Condensed Sets:

Idea: Cond Sets $\approx$ Top spaces, but categorically, $\text{Cond Sets} \cong \text{Set}$. 

Easier to add on algebraic structures.

Basic building blocks: $\text{Chaus}$ (compact Hausdorff).

We'll define a notion of covering on Chaus, giving a site, $\mathcal{S}$ then Cond sets $\approx$ Sh($\mathcal{S}$)
Def: a covery collection \( \{ X_i \rightarrow Y \} \), \( i \in I \), is a finite collection of Chaus \( X_i \) mapping to \( Y \) with \( \bigsqcup X_i \rightarrow Y \).

Strong enough so that you can work locally & that lets you something.

Not too strong so that invariants you care about still see sheaves.

Ex: \( \Omega \) top space, \( f(x) = \text{Cont}(x, \Omega), x \in \Omega \), is a sheaf of sets.
Ex: In a suitable famism, cohomology of (Haus spacing) also satisfies descent for this topology.

Strong enough?

Gleason: The category (Haus "has enough projective":

\[ T \text{ "projective" } \quad \text{with} \quad \exists \text{ proj } T \]

"enough" \[ \exists \text{ proj } T \text{ for all } X \]
Proof: \( S : \text{set} = \beta S \) is a projective object of \( \mathcal{C} \).

\[ \beta S = \lim_{S \to S_i} S_i \]

\( S_i \) finite

\[ \beta S \to Y \]

Enough: \( \beta S \to X \)

\( S \to X \)

\( \beta S \to X \)

\( S \to X \)
\( \Rightarrow \) a sheaf on \( \mathbb{C} \)-Haus is uniquely determined by its restriction to \( \beta_5 \)'s.

\( \beta_5 \to \beta_5 \times \beta_5 \leftarrow \text{closed subset of } \beta_5 \times \beta_5. \)

\( \bigtriangledown \) product of \( \beta_5 \)'s is not a \( \beta_5. \)

\( = \)

\( f(x) \in \operatorname{eq} \left( f(\beta_5) \equiv f(\beta_5') \right). \)
Moreover, on $\beta S'$, the topology is such:

stereotopic condition \( \iff \mathcal{F}(\beta S' + \beta S') \)

\( \subseteq \mathcal{F}(\beta S') \times \mathcal{F}(\beta S') \)

\( \mathcal{F}(\beta) = \emptyset \)

The: Projective Claws (\( \cong \)) retracts of $\beta S'$.

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"extraneously disconnected claw spaces"
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"totally disconnected (\( \cong \)) "paracomp""
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Claus \subset \text{profinite sets} \subset \beta \text{S's}

Any one of these can be taken as the define site for \text{Cond(Sets)}.

Want to define: \text{Cond(Sets)} = \text{Sh(\text{profinite sets})}

But we're not allowed, because \text{profinite sets} is not a small category.

[Hom Sets not sets]
We need to consider a truncated version of C(k, n), where we bound the cardinality.

**Def:** A cardinal \( k \) is a **strong limit cardinal** if

\[
\exists \lambda < K \implies 2^\lambda < K
\]

It's easy to make examples:

\[
2^{\lambda+1} = 2^{\lambda+1} = 2^\lambda
\]

\[
2^\lambda = \bigcup_{\alpha < \lambda} 2^\alpha
\]

This is a strong limit cardinal.
For all practical purposes, can take $k = J_0$.

Def: Category of $k$-condensed sets =
$\mathcal{Sh}(k\text{-small prof. set}_k)$

Baez-Dolan: $k$ strongly inaccessible

Conclude: take $\bigcup_{k} \mathcal{Sh}(k\text{-small } \mathcal{B}_k)$
$k \times k'$, there is a pullback map

$k$-small $\mathcal{B}S'$'s $\rightarrow k'$-small $\mathcal{B}S'$'s
given by the fully faithful inclusion

$\Rightarrow k$-Cond $\mathcal{B}S$'s $\rightarrow k'$-Cond $\mathcal{B}S$'s
also fully faithful.

$X \rightarrow ( \prod_{T \text{ k-small}} \ker(T') )$
What do we need to check?

RK: There is an alternative def'n of $\text{Cod}(\text{Sets})$.

$$\text{Cod}(\text{Sets}) \subseteq \text{PSh}(\beta S')$$

is the full subcategory defined by the sifted colimits of $\beta S'$.

Ideal of filtered colims

& reflective equalizers

$\text{Cod}(\text{Sets})$ satisfies Giraud's axioms except for the small generator condition.
So:

1) Finite lim's & colims: as in sets (Topos)

2) Arb limits & sifted colims:
   as in sets, (Alg Thy)

(come from compact proj generators)

\( T : \text{extr. disconnected} \Rightarrow \text{Hom}(T, -) : \text{card}(\text{sets}) \rightarrow \text{Set} \)
Pass to Condensed Ab gps.

"Ab gp object in CondSets a tree, but with sheaves of ab gps.

\[
\Rightarrow \text{Cond}(Ab) \text{ is an abelian category such that arbitrary lim's & colim's interact exactly as in ab gps.}
\]

\[
\Rightarrow \text{exact, if exact}
\]
Cond($\text{Ab}$) is an example of an ab. cat. with enough compact projectives.

$M$ cpt proj $\iff$ Hom($M$, -) : $A \to \text{Ab}$ comm w/ all lims & colims

"enough": $\forall A, A, \exists$ cpt proj $P_i \in A, \bigoplus_{i \in I} P_i \to M$

$P = \mathbb{Z} [R^5]$
Rk: A6 cat will enough spot proj
join by a single spot proj

\[ = \text{Mod}_R \, r \, m. \]
How to control limits in $\text{Gedsets}$?

Suppose we have a diagram. Put it in some common $K'$-Cond $\text{Set}$. Let

$$X^{K'}(T) = \lim_{T \to \mathbb{T}_K} X(T_{K'})$$

If we can guarantee that the colim is $1\text{diagram}$-filtered, then this factor commutes with the limit over the diagram.

\text{Lemma:} Let $\text{cof}(K)$ be the cofinality of $K$:

$$2 < \text{cof}(K) \Rightarrow \text{as } d-$$
indexed colim of $\kappa$-small sets is still $\kappa$-small.

Claim 1: $\text{cof}(K)$ can be made arbitrary (e.g.

Claim 2: Category of

$$T \to T'$$

$T_0 \leq \text{small}$ is

$\text{cof}(K)$-filtered.

It's enough to show a $\text{cof}(K)$-small limit of $\kappa$-small

Chau's is $\kappa$-small.
Enough to bound \( \prod_{i \in I} x_i \)

\[
\prod_{i \in I} k_i \leq \prod_{i \in I} 2^{k_i} = 2^{\sum_{i \in I} k_i} < 2^n, \quad \forall k
\]

if \( k_i \) are K-small.

Relation to top spaces.

First, \( \text{Cont}(-, X) \) is a sheaf

\[
\text{Chaus} \subseteq \text{CondSets}.
\]
Claim: CHaus is exactly the full subcategory of condenser sets.

Pf. Each object of CHaus is qc because of the

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If X is qc, it admits a section from a

\[ \exists \sigma : \text{CHaus} \rightarrow \text{Chaus space} \]

\[ \sigma_c \text{?} \rightarrow y \rightarrow y' \] and coarsely.

\[ \downarrow \]

\[ y' \rightarrow X \in \text{CHaus} \]
Quasi-separated condensed sets:

"Hausdorff"

Claim: A condensed set is quasi-separated (\(=\)) if it is a filtered colimit of Hausdorff spaces along a certain relation.  

\(\text{sets} \quad \subset \quad \text{Haus} \quad = \quad \text{X} \ni \) the quotient of \( \varphi \) by a certain relation.  

Haus.
\[ X = \lim_{K \to X} K \]

Addendum: Have \(\lim_{i \in I} K_i, \lim_{j \in J} K_j\)

Due to translation issues.
\[
\lim_{I \to \infty} \operatorname{Hom}(K_i, K_j)
\]

\( Q_{\text{sep}} \text{ cord sets} \)

\( \subseteq \text{Ind} \left( \text{CH}_{\omega_1} \right) \)

spanned by those with injective transfinite maps.

= same as Willard's "comparable spaces"

\( \text{Cor.} \quad \text{CGWH by spaces} \times I \)

\( q \)-sep (cord sets) \( \text{TH cont.}(I) \)
pf: \(\text{Cont}^+ (\mathbb{C}^n) \cong \bigoplus \text{subspaces of } \mathbb{C}^n \text{'s Haus subspaces,}
\)
\[
\& \quad T \rightarrow X
\]
\[
\text{has closed image.}
\]
\[
\text{Cont}^+ (\mathbb{C}^n) = \bigcup \text{cont} / (-, k)
\]
\[
k \in \mathbb{R}
\]
\[
\text{Choose}
\]
\[
\text{A basic. Cont set } = \bigcup \text{all cha subspace}
\]

\(\bigcirc\) It's not enough to take any cha subspace filtered union to \(X\).
filtered data of GTA spaces

Under drug closed eyes

preteen watshle se

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include, dis...
& pushout of other open spheres $T$ map is an inclusion.

exists left adjoint to the inclusion

$a_{\text{Sep}} \preceq \text{CondSets}$

"Sepification"

$x \mapsto x^{a_{\text{sep}}}$

How to build it?
\[ \begin{align*}
4 \cdot K_i &\quad K_i \in \text{Cl} \{ x \} \\
\times &\quad \\
\times &\quad \Xi \frac{\Xi K_i}{\sim},
\end{align*} \]

\[ \sim \subset \frac{\Xi K_i \times K_j}{\sim j} \]

We can just collapse \( \sim \) by taking the smallest closed end of \( \text{Cl} \{ x \} \) generated by it.

Fact: \( X \uparrow \downarrow \text{ appearance} \)
preserves finite products!

\[
=) \text{ induces analogs for } C \text{ and } A_5, \text{ and } \text{RGs},
\]

The above was about Hausdorff phenomena, but it's very important that we have lots of non-Hausdorff phenomena too.
E.g.: \( \text{cond } Ab \)

direct map

\[ Q \rightarrow \mathbb{R} \]

in Top as sets, this may be both a monomorphism & an epimorphism. But (obviously) not an iso.

When we pass to \( \text{cond } (Ab) \) something different must happen: \( \text{cond } Ab \) is an \( Ab \) cat.

\[ Q \rightarrow \mathbb{R} \]
A highly non-Handoff
ci-densed At gp.

**Remark:**

\[ R^S \rightarrow R \rightarrow R/R^S \]

so non-Handoff that

\[ (R/R^S)(\varnothing) = 0 \]

\[ X(\varnothing) : \text{underlying set of a card set} \]

\[ X \]
$e = \chi_E$ for $E \neq \emptyset$

But, e.g.,

$$(\pi(\eta_{12}))_{\mathfrak{S}_4} \neq 0$$

$$(\pi(\eta_{12}))_{\mathfrak{S}_4} \neq \chi_{\mathfrak{S}_4}$$

$$(\pi(\eta_{12}))_{\mathfrak{S}_4} \neq \chi_{\mathfrak{S}_4}$$

$$\chi_{\mathfrak{S}_4} = \chi_{\mathfrak{S}_4}$$
Prop: $X$ typ space

$\Rightarrow \text{Cont}(-, x) \Rightarrow \text{a sheaf on } \text{Haus.}$

When is it a Cond set
(i.e. when is it the case
that $\text{Cont}(-, x)$ is
a small colim of
representables?)

Ans: $\Leftarrow \Rightarrow X$ is $T_1$.

(all pts are closed).

Proof in Releś, notes (undirected).