Can condensed sets, being built from profinite sets, possibly be a good language to talk about "real" stuff?

\[ S \rightarrow [0,1] = S/R. \]

(profinite) (or extra-dir.) Card Set

\[ [0,1] = \left\{ \text{decimal expansion} \right\}/\sim. \]

\[ 0.395\ldots \]

\[ 0.0999\ldots = 0.1000\ldots \]

\[ \prod \{0,1,\ldots,9\}. \text{ profinite.} \]

More fancy: Any \( x \in \mathbb{R} \) can be written
\[ a_n \rightarrow \sum_{n \rightarrow -\infty} \frac{1}{10^n} a_n \in \mathbb{Z} \text{ (bounded, or not grow too fast).} \]

\[ Z(T)_{\text{conv}} = \left\{ f(T) = \sum_{n \rightarrow -\infty} x_n T^n \in Z(T) \mid f \text{ converges on } \mathbb{R}_{\mathbb{Z}} \right\} \]

\[ \mathbb{R} = \frac{Z(T)_{\text{conv}}}{(T - \frac{1}{10})} \]

\[ A: \text{ Yes, it works. But it takes some work.} \]

Goal: Define an analogue of category of solid \( O_{\mathbb{P}} \)-vector spaces for \( \mathbb{R} \).
Recall: \( R^0 = 0 \), so
\[
\text{Mod}_R(\text{Solid}_Z) = 0
\]

The formalism should be similar:
full subcat. of condensed \( R \)-vector spaces
specified by a functor.

\[
L : \text{Cord}(R)^{op} \longrightarrow \text{Cord}(R)
\]

\[
R[S] \longrightarrow L(R[S])
\]

satisfying Deligne's axioms.

What should \( L(R[S]) \) be?

Some descriptions of \( \Omega_s[S] \):
\[
S = \lim_{\leftarrow} S_i
\]

1) \( \left( \lim_{\rightarrow} \mathbb{Z}_p[S_i] \right) \left[ \frac{1}{p} \right] \).
2) \[ \text{Ham}(C(S, \mathbb{Z}_p), \mathbb{Z}_p) \left[ \frac{1}{p} \right] \]
\[ \text{dual of continuous functions.} \]
\[ \text{measures on } S. \]

If \( V \in \text{Solid } \mathcal{Q}_p \),

\[ f : S \rightarrow V \]
\[ \forall \mu \in \mathcal{M}(S, \mathbb{Q}_p) \]
\[ \text{can define } \int f \mu \in V. \]

\( \delta_S \in \mathcal{M}(S, \mathbb{Q}_p) \) \( \text{Dirac measures} \)

\( \sim \) \text{ First guess for } L(\mathcal{R}[S]):

Take \( \mathcal{M}(S, \mathbb{R}) \) \( \text{signed Radon measures} \).
\[ \text{Hom} \left( C(S, \mathbb{R}), \mathbb{R} \right) \]

Prop'n. \[ W(S, \mathbb{R}) = \bigcup_{c \geq 0} \lim_{i \to \infty} R(S, i) \]

is a "Smith space".

Def'n. A Smith space is a compactly generated top. \( \mathbb{R} \)-vector space \( W \) s.t.

\[ W = \bigcup_{c \geq 0} c \cdot K, \quad K \subseteq W \]

compactly convex subet.

Def'n. A condensed \( \mathbb{R} \)-vector space \( V \)

is \( \mathcal{U} \)-complete if it is pre-separated.
and for all prof. sets $S$, \( f: S \to V \)

(Rec. unique) \[
\exists \text{ extension to } \tilde{f}: \mathcal{M}(S, \mathbb{R}) \to V.
\]

Prop. Any Banach space, in fact any complete locally convex R-v.s. $V$, is $\mathcal{M}$-complete.

Proof. Quasi-separated easy.

Take $f: S \to V$, $\mu \in \mathcal{M}(S, \mathbb{R})_{\leq 1}$

\[
\lim_{i \to \infty} \mathcal{M}(S_i, \mathbb{R})_{\leq 1}
\]

Pick any lift

\[
\tilde{f}_i: S_i \to S \text{ of } \pi_i: S \to S_i.
\]

Can define a net in $V$ param. by $i$'s.
\[ v_i = \sum_{s \in S_i} f(t_i(s)) \mu(\pi_i^{-1}(s)) \in V. \]

want to see that this is a Cauchy net, so pick abs. convex open nbhd \( U \) of \( 0 \).

\( \forall i \) large \( \rightarrow f(t_i(s)) - f(t'_i(s)) \in U \)

for any other choice \( t'_i : S_c \rightarrow S \).

\[ \sum_{s \in S_i} \left| \mu(\pi_i^{-1}(s)) \right| \leq 1 \]

\( U \) abs. convex.

\[ \rightarrow \sum_{s \in S_i} \left( f(t_i(s)) - f(t'_i(s)) \right) \mu(\pi_i^{-1}(s)) \in U \]

\( \forall v_i \) indep' of choice "up to \( U \)", so give net.

\( \forall U \) gets unique limit \( v \in V \).

as map of underlying \( R \)-v.s.

\[ W(S|R) \rightarrow V. \]

Check: This is continuous. \( \square \).
Prop' 1. $V \in \text{Core}(\mathcal{R})$ is weak-compact if it is a finite union of Smith spaces.

Proof. Given any $f: S \to V$ perfect

$$W(S, \mathcal{R})$$


equality: image of

$$W(S, \mathcal{R}) \to V$$

is a Smith space.

$$U \subseteq \mathcal{R}_0$$

compact, i.e. convex.

Then image = $U \cap K$ Smith space.

Prop' 2. Smith spaces are anti-equivalent to

(Smith '50s)

Borel spaces.

$$W \mapsto \overline{\text{Han}}(W, \mathcal{R}) \quad V \mapsto \text{Han}(V, \mathcal{R}).$$
\[ C(S, R) \rightarrow \mathcal{M}(S, R) \quad \text{or} \]
\[ \mathcal{M}(S, R) \rightarrow C(S, R) \quad \text{or} \]

In general, any Banach \( V \) admits an inclusion of image \( 0 \rightarrow V \rightarrow C(S, R) \rightarrow C(S', R) \).

Reduce to previous case, using Hahn-Banach.

\((e.g. \ S \rightarrow V^* \quad \text{compact by Banach-Alaoglu}) \quad \Box \)

**\( \otimes \)-Products**

**Prop`.** For \( V, V' \in \text{Cond}(R) \) \( \mathcal{M} \)-complete, there is \( \mathcal{M} \)-complete \( V \otimes V' \in \text{Cond}(R) \) representing bilinear maps:

\( (\mathcal{M} \text{- completion of } V \otimes V') \).

In fact, if \( \mathcal{M} \)-complete \( V_j \rightarrow \text{Cond}(R) \)
In a left adjoint, unique colimit-pres.

\[ \mathcal{R}(S) \to \mathcal{U}(S, R) \]

Sketch. Need to show existence of left adj. in general.

\[ \Theta(R_i) \to \Theta(R_{ij}) \to V \to 0 \]

\[ \mathcal{M}(T_j, R) \to \mathcal{M}(S_j, R) \to \tilde{V} \to 0 \]

Classical $\otimes$-products of Banach.

- proj $\otimes$-product

\[ V \otimes W: \text{ repr. bilinear maps} \]

$l^\infty$-bounded sums of tensors
- injective $\otimes$-product

\[ V \otimes W : \]
\[ C(S, \mathbb{R}) \otimes C(T, \mathbb{R}) = C(S \times T, \mathbb{R}). \]

"$\ell^\infty$-bounded sums of tensors".

\[ \text{Prop'n.} \quad V \otimes W = V \underline{\otimes} W. \]

(Computation similar to p-adic case.)

\[ \text{Prop'n.} \quad V, W \rightarrow \text{dual Smith space } V^* \otimes W^*. \]

The \( V^* \otimes W^* \) is a Smith space, dual is \( V \otimes W \) (if one of \( V, W \) satisfies the "approximation property").

Projective objects in Smith spaces:

\[ S \text{ ext. disc. } \rightarrow \mathcal{U}(S, \mathbb{R}) \text{ projective. } \]
(Dually: $C(S, R)$ injective.)

Open Question. Is any injective Banach space of this form?

Bad News. If $S_1, S_2$ ext. disc., infinite, (Cembranos' 80)
then $M(S_1, R) \otimes M(S_2, R)$ is never

\[ M(S_1 \times S_2, R) \]

projective again.

So far, we neglected non-$qs$ spaces by first. If want nice abelian category, need to allow them.

Bad News. The category of ill-complete card. $R$-v.s. is not stable under extension.
In fact, there are nonsplit extensions

\[ 0 \to R \to \varpi \to \mathcal{H}(S; R) \to 0. \]

For any \( S = \mathcal{N} \cup \text{poly} \).

Ribe extension. For finite set \( S \)
\((\text{poly})\).

\[ 0 \to R \to \varpi_S \to \mathcal{H}(S) \to 0. \]

\[ \{ (x_s)_{s \in S}, y \} \]

\[ \varpi_S = \bigcup_{c > 0} \{ (x_s)_{s \in S}, y \} \Bigg| (x_s)_{s \in S} \mid \sum |x_s| \leq c, \]

\[ |y - \sum x_s \log |x_s| | \leq c \frac{\varpi}{2}. \]

Not additive, but locally almost linear

\[ |x \log x + y \log y - (xy) \log(x+y)| \leq 2(|x| + |y|). \]
\[ \xi_{s, \leq c} + \xi_{s, \leq c} \leq \xi_{s, \leq 4c}. \]

For infinite \( s \), can define

\[ \lim_{s \to \infty} \xi_{s, \leq c} = \bigcup_{c=0}^{\infty} \xi_{s, \leq c} \to W(SR) \to 0 \]

In fact, \( \text{Ext}'(V, \mathbb{R}) \) for Borel \( V \).

\[ \exists \text{locally almost linear maps } V \to \mathbb{R} \]

\[ \text{entropy, } x \to x \log |x| \]

non-locally causes out's of locally
Definition. Let $0 < p \leq 1$. A $p$-Banach vector space $V$ is a top. $R$-v.s. s.t. there exists a $p$-norm

$$\|\cdot\| : V \to R_{\geq 0}$$

such that:

1. $\|v\| \to 0 \iff v \to 0$
2. $\|v + w\| \leq \|v\| + \|w\|
3. $\{v | \|v\| < s\} \subset V$

Definition: the $p$-norm defines a top. of $V$.

5. $\|av\| = |a|^p \|v\|$ for $a \in R$ and $v \in V$

Example. $l^p(\mathbb{N}) = \{(a_n) | \sum |a_n|^p < \infty\}$

Not locally convex for $p < 1$. Satisfies usual triangle ineq.
Remak. \[ \text{If } p' < p \]
\[ p - \text{Banach} \implies p' - \text{Banach} \]
\[ \| \cdot \| \implies \| \cdot \|^{p'/p} \]

Defn. Quasi Banach = \text{p-Banach for some } p > a

Thm (Kalton '80s). An ext. of \text{p-Banach is } \text{p'-Banach for all } p' < p.

\"<p - \text{Banachs are stable under extensions}\".

This suggests following modification of
\[ W(S, \mathbb{R}) = \bigcup_{c > 0} \lim_{n \to \infty} R[S, \mathbb{R}]^c_{c^q} \leq c \]

Defn. \text{p-measures } 0 < p \leq 1
\[ \mathcal{W}_p(S, \mathbb{R}) = \bigcup_{p=1}^{\infty} \lim_{\epsilon \to 0} \mathcal{R}^*_\epsilon \mathcal{W}_p(S, \mathbb{R}). \]

\[ \mathcal{W}_{<p}(S, \mathbb{R}) = \bigcup_{p' < p} \mathcal{W}_{p'}(S, \mathbb{R}). \]

\[ \mathcal{R}(S) \text{ "sum of Dirac measures".} \]

**Theorem.** \[ \mathcal{R}(S) \rightarrow \mathcal{W}_{<p}(S, \mathbb{R}) \]

defines a functor \( L \) satisfying Dustin's axioms. In particular:

**Defn.** A compact \( \mathbb{R} \)-vector space \( V \)

is \( p \)-**liquid** \((0 < p \leq 1)\) if the following generic condition hold:

1) For all \( p' < p \), all \( f: S \rightarrow V \),

\[ \exists \! \xi_p: \mathcal{W}_{p'}(S, \mathbb{R}) \rightarrow V \]

is unique.
2) For all \( f : S \to V \), 
\[ \exists ! \tilde{f} : \mathcal{M}_p(S, R) \to V. \]

3) \( V \) can be rewritten as a colimit of maps

\[ \mathcal{M}_p(T_j, R) \to \mathcal{O} \mathcal{M}_p(S_i, R). \]

\[ \cup \text{ left adj. i} \]

\[ \mathcal{L}_{\mathcal{M}}_p(R) \leq \mathcal{C}_{\text{od}}(R), \]

\[ \otimes_p \] everything passes to derived categories,

\[ p \text{-liquid} \Rightarrow \text{p-Banach}. \]

Need to prove that for all p-Banach V,

\[ 0 < p' < p \leq 1, \]

\[ \text{RHom}_p(\mathcal{M}_p(S, R), V) \overset{\text{need to compute this}}{\longrightarrow} \text{p-Banach}. \]
\[
\text{Breen-Deligne resolution.}
\]
\[
\begin{array}{c}
\mathbb{R} \text{Haus} \left( \mathbb{R}[\mathcal{S}], V \right) \\
\mathbb{R} \text{Haus} \left( \mathcal{C}(S, V) \right)
\end{array}
\]
\[
\text{Real Problem. If } V \text{ - Banach, } p < 1,
\]
\[
T \in \text{Chaus}
\]
\[
H^i \text{ (T, V)} = ???
\]
probably nonzero for all i. Expect that one cannot control this.
You really have to resolve by \( R \langle T \rangle \)’s, with \( T \) profinite.

'Explicitly resolve \( R \)-v.s by (free \( R \)-v.s of)

profinite sets.'

Way out: Write \( Q^p \)

\[ R = \mathbb{Z}(T) \text{conv} / (10T - 1) \text{.} \]

\( \text{Union of profinite things.} \)

Define an analogous liquid theory over

\( \mathbb{Z}(T) \text{conv} \), then execute the

intended strategy.

Liquid \( Q \)-vector spaces also make sense.

all said. \( Q \)-v.e. "genuine"
\[ \text{liquid} \quad 0 < p < \infty \quad p = \infty \]

\[ p = c \]