in the proof of Theorem 8.11.

Part II.
Week 2

9. Mackey functors and Dress inductions

We now introduce Mackey functors. We will present it in a modern manner, but one of the canonical classical references for this is chapter 6 of [Die79]. We will first informally recall the classical notions of Mackey functors and Green functors, not because we will be using them, but rather to show to the reader that categorifying it will be seen to recover successfully all the defining features from the classical setting.

Recollections 9.1 (Classical Mackey and Green)**.** A *G–Mackey functor* $\underline{M}$ is the datum of an abelian group $M(H)$ for each $H \leq G$ such that for any subconjugation $K \leq H$ there are homomorphisms $\operatorname{Res}^H_K \colon M(H) \to M(K)$ and any subgroup $L \leq K$, a homomorphism $\operatorname{Ind}^K_L \colon M(L) \to M(K)$. Besides satisfying the usual transitivity relations $\operatorname{Ind}^G_H \operatorname{Ind}^H_K = \operatorname{Ind}^G_K$ and $\operatorname{Res}^H_K \operatorname{Res}^G_H = \operatorname{Res}^G_K$, these have to satisfy the following double–coset formula: for any subgroups $H, K \leq G$, there is an identification of maps $M(H) \to M(K)$

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H = \bigoplus_{g \in K \backslash G / H} \operatorname{Ind}^K_{K \cap {}^g H} \operatorname{Res}^H_{K \cap {}^g H}$$

Note here that since we have a pullback square of finite G–sets

$$\begin{array}{ccc} \coprod_{g \in K\backslash G/H} G/K \cap^g H & \longrightarrow & G/H \\ \big\downarrow & \lrcorner & \big\downarrow \\ G/K & \longrightarrow & G/G \end{array}$$

we are really saying that whenever we have such pullbacks, we get a commuting

$$\begin{array}{ccc} \bigoplus_{g \in K\backslash G/H} M(G/K \cap^g H) & \xleftarrow{\operatorname{Res}^H_{K \cap^g H}} & M(H) \\ {\scriptstyle \oplus \operatorname{Ind}^K} \big\downarrow & & \big\downarrow {\scriptstyle \operatorname{Ind}^G_H} \\ M(K) & \xleftarrow[\operatorname{Res}^G_K]{} & M(G) \end{array}$$

In the literature, such conditions are sometimes called *Beck–Chevalley conditions*.

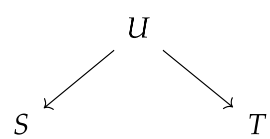
There is a natural symmetric monoidal structure on the 1–category of Mackey functors, under which the commutative algebra objects are called *G–Green functors*. Concretely speaking, these are Mackey functors $\underline{R}$ which levelwise have commutative ring structures, the restriction maps Res^G_H are ring maps, and which satisfy the so–called *Frobenius reciprocity*, ie. for $H \leq K$ and $x \in R(H), y \in R(K)$, we have

$$y \cdot \mathrm{Ind}^K_H x = \mathrm{Ind}^K_H(\mathrm{Res}^K_H y \cdot x) \in R(K)$$

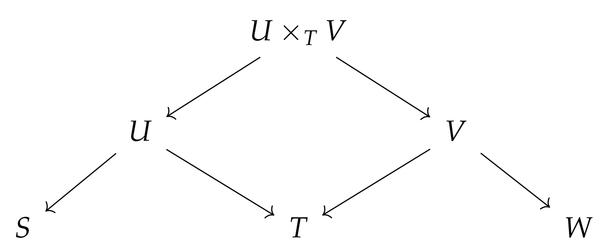
One of the most important examples of Green functors is the Burnside ring Mackey functor $\underline{A}$: $A(H)$ is given by the commutative ring generated by transitive G–orbits, addition given by disjoint unions, and multiplication given by products of G–sets. In fact, this is the unit object in the symmetric monoidal structure on Mackey functors alluded to above, and consequently all Mackey functors are modules over $\underline{A}$.

Having seen the classical picture, we now return to the functorial mode of living.

Construction 9.2 (Lindner)**.** We will not belabour the details here but we will write $\mathrm{Span}(G)$ for the following 1–category: the objects are finite G–sets and morphisms between finite G–sets S and T are isomorphism classes of spans



where U is another finite G–set and the two legs are G–equivariant maps. Compositions are given by taking by taking pullbacks



This 1–category is semiadditive where $\emptyset$ is the zero object and the biproduct is given by disjoint unions of finite G–sets; it admits a symmetric monoidal structure given by taking cartesian product of finite G–sets which we will write as $\mathrm{Span}(G)^{\otimes}$: we have opted for this notation since the cartesian product of sets is *not* the categorical product of this category and so this structure is *not* the cartesian symmetric monoidal structure. Moreover, for $K, H \leq G$ and a subconjugation ${}^g(-)\colon K \hookrightarrow H$, there is a product–preserving functor $i^H_K\colon \mathrm{Span}(K) \to \mathrm{Span}(H)$ sending orbits K/L to $H/{}^gL$.

The insight of Harald Lindner from 1976 was that Mackey functors can be encoded by this category. For ∞-categories, we have to get rid of the words "isomorphism classes of", and instead consider the groupoid of spans $((\mathrm{Fin}_G)_{/S,T})^{\simeq}$ as the mapping space between S and T, thus making $\mathrm{Span}(G)$ into a $(2,1)$-category. The details of this construction, in much greater generality, were worked out in [Bar17]. We can now define:

Definition 9.3. Let $\mathcal{C}$ be a semiadditive ∞–category. The ∞–category of *$\mathcal{C}$–valued G–Mackey functors* $\mathrm{Mack}_G(\mathcal{C})$ is defined as $\mathrm{Fun}^{\times}(\mathrm{Span}(G), \mathcal{C})$ where $\mathrm{Fun}^{\times}$ denotes product–preserving functors.

If $\mathcal{C}$ furthermore is given a symmetric monoidal structure which commutes with colimits in each variable, then $\mathrm{Mack}_G(\mathcal{C})$ can be equipped with the Day convolution symmetric monoidal structure. In this case, objects in $\mathrm{CAlg}(\mathrm{Mack}_G(\mathcal{C})^{\otimes})$ are called *$\mathcal{C}$–valued G–Green functors*.

Remark 9.4 (Mackey structures and properties). We can collect a wealth of structures and properties on $\mathrm{Mack}_G(\mathcal{C})$, categorifying the features of classical Mackey functors, which we record in the following list:

1. By virtue of the semiadditivity of $\mathcal{C}$, limits and colimits in $\mathrm{Mack}_G(\mathcal{C}) = \mathrm{Fun}^{\times}(\mathrm{Span}(G), \mathcal{C})$ are computed pointwise.

2. The maps i_H^G from Construction 9.2 gives rise to the following adjunctions

$$\mathrm{Mack}_G(\mathcal{C}) \underset{\mathrm{Coind}_H^G}{\overset{\mathrm{Ind}_H^G}{\leftleftarrows}} \xrightarrow{\mathrm{Res}_H^G} \mathrm{Mack}_H(\mathcal{C})$$

   where we have called $\mathrm{Res}_H^G := (i_H^G)^*$, $\mathrm{Ind}_H^G := (i_H^G)_!$, and $\mathrm{Coind}_H^G := (i_H^G)_*$. By looking at the appropriate Kan extension formulas, one sees that there is a canonical equivalence $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$.

3. When $\mathcal{C}$ has a symmetric monoidal structure commuting with colimits in each variable, the map Res_H^G refines to a symmetric monoidal functor, and so Coind_H^G refines to a lax symmetric monoidal functor.

4. For subgroups $H, K \leq G$, we have a double–coset decomposition of functors $\mathrm{Mack}_K(\mathcal{C}) \to \mathrm{Mack}_H(\mathcal{C})$

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G \simeq \bigoplus_{g \in H\backslash G/K} \mathrm{Ind}_{H \cap {}^g K}^H \circ \mathrm{Res}_{H \cap {}^g K}^{H^g \cap K} \circ \mathrm{Res}_{H^g \cap K}^K$$

   where $H^g := g^{-1}Hg \leq G$.

5. When $\mathcal{C}$ is presentable and is given the structure of a closed presentably symmetric monoidal structure, the Day convolution $\mathrm{Mack}_G(\mathcal{C})^{\otimes}$ will also be closed

symmetric monoidal. In this case, writing $\underline{\mathrm{Hom}}_G$ for the internal hom in $\mathrm{Mack}_G(\mathcal{C})$, it will be compatible with restrictions in that

$$\mathrm{Res}^G_H \underline{\mathrm{Hom}}_G(-,-) \simeq \underline{\mathrm{Hom}}_H(\mathrm{Res}^G_H -, \mathrm{Res}^G_H -)$$

It is then an easy adjunction consequence that we have the *Frobenius reciprocity* relations:

$$Y \otimes \mathrm{Ind}^G_H X \simeq \mathrm{Ind}^G_H(\mathrm{Res}^G_H Y \otimes X) \qquad Y \otimes \mathrm{Coind}^G_H X \simeq \mathrm{Coind}^G_H(\mathrm{Res}^G_H Y \otimes X)$$

This list is very much not exhaustive and we reserve a much more comprehensive and complementary enumeration in Remark 10.5 in the case when $\mathcal{C} = \mathrm{Sp}$.

We end this section by recording Dress' famous *induction theorem* [Dre75]. This was a breakthrough that neatly captured many famous results in representation theory, and as with many great ideas, in hindsight it is surprisingly easy.

Lemma 9.5 (Abstract splitting)**.** *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞–category and $R \in \mathrm{CAlg}(\mathcal{C}, \otimes, \mathbb{1})$. Let M be an R–module such that we have a commuting diagram*

$$\begin{array}{ccc} \mathbb{1} & & \\ \downarrow & \overset{1}{\searrow} & \\ M & \longrightarrow & R \end{array}$$

For any R–module N, we then have a retraction of R–module $\mathrm{id}_N \colon N \to M \otimes_R N \to N$.

Theorem 9.6 (Higher Dress induction)**.** *Let $\mathcal{F}$ be a family of subgroups in G and $(\mathcal{C}, \otimes, \mathbb{1})$ be a presentably symmetric monoidal semiadditive ∞–category. Let $\underline{R} \in \mathrm{CAlg}(\mathrm{Mack}_G(\mathcal{C})^{\otimes})$ and suppose that the unit $\underline{\mathbb{1}} \xrightarrow{1} \underline{R}$ factors through the $\underline{R}$–module map $\mathrm{Ind}^G_{\mathcal{F}} \colon \bigoplus_{H \in \mathcal{F}} \mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{R} \to \underline{R}$. For any $\underline{R}$–module $\underline{M}$, the canonical map*

$$\mathrm{Ind}^G_{\mathcal{F}} \colon \bigoplus_{H \in \mathcal{F}} \mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{M} \longrightarrow \underline{M}$$

then admits a splitting natural in $\underline{M}$.

Remark 9.7. The formulation above is nowhere near optimal since we have mainly used the adjective "presentably symmetric monoidal" for convenience. One can of course ask for much less and still the same manoeuvres hold. In particular, setting $\mathcal{C}$ to be the 1–category of abelian groups with its tensor product, the above recovers Dress' original statement for Mackey functors valued in abelian groups.

Exercise 9.8.

1. Let N be an abelian group. Convince yourself that you can always build a G–Mackey functor $\underline{M}$ which is zero everywhere except at G, where it evaluates to $M(G) = N$. Show however that it is not in general possible to get a G–Mackey functor which is zero everywhere except at e where it evaluates to $M(e) = N$. Can you find a condition on N and G where this can be done?

2. Prove the Frobenius reciprocity law from Remark 9.4 (5).

3. Prove the abstract splitting lemma Lemma 9.5. Deduce Dress' induction Theorem 9.6 from this.

10. Genuine equivariant spectra

Before talking about genuine G–spectra, we should say a few words first about *genuine G–spaces*. Recall first that the orbit category of G, $\mathcal{O}_G$, is the 1–category whose objects are transitive G–sets G/H and G–equivariant maps between these.

Definition 10.1. The ∞–category of *genuine G–spaces* $\mathcal{S}_G$ is defined as $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$.

Remark 10.2. In particular, the datum of a genuine G–space contains much more structure than just a space with a G–action. The latter kind is what is called Borel equivariant G–spaces whose ∞–category is given by $\mathrm{Fun}(BG, \mathcal{S})$. The reader might want to consult §4 for more on these kinds of categories. In fact, since there is a fully faithful inclusion $i\colon BG \hookrightarrow \mathcal{O}_G^{\mathrm{op}}$ given by $* \mapsto G/e$, we get the Bousfield (co)localisation

$$\mathcal{S}_G \xrightarrow{\mathrm{Bor}:=i^*} \mathrm{Fun}(BG, \mathcal{S}), \qquad EG_+\otimes - := i_! \colon \mathrm{Fun}(BG,\mathcal{S}) \to \mathcal{S}_G, \qquad F(EG_+, -) := i_* \colon \mathrm{Fun}(BG,\mathcal{S}) \to \mathcal{S}_G$$

Here we have included the classical notations $EG_+ \otimes -$ and $F(EG_+, -)$ for the benefit of the reader who might need to navigate older literature. These correspond to the cofibrant and fibrant replacements, respectively, when one takes (co)limits. In any case, conceptually this is also a reasonable notation since one can check for example that the idempotent endofunctor on $i_*i^*\colon \mathcal{S}_G \to \mathcal{S}_G$ is given by $X \mapsto F(EG_+, X)$ where $F(-,-)$ is the internal hom in $\mathcal{S}_G$. Additionally, for those coming from a more model categorical background, we should also note that both $\mathcal{S}_G$ and $\mathrm{Fun}(BG, \mathcal{S})$ come from the same topological category, ie. $\mathrm{Fun}(BG, \mathrm{Top})$. They differ, however, in the model structures used: in the latter, an equivariant map of G–topological spaces $f\colon X \to Y$ is a weak equivalence if f is a weak equivalence; in the former, such a map is a weak equivalence if $f^H\colon X^H \to Y^H$ is a weak equivalence for all $H \leq G$.

Construction 10.3 (Universal spaces for families)**.** A *family* $\mathcal{F}$ associated to the finite group G is a collection of subgroups of G that is closed under subconjugations. Then $E\mathcal{F} \in \mathcal{S}_G$ is the G–space defined using its fixed points as

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F} \end{cases}$$

Now, write S^0_G for the genuine G–space which has S^0 at every fixed point. We can then define an associated pointed G–space $\widetilde{E\mathcal{F}}$ by the cofibre sequence

$$E\mathcal{F}_+ \longrightarrow S^0_G \longrightarrow \widetilde{E\mathcal{F}}$$

On fixed points, this G–space contains the complementary datum:

$$\widetilde{E\mathcal{F}}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ S^0 & \text{if } H \notin \mathcal{F} \end{cases}$$

One should think of these as characteristic functions: $E\mathcal{F}$ holds only information of fixed points inside $\mathcal{F}$ whereas $\widetilde{E\mathcal{F}}$ only holds information for those outside of $\mathcal{F}$.

Definition 10.4. The ∞–category of *genuine G–spectra* Sp_G can be defined simply as $\mathrm{Mack}_G(\mathrm{Sp})$.

This is an extremely highly structured ∞–category and so unsurprisingly it participates in many useful adjunctions. Moreover, at least three distinct "fixed points" can be extracted out of a genuine G–spectrum, all of them useful in their own ways, and they are: genuine fixed points $(-)^G$, homotopy fixed points $(-)^{hG}$, and geometric fixed points $(-)^{\Phi G} = \Phi^G(-)$.

We enumerate all these structures here without providing any proofs. As far as we know, all the following statements can be obtained using the Mackey functor description and the keen reader may want to try to prove these for themselves or refer to [Bar17; Nar17; Wil17] for proofs.

Remark 10.5 (Structures and properties of genuine G–spectra)*.* We will collect all the adjunctions in one diagram at the end of this list.

1. There is a functor $(-)^G\colon \mathrm{Sp}_G \to \mathrm{Sp}$ called the *genuine G–fixed points*. Viewing genuine G–spectra as spectral G–Mackey functors, this functor is concretely given by evaluating on the orbit G/G. This functor commutes with all limits and colimits. We can also define $(-)^H\colon \mathrm{Sp}_G \to \mathrm{Sp}$ as the composition $(-)^H \circ \mathrm{Res}^G_H$ (cf. point 4 for the restriction functor). From the Mackey functor point of view, it is clear that $\{(-)^H\}_{H\leq G}$ is jointly conservative since a Mackey functor is zero if and only if the datum at G/H is zero for all $H \leq G$.

2. There is a canonical presentably symmetric monoidal structure on Sp_G. Viewed as Mackey functors, this symmetric monoidal structure is constructed as the Day convolution (cf. [BGS20]). Since $\mathrm{Sp}^{\otimes}$ is the tensor unit in $\mathrm{CAlg}(\mathrm{Pr}^{\otimes}_{L,\mathrm{st}})$, we obtain a unique symmetric monoidal colimit–preserving functor

$$\mathrm{infl}_G\colon \mathrm{Sp} \longrightarrow \mathrm{Sp}_G$$

It turns out that the right adjoint of this is given precisely by the genuine fixed points $(-)^G$. Consequently, $(-)^G$ canonically refines to a lax symmetric monoidal functor.

3. There is an adjunction $\mathbb{S}_G[-] = \Sigma^\infty_{G,+} \colon \mathcal{S}_G \rightleftharpoons \mathrm{Sp}_G : \Omega^\infty_G$ generalising the usual adjunction in the nonequivariant setting. The description of Σ^∞_G is *not* as easy as one might expect in that it does not commute with genuine fixed points, although there is a formula. This is called the *tom–Dieck splitting*, and it says that for $X \in \mathcal{S}_G$, we have an equivalence

$$(\mathbb{S}_G[X])^G \simeq \bigoplus_{(H)\leq G} \mathbb{S}[(X^H)_{hW_GH}]$$

where $W_GH \coloneqq N_GH/H$ is the Weyl group of H in G.

4. There is a unique symmetric monoidal colimit–preserving functor $\Phi^G \colon \mathrm{Sp}_G \to \mathrm{Sp}$ satisfying $\Phi^G\mathbb{S}_G[-] \simeq \mathbb{S}[(-)^G] \colon \mathcal{S}_G \to \mathrm{Sp}$. This is called the *geometric G–fixed points*, and it has a fully faithful right adjoint which we denote by Ξ^G (following [Gla17]). From the Mackey functors point of view, for $X \in \mathrm{Sp}$, the genuine G–spectrum $\Xi^G X$ is given by

$$(\Xi^G X)^H \simeq \begin{cases} X & \text{if } H = G \\ 0 & \text{if } H \lneq G \end{cases}$$

For a genuine G–spectrum X, we will also write $\Phi^H X \in \mathrm{Sp}$ for $\Phi^H \operatorname{Res}^G_H X$.

5. There is a symmetric monoidal colimit–preserving functor $\operatorname{Res}^G_H \colon \mathrm{Sp}_G \to \mathrm{Sp}_H$ called *restriction*. On Mackey functors, this is given by only remembering the data of genuine fixed points on subgroups of H. This functor has both a left and a right adjoint. These are called *induction* Ind^G_H and *coinduction* Coind^G_H respectively. Since Res^G_H was symmetric monoidal, by general nonsense, Coind^G_H is lax symmetric monoidal.

   Moreover, there is a canonical natural transformation $\operatorname{Ind}^G_H \Rightarrow \operatorname{Coind}^G_H$ which is an equivalence. Classically, this is termed the *Wirthmüller isomorphism*. These functors also satisfy

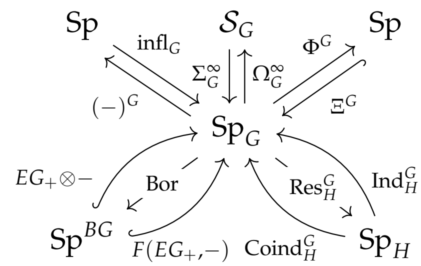
$$\operatorname{Ind}^G_H \operatorname{Res}^G_H X \simeq X \otimes \mathbb{S}_G[G/H] \simeq F(\mathbb{S}_G[G/H], X) \simeq \operatorname{Coind}^G_H \operatorname{Res}^G_H X$$

   Finally, they also interact nicely with the symmetric monoidal structures via the so–called *Frobenius reciprocities* (cf. Remark 9.4 (5)):

$$X \otimes \operatorname{Ind}^G_H Y \simeq \operatorname{Ind}^G_H(\operatorname{Res}^G_H X \otimes Y) \quad \text{and} \quad X \otimes \operatorname{Coind}^G_H Y \simeq \operatorname{Coind}^G_H(\operatorname{Res}^G_H X \otimes Y)$$

6. There is a functor $\mathrm{Bor} \colon \mathrm{Sp}_G \to \mathrm{Sp}^{BG}$ which is sometimes called *Borellification*. This forgets the structures of all the other fixed points and only remembers the underlying spectrum with its G–action. This has both a fully faithful left and right adjoint written as $EG_+ \otimes -$ and $F(EG_+, -)$. The justification for this notation is the same as the one we gave in Remark 10.2. In particular, for instance, for $X \in \mathrm{Sp}_G$, we have $EG_+ \otimes \mathrm{Bor}(X) \simeq EG_+ \otimes X = \mathbb{S}_G[EG] \otimes X$.

As promised at the beginning of this remark, we summarise all the adjunctions in this pretty G–flower.



Before ending this section, we record the following important observation that the geometric fixed points are an alternative to the genuine fixed points when checking that something is zero.

Proposition 10.6 (Geometric fixed point detection)**.** *Suppose* $\Phi^H M \simeq 0 \in \mathrm{Sp}$ *for all* $H \leq G$. *Then* $M \simeq 0$.

Proof. We prove by induction on G. By induction we have that $\mathrm{Res}^G_H M \simeq 0$ for all $H \lneq G$. Hence by Remark 10.5 (4), we se that $M \simeq \Xi^G \widetilde{M}$ for some $\widetilde{M} \in \mathrm{Sp}$. But then $0 \simeq \Phi^G M \simeq \Phi^G \Xi^G \widetilde{M} \simeq \widetilde{M} \in \mathrm{Sp}$, and so indeed $M \simeq 0$ as desired. □

Exercise 10.7.

1. By only using the adjunctions above, show that

$$(\mathrm{Coind}^G_H(-))^G \simeq (-)^H \qquad \Phi^G \mathrm{Ind}^G_H \simeq 0 \qquad \Phi^G(EG_+) \simeq 0$$

2. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is such that $R^G \simeq 0$, then $R \simeq 0$. **Hint:** use one of the identities in the preceding problem.

3. Show that $\{G/H\}_{H \leq G}$ form a (finite) set of compact generators for Sp_G.

11. Abstract nilpotence technology

This section will summarise the key abstract points of the influential papers [Mat16; MNN17] which in turn were built on top of the insights of Daniel Quillen, Ethan Devinatz, Mike Hopkins, Jeff Smith, and others in their groundbreaking work on nilpotence phenomena in equivariant and chromatic stable homotopy theory. Throughout this section, the following assumptions will be in force.

Hypotheses 11.1 ([MNN17, Hypotheses 2.26])**.** $(\mathcal{C}, \otimes, \mathbb{1})$ is a presentably symmetric monoidal stable ∞–category and $A \in \mathrm{Alg}(\mathcal{C}^{\otimes})$ which satisfy the following:

1. The unit $\mathbb{1}$ is compact,

2. The object A is dualisable in $\mathcal{C}$,

3. $\mathcal{C}$ is generated as a localising subcategory by dualisable objects.

Definition 11.2. We say that an object $M \in \mathcal{C}$ is *A–acyclic* if $A \otimes M \simeq 0$.

Here are the three important abstract notions:

Definition 11.3. Suppose we are in the setting of Hypotheses 11.1. We define three stable subcategories as follows:

1. Let $\mathcal{C}_{\Gamma_A} \subseteq \mathcal{C}$ be the smallest localising subcategory of $\mathcal{C}$ containing $A \otimes X$ for arbitrary dualisable $X \in \mathcal{C}$. This is called the full subcategory of *A–torsion objects*. By construction, $\mathcal{C}_{\Gamma_A}$ is presentable and the inclusion $\mathcal{C}_{\Gamma_A} \subseteq \mathcal{C}$ preserves colimits, and so by the adjoint functor theorem, this admits a right adjoint which we denote by $\Gamma_A \colon \mathcal{C} \to \mathcal{C}_{\Gamma_A}$ called *A–acyclisation*;

2. Let $\mathcal{C}_{\Lambda_A} \subseteq \mathcal{C}$ be the full subcategory of *A–complete objects*, ie. those objects $Y \in \mathcal{C}$ such that for any A–acyclic $M \in \mathcal{C}$, we have $\mathrm{map}_{\mathcal{C}}(M, Y) \simeq 0$. This is clearly a full subcategory closed under limits, and is thus in particular a thick subcategory. Moreover, this can be viewed as a localisation of $\mathcal{C}$ at a set of morphisms and so the inclusion $\mathcal{C}_{\Lambda_A} \subseteq \mathcal{C}$ has a left adjoint which we denote $\Lambda_A \colon \mathcal{C} \to \mathcal{C}_{\Lambda_A}$ called *A–completion*;

3. Let $\mathcal{C}_{\mathrm{Nil}_A} \subseteq \mathcal{C}$ be the smallest thick $\otimes$–ideal in $\mathcal{C}$ containing A. This is called the full subcategory of *A–nilpotent objects*.

Observation 11.4. Importantly, if $M \in \mathrm{Mod}_A$, then M is A–torsion, A–complete, and A–nilpotent. And so all the subcategories introduced above can be thought of as different generalisations of the underlying objects of A–modules. This is a simple observation whose proof we defer to the exercises at the end.

Using this observation, we can prove the following.

Lemma 11.5. *Under Hypotheses 11.1, $A^\vee \in \mathcal{C}_{\mathrm{Nil}_A}$ where $A^\vee$ is the dual of A.*

Construction 11.6 (Adams towers and the A–acyclisation formula, [MNN17, Cons. 3.4, Prop. 3.5])**.** Let $I := \mathrm{fib}(\mathbb{1} \xrightarrow{1} A)$. From this we may consider the infinite filtration

$$\cdots \to I^{\otimes 3} \to I^{\otimes 2} \to I \to \mathbb{1}$$

called the *Adams tower of A*, where note that $\mathrm{cofib}(I^{\otimes k} \to I^{\otimes(k-1)}) \simeq I^{\otimes(k-1)} \otimes A$. Note also that since A was dualisable by Hypotheses 11.1, each of the terms in the Adams tower is also dualisable. We now define U_A to be the colimit of the tower's dual, ie.

$$U_A := \mathrm{colim}\left[\mathbb{1} \to I^\vee \to (I^\vee)^{\otimes 2} \to (I^\vee)^{\otimes 3} \to \cdots\right]$$

and define also

$$V_A := \mathrm{fib}(\mathbb{1} \to U_A)$$

Now by Lemma 11.5 we know that $A^\vee$ is A–nilpotent, and on the other hand, the fibres $V_k := \mathrm{fib}(\mathbb{1} \to (I^\vee)^{\otimes k})$ are built in finite steps from terms like $A^\vee \otimes (I^\vee)^{\otimes i}$. Hence in total we see that V_k is A–nilpotent for all k. A fact that we will not prove in these notes is that we have the formula for A–acyclisations given by

$$\Gamma_A(X) \simeq V_A \otimes X \tag{11}$$

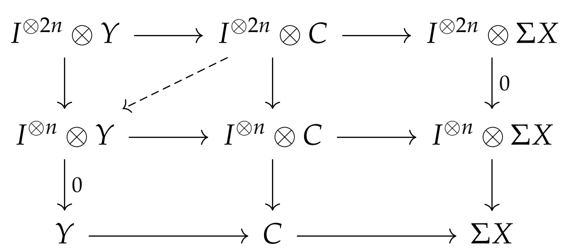
and the adjunction counit is given by applying $- \otimes X$ to $V_A \to \mathbb{1}$. In particular, since $\mathcal{C}_{\Gamma_A} \subseteq \mathcal{C}$ is a Bousfield colocal subcategory, X is A–torsion if and only if $V_A \otimes X \to X$ is an equivalence.

Another important reason to have introduced the Adams towers is that it provides a valuable alternative description of nilpotence which perhaps hews closer to and explains the word "nilpotence". The proof is extremely instructive as it exhibits many of the important standard manoeuvres in studying nilpotent phenomena.

Proposition 11.7 ([MNN17, Prop. 4.7])**.** *Let $A \in \mathrm{Alg}(\mathcal{C})$. An object $M \in \mathcal{C}$ is A–nilpotent if and only if for all $n \gg 0$, the maps $I^{\otimes n} \otimes M \to M$ in the Adams tower are nullhomotopic.*

Proof. Suppose that M is A–nilpotent. Define $\mathcal{J} \subseteq \mathcal{C}$ to be the full subcategory of $\mathcal{C}$ on those objects X such that $I^{\otimes n} \otimes X \to X$ is nullhomotopic for $n \gg 0$. We claim that $\mathcal{J}$ is a thick $\otimes$–ideal which contains A. Given this, by minimality of $\mathcal{C}_{\mathrm{Nil}_A}$ as a full subcategory having these properties, we see that $M \in \mathcal{C}_{\mathrm{Nil}_A} \subseteq \mathcal{J}$ as was to be shown.

Now to prove the claim: that it is a $\otimes$–ideal and is closed under retracts is clear, and so we are left to show that this is closed under taking cofibres. Suppose $X \to Y$ is a map in $\mathcal{J}$ and write $C := \mathrm{cofib}(f)$. Let $n \gg 0$ be such that $I^{\otimes n} \otimes X \to X$ and $I^{\otimes n} \otimes Y \to Y$ are nullhomotopic. Consider now the solid diagram of horizontal cofibre sequences



We claim that the middle vertical composition is nullhomotopic. This is simply because the top right square ensures that we have the dashed map factoring the middle top vertical map. Hence, the middle vertical composite map factors through the null map $I^{\otimes n} \otimes Y \to Y$, whence its nullity as required.

Now for the converse, suppose for all $n \gg 0$, the maps $I^{\otimes n} \otimes M \to M$ are nullhomotopic. Recall first that the cofibres of $I^{\otimes k} \otimes M \to I^{\otimes (k-1)} \otimes M$ are A–modules, and so A–nilpotent. Since $\mathcal{C}_{\mathrm{Nil}_A}$ is thick, we see by induction that the cofibre

$Q := \mathrm{cofib}(I^{\otimes n} \otimes M \to M)$ is A–nilpotent also. But on the other hand, $I^{\otimes n} \otimes M \to M$ is nullhomotopic and so we get that $Q \simeq I^{\otimes n} \otimes M \oplus \Omega M$. Therefore, since $\mathcal{C}_{\mathrm{Nil}_A}$ is closed under retracts, ΩM, and so also M, is in $\mathcal{C}_{\mathrm{Nil}_A}$ as desired. □

The following relates the notion of nilpotence with torsionness and completeness. In fact, these notions are even more interrelated than we will be able to cover here, and for more details we refer the reader to [MNN17; BHV18]. For example, the map $\mathcal{C}_{\Gamma_A} \hookrightarrow \mathcal{C} \xrightarrow{\Lambda_A} \mathcal{C}_{\Lambda_A}$ is always an equivalence: this says that the A–torsions and A–completes are two different ways that an abstract ∞–category sits inside $\mathcal{C}$.

Proposition 11.8. *The inclusion $\mathcal{C}_{\mathrm{Nil}_A} \subseteq \mathcal{C}$ factors through $\mathcal{C}_{\Gamma_A} \cap \mathcal{C}_{\Lambda_A}$. That is, A–nilpotent objects are both A–torsion and A–complete.*

Having explained the general setup, we now discuss how nilpotence can be a very useful notion in attacking descent problems, and for that we will need the following

Definition 11.9 ([Mat16, Def. 3.18])**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be as in Hypotheses 11.1 and let $A \in \mathrm{CAlg}(\mathcal{C})$. We say that A is *descendable* if $\mathcal{C}_{\mathrm{Nil}_A} = \mathcal{C}$, ie. $\mathbb{1} \in \mathcal{C}_{\mathrm{Nil}_A}$.

The following is a prototypical and instructive example of a thick $\otimes$-ideal argument whose proof we defer to the exercise. It should give the reader a flavour of how one uses the definition of nilpotence above.

Proposition 11.10 (Thick ideal membership descent via descendability, variation on Proposition 11.12)**.** *Suppose $A \in \mathrm{CAlg}(\mathcal{C})$ is descendable and let $\mathcal{I} \subseteq \mathcal{C}$ be a thick tensor ideal. For any $M \in \mathcal{C}$, $M \in \mathcal{I}$ if and only if $M \otimes A \in \mathcal{I}$.*

One advantage of this definition of descendability is that it is easy to prove, by a straightforward unravelling of definitions, statements of the following form. The slogan here is that "nilpotence has very nice categorical permanence properties".

Proposition 11.11 (Abstract nilpotence functoriality, [MNN17, Cor. 4.13])**.** *Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ be presentably symmetric monoidal stable ∞–categories and $f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a lax symmetric monoidal functor. For $A \in \mathrm{Alg}(\mathcal{C})$, we have $f(\mathcal{C}_{\mathrm{Nil}_A}) \subseteq \mathcal{D}_{\mathrm{Nil}_{f(A)}}$. In particular, descendability of algebra objects are preserved under strong symmetric monoidal functors.*

Descendability will not play any role elsewhere in these notes, but we felt that we cannot exposit on the theory of nilpotence without mentioning it since it plays such a central unifying role in the general theory. We now return to the main stream.

Proposition 11.12 (Thick ideal membership descent via nilpotence, [MNN17, Prop. 4.16])**.** *Suppose $M \in \mathcal{C}$ is A–nilpotent and let $\mathcal{I} \subseteq \mathcal{C}$ be a thick tensor ideal. In this case, $M \in \mathcal{I}$ if and only if $M \otimes A \in \mathcal{I}$.*

Remark 11.13. In the last few results, we have shown how useful the notion of nilpotence is, and how easy it is to transfer this property around under categorical operations once we have it. On the other hand, while torsionness and completeness are very interesting properties, they do not enjoy nearly as much categorical closure properties and so are in general quite hard to deal work with. In this perspective, the punchline of Proposition 11.8 is then that if we can show that something is nilpotent, then we have the luxury of dragging along its torsionness and completeness under various categorical operations which might have otherwise destroyed these two properties.

We end this section with a powerful abstract criterion in showing A–nilpotence in the presence of unital ring structures. This will be important in the next section when we specialise the nilpotence technology to the equvariant setting.

Theorem 11.14 ([MNN17, Thm. 4.19])**.** *Under Hypotheses 11.1, and for $R \in \mathcal{C}$ be a homotopy unital algebra, we have that R is A-torsion if and only if it is A-nilpotent.*

Proof. Being A-nilpotent always implies A-torsion by Proposition 11.8. The interesting direction is deferred to an exercise at the end. □

Remark 11.15. We rehash [MNN17, Rmk. 4.20] here, namely, that the theorem above is a souped up version of the following simple observation: if a classical discrete ring R is p–power torsion, then there exists a uniform n such that $p^n \cdot R = 0$. In this way, one slogan for nilpotence is that it generalises the niceness coming from "uniformly bounded p–torsion". From this point of view, we should interpret the theorem as saying that, morally, checking p–power torsionness is easier than showing p^n–power torsion for a specific n, and in the case of unital rings, satisfying the easier condition implies that we already have p^n–power torsion for some n.

Exercise 11.16.

1. Show Observation 11.4. Hence, deduce Lemma 11.5.
2. Prove Proposition 11.8. **Hint:** *you will need Hypotheses 11.1 (3) to prove the torsion statement, and for the completeness statement, you may apply Proposition 11.7.*
3. Prove Proposition 11.10. **Hint:** *for the interesting direction, show that the full subcategory on objects X such that $X \otimes M \in \mathcal{I}$ is a thick $\otimes$–ideal.*
4. Prove Theorem 11.14. **Hint:** *use Construction 11.6 and the compactness of the unit $\mathbb{1}$. It will be helpful to show along the way that in a stable ∞–category $\mathcal{D}$, if a map $f\colon X \to Y$ is nullhomotopic, then X is a retract of* $\mathrm{fib}(f)$*.*

12. Nilpotence in the equivariant setting

We will now explore nilpotence phenomena in the setting $(\mathcal{C}, \otimes, \mathbb{1}) = (\mathrm{Sp}_G, \otimes, \mathbb{S}_G)$ of genuine G–spectra. Let $\mathcal{F}$ be a family of subgroups for the finite group G. Recall also from Remark 10.5 (5) that $G/H_+ = \mathrm{Coind}_H^G \mathrm{Res}_H^G \mathbb{S}_G = F(G/H_+, \mathbb{S}_G)$ is naturally an object in $\mathrm{CAlg}(\mathrm{Sp}_G)$.

Definition 12.1 ($\mathcal{F}$-nilpotence, [MNN17, Def. 6.36])**.** Let $A_{\mathcal{F}} := \prod_{H\in\mathcal{F}} F(G/H_+, \mathbb{S}_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$. We define the ful subcategory of *$\mathcal{F}$–nilpotent objects* $(\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}} \subseteq \mathrm{Sp}_G$ to be the full subcategory of $A_{\mathcal{F}}$–nilpotent objects. This is equivalent to the smallest thick $\otimes$-ideal in Sp_G containing $\{G/H_+\}_{\mathcal{F}}$.

Morally, substantiated by Proposition 11.7, we can think of $\mathcal{F}$–nilpotent objects as those G–spectra that are built in finitely many steps from objects of the form $G/H_+ \otimes X = \mathrm{Ind}_H^G \mathrm{Res}_H^G X$ where $H \in \mathcal{F}$. That is, they are determined, in a finitary way, only by data from subgroups inside the family $\mathcal{F}$.

We now state a couple of facts without further comments and we refer the reader to [MNN17, Props. 6.5 & 6.6] for details.

Proposition 12.2 (Formulas for equivariant localisations)**.** *Let $\mathcal{F}$ be a family. We have the following formulas for $\mathcal{F}$–acyclisation and $\mathcal{F}$–completion:*

$$\Gamma_{A_{\mathcal{F}}} X \simeq E\mathcal{F}_+ \otimes X \quad \textit{and} \quad \Lambda_{A_{\mathcal{F}}}(X) \simeq F(E\mathcal{F}_+, X)$$

The following is then an immediate corollary of the facts above and Proposition 11.8.

Corollary 12.3 ([MNN19, Prop. 2.12])**.** *Let $M \in \mathrm{Sp}_G$ be $\mathcal{F}$-nilpotent. Then it is $\mathcal{F}$-torsion and $\mathcal{F}$-complete, that is, the maps*

$$E\mathcal{F}_+ \otimes M \longrightarrow M \qquad M \longrightarrow F(E\mathcal{F}_+, M)$$

are equivalences.

The following is also not hard, and we will prove it in an exercise below.

Proposition 12.4 (Calculus of $\mathcal{F}$-nilpotence, [MNN17, Prop. 6.38])**.** *Let $\mathcal{F} \subseteq \mathcal{F}'$ be families of subgroups of G. Let $H \leq G$ and let $\mathcal{F}_H$ denote the family of subgroups of H which also belong to $\mathcal{F}$ (so $\mathcal{F}_H$ can be viewed both as a family for H and for G).*

1. *We have an inclusion $(\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}} \subseteq (\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}'}}$;*
2. *If $X \in (\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}}$ then $\mathrm{Res}_H^G X \in (\mathrm{Sp}_H)_{\mathrm{Nil}_{\mathcal{F}_H}}$;*
3. *If $Y \in (\mathrm{Sp}_H)_{\mathrm{Nil}_{\mathcal{F}_H}}$ then $\mathrm{Ind}_H^G Y \in (\mathrm{Sp}_H)_{\mathrm{Nil}_{\overline{\mathcal{F}}}}$ for any $\overline{\mathcal{F}}$ containing $\mathcal{F}_H$. In particular, $\mathrm{Ind}_H^G Y \in (\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}}$.*

The following pair of results are now the culmination of all the work we did in these notes on the nilpotence technology in equivariant stable homotopy theory.

Lemma 12.5 (Proper $\mathcal{F}$-nilpotence reduction, [MNN17, Prop. 6.40])**.** *$X \in \mathrm{Nil}_{\mathcal{F}}$ if and only if for all $H \notin \mathcal{F}$ the restriction $\mathrm{Res}_H^G X \in \mathrm{Sp}_H$ is $\mathcal{P}_H$-nilpotent.*

Proof. The only if direction is by Proposition 12.4 (1, 2), noting that if $H \notin \mathcal{F}$ then $\mathcal{F}_H \subseteq \mathcal{P}_H$. For the other direction, suppose without loss of generality that $G \notin \mathcal{F}$, otherwise there is nothing to do. By hypothesis we get that $X \in \mathrm{Sp}_G$ is $\mathcal{P}_G$-nilpotent. Thus by Proposition 11.12, it suffices to show that $X \otimes A_{\mathcal{P}_G} \in \mathrm{Nil}_{\mathcal{F}}$. By induction on $|G|$ we may assume that for all $H \lneq G$, $\mathrm{Res}^G_H X$ is $\mathcal{F}_H$-nilpotent. Hence by Proposition 12.4 (3) we get that

$$X \otimes A_{\mathcal{P}_G} \simeq \bigoplus_{H \lneq G} \mathrm{Ind}^G_H \mathrm{Res}^G_H X \in (\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}}$$

as required. □

We may use this reduction step to deduce the following important theorem. In the presence of unital ring structures, this is one of the first things to try when showing that something is nilpotent, and this is also the criterion used in [CMN+20] in proving some of the main results.

Theorem 12.6 (Geometric fixed criterion for ring nilpotence, [MNN17, Thm. 6.41])**.** *Let $R \in \mathrm{Sp}_G$ be a homotopy commutative unital ring spectrum. Then $R \in (\mathrm{Sp}_G)_{\mathrm{Nil}_{\mathcal{F}}}$ if and only if for all $H \notin \mathcal{F}$, $\Phi^H R \simeq 0 \in \mathrm{Sp}$.*

Exercise 12.7.

1. Show the equivalence in Definition 12.1.

2. Prove Proposition 12.4. **Hint:** *this is a straightforward consequence of the functoriality of nilpotence and the double–coset formula Remark 9.4 (4).*

3. Prove Theorem 12.6. **Hint:** *you might want to recall Theorem 11.14.*

13. K–theory and localising invariants

Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cat}^{\mathrm{perf}}_\infty$ and suppose $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a null–composing sequence, so that we have the datum of a commuting square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{D} \\ \downarrow & \equiv & \downarrow p \\ 0 & \longrightarrow & \mathcal{E} \end{array}$$

Since $0 \in \mathrm{Cat}^{\mathrm{perf}}_\infty$ is the zero object, the space of such commuting squares is contractible if it is non–empty. Hence, we will from now on denote such a datum merely as the sequence

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

Definition 13.1. Such a null–composing sequence is said to be a *Verdier sequence* or an *exact sequence* if it is both a fibre and a cofibre sequence. If p has both a left and a right adjoint, then we say that it is a *split* Verdier/exact sequence.

Remark 13.2. The reader is referred to [CDH+20, App. A] for a comprehensive treatment on Verdier sequences, but here is brief summary of some of the important points about sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ which compose to zero:

1. If it is a fibre sequence, then the map $\mathcal{C} \to \mathcal{D}$ is fully faithful;

2. It is a cofibre sequence if and only if the map $\mathcal{D} \to \mathcal{E}$ is a Verdier quotient. In particular, it is essentially surjective;

3. Suppose it is a Verdier sequence. Then i has a left (resp. right) adjoint if and only if p has a left (resp. right) adjoint. In these cases, the left (resp. right) adjoint of p are necessarily fully faithful, so that p is then a Bousfield colocalisation (resp. localisation). As such, the splitness condition in the definition of split Verdier sequences actually can be completed to a diagram of adjoints

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "i"] & \mathcal{D} \arrow[r, two heads, "p"] \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] & \mathcal{E} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

Such a situation is also known as a *stable recollement.*

Definition 13.3. Let $\mathcal{C}$ be a presentable stable ∞–category. A functor $E\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{C}$ is said to be an *additive invariant* (resp. a *localising invariant*) if it sends split Verdier sequences (resp. Verdier sequences) to fibre sequences in $\mathcal{C}$.

Example 13.4 (Algebraic K–theory). Segal–Waldhausen's algebraic K–theory constructed via the $S_\bullet$– or Q–construction is an example of a localising invariant. In fact, it satisfies the universal property of being the initial additive invariant under the groupoid core functor $(-)^{\simeq}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$. That is, it is the initial additive invariant receiving a transformation $(-)^{\simeq} \Rightarrow \mathrm{K}(-)$ and it can moreover be delooped to take values in connective spectra. Additionally, and very importantly from its construction via the $S_\bullet$– or Q–construction, K–theory is also in fact a localising invariant[12]. Moreover, by virtue of the transformation $(-)^{\simeq} \Rightarrow \mathrm{K}$, it makes sense for us to speak of the class $[X] \in \mathrm{K}(\mathcal{C})$ of an object $X \in \mathcal{C}$.

Observation 13.5 (Waldhausen's splitting). Write $s\colon \Delta^0 \to \Delta^1$ and $t\colon \Delta^0 \to \Delta^1$ for the inclusion of the source and the target, respectively. One of Waldhausen's many key

[12]See [BGT13] for the original treatment of all these in the ∞–categorical setting. Beware, however, that there they also imposed the condition of preserving filtered colimits in the definition of additive/localising invariants. This is however a technical point that can be avoided, and is dealt with in the upcoming work [CDH+].

original insights is the following: for $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, we have the split Verdier sequence

$$\mathcal{C} \xrightarrow{s_*} \mathcal{C}^{\Delta^1} \xrightarrow{t^*} \mathcal{C} \qquad (s^*,\ \mathrm{fib}\colon \mathcal{C}^{\Delta^1} \to \mathcal{C};\quad t_!,\ c\colon \mathcal{C} \to \mathcal{C}^{\Delta^1})$$

where c takes X to $X \xrightarrow{=} X$. This means that for any additive $F\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$, we have the split fibre sequence of spectra

$$F(\mathcal{C}) \xrightarrow{F(s_*)} F(\mathcal{C}^{\Delta^1}) \xrightarrow{F(t^*)} F(\mathcal{C}) \qquad (F(\mathrm{fib})\colon F(\mathcal{C}^{\Delta^1}) \to F(\mathcal{C}),\quad F(c)\colon F(\mathcal{C}) \to F(\mathcal{C}^{\Delta^1}))$$

which yields the equivalence of spectra

$$F(\mathrm{fib}) \times F(t^*)\colon F(\mathcal{C}^{\Delta^1}) \xleftrightarrow{\simeq} F(\mathcal{C}) \times F(\mathcal{C}) : F(s_*) + F(c)$$

This is of foundational importance as we will see in the next two basic results which may be seen as "trickling down" the additivity property through various levels of decategorifications.

Proposition 13.6. *Let $F\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ be an additive invariant. If we have a cofibre sequence $\alpha \Rightarrow \beta \Rightarrow \gamma$ of maps $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, then we have an equivalence of morphisms $F\beta \simeq F\alpha \oplus F\gamma\colon F\mathcal{C} \to F\mathcal{D}$.*

Proof. The key for these kinds of statements is that both natural transformations

$$(\beta \Rightarrow \gamma), (\alpha \oplus \gamma \Rightarrow \gamma)\colon \mathcal{C} \longrightarrow \mathcal{D}^{\Delta^1}$$

have the same fibres, ie. γ. Hence, applying F and postcomposing further with the equivalence $F(\mathrm{fib}) \times F(t^*)\colon F(\mathcal{D}^{\Delta^1}) \xrightarrow{\simeq} F(\mathcal{D}) \times F(\mathcal{D})$ from Observation 13.5 yields that the two morphisms

$$F(\beta \Rightarrow \gamma),\ F(\alpha \oplus \gamma \Rightarrow \gamma)\colon F(\mathcal{C}) \longrightarrow F(\mathcal{D}^{\Delta^1})$$

are equivalent. Finally, postcomposing now these two equivalent morphisms with $F(s^*)\colon F(\mathcal{D}^{\Delta^1}) \to F(\mathcal{D})$ shows that we have an equivalence of morphisms

$$F(\beta) \simeq F(\alpha \oplus \gamma) \simeq F(\alpha) \oplus F(\gamma)\colon F(\mathcal{C}) \longrightarrow F(\mathcal{D})$$

as was to be shown. □

Using the same kind of tricks, we may also prove the following, which we have left as an instructive exercise to the reader.

Proposition 13.7. *If $X \to Y \xrightarrow{f} Z$ is a fibre sequence in $\mathcal{C}$, then $[Y] = [X] + [Z] \in \mathrm{K}(\mathcal{C})$. In particular, $-[X] = [\Omega X] = [\Sigma X] \in \mathrm{K}(\mathcal{C})$.*

Exercise 13.8.

1. Prove Proposition 13.7. **Hint:** *the key trick is very similar to the proof of Proposition 13.6. We highly recommend working this out as a didactic exercise.*

2. This question is about a toy model of an important technique from [CMN+20] in showing vanishing geometric fixed points. It might be beneficial first to review Lemma 4.7 and §5. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Recall that we can define a genuine C_p–spectrum $K_{C_p}(R)$ by considering the image of the ∞–category Perf_R with the trivial G–action in the following composition

$$\mathrm{Fun}(BC_p, \mathrm{Cat}_\infty^{\mathrm{perf}}) \hookrightarrow \mathrm{Mack}_{C_p}(\mathrm{Cat}_\infty^{\mathrm{perf}}) \xrightarrow{\mathrm{K}} \mathrm{Mack}_{C_p}(\mathrm{Sp}) = \mathrm{Sp}_{C_p}$$

Let $\mathrm{ass}_p \colon \mathrm{K}(R)_{hC_p} \simeq \mathrm{colim}_{BC_p} \mathrm{K}(R) \to \mathrm{K}(\mathrm{colim}_{BC_p} \mathrm{Perf}_R) \simeq \mathrm{K}(R[C_p])$ be the colimit–interchange map (which is sometimes known as the assembly map, hence the notation). Show that if this assembly map is an equivalence and if $\mathrm{K}(R^{tC_p}) \simeq 0$, then $\Phi^{C_p}\mathrm{K}_{C_p}(R) \simeq 0$.

14. Picard spectra

Definition 14.1. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a small symmetric monoidal stable ∞–category. An object $X \in \mathcal{C}$ is said to be *invertible* if there exists a $Y \in \mathcal{C}$ such that $X \otimes Y \simeq \mathbb{1}$. Define $\mathcal{P}ic(\mathcal{C})$ to be the subspace of $\mathcal{C}^{\simeq}$ on those objects which are invertible.

Remark 14.2. We will not argue for this here, but $\mathrm{Pic}(\mathcal{C})$ is naturally a group–like $\mathbb{E}_\infty$–space using $\otimes$ as the addition, $\mathbb{1}$ as the zero, and the inverses of elements given by the inverse of an invertible object. Consequently, via Segal's equivalence $\mathrm{CGrp}(\mathcal{S}) \simeq \mathrm{Sp}_{\geq 0}$, it can be viewed as an object in $\mathrm{Sp}_{\geq 0}$. We write $\mathrm{pic}(\mathcal{C})$ for the corresponding connective spectrum.

Fact 14.3. The inverse Y of an invertible object X is unique and exhibits the dual $X^\vee$ of X.

The following basic fact will be proved in the exercise.

Proposition 14.4. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. The homotopy groups of $\mathrm{pic}(R) := \mathrm{pic}(\mathrm{Perf}_R) \in \mathrm{Sp}_{\geq 0}$ are computed as follows:*

$$\pi_n \mathrm{pic}(R) \cong \begin{cases} \pi_{n-1}R & n \geq 2 \\ (\pi_0 R)^\times & n = 1 \end{cases}$$

Here is another fact which is good to know.

Proposition 14.5. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal stable ∞–category whose unit is compact and where $\otimes$ commutes with colimits in each variable. In this case, any dualisable object is compact.*

Proof. Let $X \in \mathcal{C}$ be dualisable and let $\partial\colon I \to \mathcal{C}$ be a filtered diagram in $\mathcal{C}$. To see that X is compact, consider the following:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(X, \operatorname*{colim}_I \partial_i) &\simeq \mathrm{Map}_{\mathcal{C}}(\mathbb{1}, DX \otimes \operatorname*{colim}_I \partial_i)\\
&\simeq \mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \operatorname*{colim}_I DX \otimes \partial_i)\\
&\simeq \operatorname*{colim}_I \mathrm{Map}_{\mathcal{C}}(\mathbb{1}, DX \otimes \partial_i)\\
&\simeq \operatorname*{colim}_I \mathrm{Map}_{\mathcal{C}}(X, \partial_i)
\end{aligned}$$

where the first and last equivalences are by dualisability, the second and third equivalences are by our hypotheses on $(\mathcal{C}, \otimes, \mathbb{1})$. This finishes the proof. □

Exercise 14.6.

1. Show Proposition 14.4. **Hint:** *recall that* $\mathrm{map}_R(R, R) \simeq R$ *in* Mod_R.
2. Compute $\mathrm{Pic}(\mathrm{Sp})$ as $\mathbb{Z}$. **Hint:** *for* $X, Y \in \mathcal{P}ic(\mathrm{Sp})$ *such that* $X \otimes Y \simeq \mathbb{S}$, *argue first that they must be finite spectra and can without loss of generality be assumed to be connective with nonzero* π_0. *Next, consider the* $\mathbb{F}_p$–, $\mathbb{Q}$–, *and* $\mathbb{Z}$–*homologies of* $X \otimes Y$ *to argue that* $H_*(X; \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}$ *in the zeroth degree. Deduce finally that* $X \simeq \mathbb{S}$.

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