Condensed K-theory cont'd

Recall Suslin's theorem:

Thm: \( K(C^0) \rightarrow k^0 \)

\[ \text{gp cplx of category enriched in finite limits} \]
\[ \text{of } \prod_{n \geq 1} \text{ free } \]
\[ \prod_n k^0 = \{ 0 \text{ if } \}

is an iso with finite coeffs:

\( k(C^0) / n \sim k^0 / n \)

\[ \left( x \mapsto x - \frac{1}{n} \right) \]

Now \( K(C) : \text{condensed} \)

s.t. 1) \( K(C)(-) \sim K(C^0) \)
2) \( k(\mathbb{C}) = ku \)

(or \( h(k(\mathbb{C})) = ku \))

Refined justification:

\( k(\mathbb{C}) / n \) is discrete.

\[
\Rightarrow k(\mathbb{C}) \xrightarrow{\text{approx}} K(\mathbb{C}) / n
\]

\( l \text{ discrete} \)

\[
(\text{K}(\mathbb{C}) / n)^{\text{vector}}
\]

\[
k_u / n = K(\mathbb{C}) / n
\]

Lemma: A condensed spectrum \( \mathbb{X} \) is discrete.

\( \exists \) A extra. disc. \( S \).
\[ s \in S, \]

\[ \lim_{x(0) \to x(3)} x(s) = x(s), \]

\( x \) is an iso.

**Def.** An ordinary speedup

\( \leq_{\text{SOC. discrete end speed}} \)

\[ A^d(S) = \text{sheaf of \text{global sections of constant shift}} \]

\[ A \text{ on half space } S. \]

\[ \Rightarrow \text{constant shift on } U \subseteq S. \]

\[ x(\omega) \rightarrow x \]

can check Do it.
Take $X = k(c)/n$

$$\lim_{n \to \infty} k(\text{cont}(c))/n$$

Note: Things of germs of cont's functions

$\tilde{a}$ a henselian local

$\tilde{a}$ residue field $C$, $\tilde{a}/\mathbb{Q}$ EUs

(Grothendieck's rigidity thm for henselian local rings)

$= 1$ $k(R) \to k(R'/\mathbb{Q})$
No \( \mathfrak{A} \) such \( \mathfrak{B} \),

\[ \Pi \]

Prokately \( A / \mathcal{C} \)

\[ = 1 \quad k(A)^* \odot = k^{st}(A) \]

\[ = h(k(A)) \]

Some argument \( \Rightarrow \)

\[ K(A) / n \Rightarrow \]

Discrete for

any Banach \( A \)

(wher \( \mathcal{O} \))

\[ \mathfrak{m} \in \mathfrak{A} \]

e.g. \( A = \mathcal{O}_p \).
Recall: localization degree

\[ \delta_p \to \delta_p \to \sum \delta_p \]

of condensed species.

\[ \K(\delta_p) = \lim_{n \to \infty} \K(\delta_p^\text{pure}) \]

discrete.

Background:

\[ \K(Cl), \K(\delta_p), \K(\delta_p) \]

these are sing...
The same way as $\mathbb{Q}_p \otimes \mathbb{Q}_p$.

E.g., $K_2(F) = \mathbb{F}_p^\times \otimes \mathbb{F}_p^\times / \mathbb{F}_{p^2}^\times$.

But they all way to more targets, e.g. if you take these coeft's, or

the regulator maps

$\text{Kant}(\mathbb{Q}) \rightarrow \mathbb{R}$.

You'd like to know what the initial reasonable thing is that $K(CR)$ maps to.
(Related: alg cycles)

Weil Con theory

Good guess:

\[ \chi \text{ residue} \]

\[ n = 0, 1, 2, 3 \]

\[ \text{in } \mathbb{Z} \times \mathcal{O} \]

\[ E_n = \text{Ext}_{\mathcal{O}_\mathbb{H}(\mathbb{G}_a)}^{\star, \star} (\mathbb{G}_m, \mathcal{O}) \]

How to recover the arithmetic invariant? I.e. he model?
\[ f(x) + f(y) + f(z) + f(w) \]
\[ f(\xi) + f(\eta) = f(\xi' \cdot f(\eta')) \]

More ETHY

Goerss-Hopkins-Miller...

\[ E((k, \mathbb{G}) \]

\[ E_n \]

\[ E_n \text{ is a } E_{\infty} \text{ ring spectrum} \]

(analog of commutative ring in spectral)

of a very special kind:

It's even \( \left( \prod_{odd} E_n = 0 \right) \)

& periodic \( \left( E_{2n} \in E_n \text{ s.t.} \right) \)
\[ \eta_0 E = (\eta_0 E)(x^k). \]

With

\[ \eta_0 E \cong W(k)[X_1, \ldots, X_m]. \]

= complete local

rig parametery

definition

of \( G \)

coordinate,

Carries universal
deformed formal

if \( G_{(nk)} \),

no longer of \( ht \) exciting

\( n \), rather \( ht \) \( eq. \)

O.T.U.H. any even

residue map locally

\( \Psi = \text{constant?} \)
formal gp on tv

[Quillen]

& for more E-try

these agree.

On the side of spectra, this is a (e.e.) of approx functors $T$

$$S \rightarrow L_{n+1} \rightarrow L_n \rightarrow \cdots \rightarrow L_0 S \rightarrow H \Omega$$

Hopkins-Rayme

$$\left( \lim_{\to} S \right) \subset \left( \lim_{\to} L_n S \right)$$

$$\left( L_n S \right) \subset \left( E_n \right) \left( E_0 \Omega \right)$$
\[ \Pi_0(E_n) = \left[ \mathcal{W}(k)(C_{\text{top}} - \mu_{\text{top}}) \right] \]
Q: How to produce a condensed E-n spectrum E_n

s.t. \[ \prod E_n = (\mathcal{W}(k)b_{\text{cut}}) \]

holds as Cmd. as gps?

Of course, how to produce Q_2 as a condensed my?

A answer:
\[ \lim_{n \to \infty} \frac{Q_2}{E_n} \]
It turns out that for *success*,
there's one that just *p* to
(can almost see)
complete at.
There's a whole
sequence

\[ P_1, v_1, v_2, \ldots \]

\[ \rho \]

\[ \mathbb{H}_0 \]

\[ e \subset \left[ \Sigma^{+s/p}, S/p \right] \]

acts in a degree-
shifty way

\[ \downarrow \times(p) \]
To "be or not to be..."

Let's start with the following:

For example, if we had

(we're...)

...and then...
Option 2:

Logic: \((k, c) \rightarrow E(k, G)\)

works also for \(W(k, W, \ldots)\)

\(k\) a perfect \(m\)

\& \(G\) a family of
\[ ht = n \]

\[ E_n(S) = E(C(S, k), \text{ pullback of } G) \]

\[ \text{perfect } m \]

\[ E_n \rightarrow E_n \]

both are \(CP(c, \ldots, 2\)-complete\)

\[ \text{can check} \]
L_n S \in \langle E_n, E_n \otimes E_n, \ldots \rangle

\left( \bigwedge_{i} (p_{n,i} \otimes \cdots \otimes p_{n,i}) \right)

\psi(n) S \in \langle E_n \rangle

Can be more precise about how to build

\psi(n) \in \text{form}_p \text{ of } E_n.

k = \# F_p, \quad G = \text{form}_p \text{ of } \text{int}_n.
Step 1: Scream / HP

1. Parameterizing automorphisms

Product of pair $(k, G)$

$G \times \text{stab} \rightarrow G$

$k \cong k$

$\Rightarrow$

$G$

$G$: (Extended)

$\text{Mor} \text{Hom}$

$\varphi$

$F$

$\text{Gal}(\mathbb{F}/k_p)$

Units in ring of integer of division of $k_p$

$\text{in}$
Problem: For them to be true, it needs to be continuous. Hypo fixed pts on right.
H (g) \text{ Th} \text{ en} \text{ cant}

\text{H is not custom}

\text{must differ from } \text{Th}

\text{& prof by un H}

\underline{\text{What we do}}

\text{it mean that}

\text{if acts on En}

\text{condensed qf}

\text{condensed spectrum}
$g \rightarrow \text{Act}(E_n)$

$G \rightarrow \text{Act}(E_n)$

Could say same thing:
Different perspective:

Use slice topics

Cond An / Bg

X

Sheaves of guys on extr disc / Bg

S, guys on S

S extr disc push out,

X

S &-torsor

S

X

1 &-torsor
\[ S = \text{const. of}\]
\[ \text{solve}\]
\[ = \text{Spec}(C(S, k))\]

\[ S \rightarrow B G\]

\[ \text{stack of}\]
\[ \text{formal\ gps}\]
\[ \text{No to } G\text{.}\]

\[ =\) \text{classify a}\]
\[ \text{formal opp}\] on \[ S\]

\[ =\) \in (\text{Spec}(S, k), G)\]

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How to check

\[ L_k(S) \rightarrow E_n\]
Enough to check
mod $p^{(n-1)n}$

$L_n S/p^{(n-1)n} \rightarrow E_n/p^{(n-1)n}$

Can $(-\otimes E_n)^{\text{cris}}$

decide $n$

$E_n/p^{(n-1)n} \rightarrow$

$C(G_{\ell}, E_n/p^{(n-1)n})$

$E_n/p^{(n-1)n}$

\textbf{Rk}:

$! E_n \otimes_\mathbb{Z} \mathbb{Q}_\ell$
\[ E_i = \text{KUn} \]

\[ E_i \otimes E_i \text{ not } \]
\[ p\text{-complete!} \]

\[ (E_i, \text{non-commut}) \]

\[ L_n S E_n E_n \otimes E_n \]

RK: We just talked about condensed spectra with \[ \Gamma \text{-actur} \]

\[ \Gamma = \text{profinf} \]

which is what I
what is the class
w/ previous approach?

\[ \mathcal{L}(\text{core } S) = \mathcal{E}_n \]

Devissage-Kleisli: \\
Produced a sheaf \\
of Eoo-rings spectra on the site of finite Cov-pres.

\[ \text{I-sets} \]

s.t.

\[ L(\text{core } S) \]

2) Have descent SS
New X as ground

In

2 ways of seeing:

1) New X as ground

2) Direct

animals.

3) Skulk @

of this site
2) Find a sheaf on finite coals
I - sets w/ stalks X

Claim: $\alpha$ - cat of n-truncel discrete cod qamn w/ action of I

Restrictions complete

of n-truncel of sheaves of anima on site of finite I-sets

$\lim_{\leftarrow} \left( Sh(X) \xrightarrow{\alpha} Sh(I) \right) \exists \alpha(X) \xrightarrow{\alpha}$
Shears of anime vs top space $T$.

$k$: if $v \in c(I) < \infty$

(e.g. $I = [-1, 1]$), then Posteriori completion

$= \text{hyperplane}$. 

Remark: This is an attempt to make the conditional approach to the only "reasonable"

transcendental
is not a top
function.

Complete resolvent can be studied classically.

\[ K(\mathbb{E}_n) \]

\[ K(\mathbb{E}_n) \cap \mathbb{R} \]

\[ \mathbb{K}(\mathbb{E}_n) \]

\[ \mathbb{K}(\mathbb{E}_n) \cap \mathbb{R} \]

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