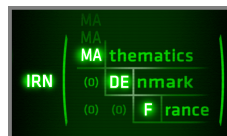# Masterclass on Arakelov Geometry 2023

University of Copenhagen,
August 28 – September 1, 2023

organised by
Nuno Hultberg and Fabien Pazuki

Program

Day:	Mon 28.08	Tue 29.08	Wed 30.08	Thu 31.08	Fri 1.09
Location:	AKB Aud 3	AKB Aud 3	HCØ Aud 8	HCØ Aud 8	AKB Aud 3
08:30-09:00	*Registration*				
09:00-09:45	**Burgos Gil**	**Freixas**	**Burgos Gil**	**Burgos Gil**	**von Pippich**
10:00-10:45	**Burgos Gil**	**Freixas**	**Burgos Gil**	**Burgos Gil**	**von Pippich**
10:45-11:15	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
11:15-12:15	**Wilms**	**Pengo**	**Eriksson**	**Shokrieh**	**Freixas**
12:15-13:30	*Lunch break*	*Lunch break*	*Lunch break*	*Lunch break*	*Lunch break*
13:30-14:15	**Freixas**	**von Pippich**	*Excursion*	**von Pippich**	
14:30-15:15	**Freixas**	**von Pippich**	*by*	**von Pippich**	
15:15-15:45	*Coffee break*	*Coffee break*	*bus*	*Coffee break*	
15:45-16:30	**von Pippich**	**Burgos Gil**	*to*	**Freixas**	
16:45-17:30	**von Pippich**	**Burgos Gil**	*Helsingør*	**Freixas**	
18:00	*Welcome drinks*		*...and bus back!*		
19:00		*Dinner*			

Map: `science.ku.dk/kontakt/find-vej/docs/NorreCampusKortDec2021.pdf`
AKB = August Krogh Building, `bio.ku.dk/om/historie/augustkrogh/`
HCØ = Hans Christian Ørsted, `nbi.ku.dk/english/www/hco/oersted/opvaekst/`

Abstracts

The set of rational solutions to the Weierstrass equation defining an elliptic curve over the rationals is an abelian group which is finitely generated and the number of independent basis points of infinite order is called the rank of the elliptic curve.

The Birch and Swinnerton-Dyer (BSD) conjecture predicts that the rank can be computed by means of the L-function of the elliptic curve. In view of the celebrated modularity theorem of Wiles et al., this L-function has a holomorphic continuation to the whole complex s-plane and satisfies a functional equation relating the value at s to the value at $2 - s$. The BSD conjecture then states that the rank of the elliptic curve equals the analytic rank of the curve, that is, the order of its L-function at $s = 1$.

By the work of Gross-Zagier and Kolyvagin we know that the BSD conjecture is true when the analytic rank of the elliptic curve equals 0 or 1. In this case the contribution of Gross and Zagier is an explicit construction of a point of infinite order on elliptic curves with analytic rank 1. This result can be deduced from the groundbreaking Gross-Zagier formula, which describes the height of Heegner points in terms of a derivative of the L-function of the elliptic curve at the point $s = 1$. More generally, Gross-Kohnen-Zagier computed the heights of Heegner points in terms of Fourier coefficients of modular forms.

The aim of the lectures is to provide the main tools for understanding the Gross-Zagier theorem, as well as related topics for actual research in Arakelov geometry.

Courses

Title: *Introduction to the theory of heights.*
Speaker: **José Burgos Gil** (ICMAT Madrid, Spain).
Abstract: In this course we will introduce the concept of height and show how Arakelov theory provides a flexible framework to define heights. As an application, we will study heights on elliptic curves and recall the proof of Mordell's theorem. If time permits, we will also discuss equidistribution properties and the proof of Bogomolov's conjecture.

Title: *Modular curves and the Gross-Zagier theorem.*
Speakers: **Anna von Pippich** (Universität Konstanz, Germany),
        Gerard Freixas (Institut de Mathématiques de Jussieu, CNRS, Paris, France).
Abstract: The goal of these lectures is to present and discuss the Gross-Zagier theorem, which is one of the available results in the direction of the conjecture of Birch and Swinnerton-Dyer. After a quick review of the basics of Riemann surfaces and elliptic curves, we will study in detail the theory of modular curves and modular forms, both from the analytic and the algebraic points of view. Combined with the formalism of heights introduced in the lectures by J. I. Burgos, this will allow us to formulate the BSD conjecture, and then conclude with the statement of the Gross-Zagier theorem and some applications.

Research Talks

Time: Monday 28.08, 11:15-12:15.   Room: AKB Aud 3.
Speaker: **Robert Wilms** (Univ. Basel, Switzerland).
Title: *On Okounkov bodies in Arakelov theory.*
Abstract: As a motivation, I will first discuss Okounkov bodies and their applications to intersection numbers, especially the proof by Jiang and Li of the reverse Khovanskii-Teissier inequality. Then, I will explain a new construction of arithmetic Okounkov bodies for hermitian line bundles on arithmetic varieties, which allows to translate the proof by Jiang and Li to Arakelov theory giving an inequality between arithmetic intersection numbers of arithmetically nef hermitian line bundles of great generality. This is work in progress.

Time: Tuesday 29.08, 11:15-12:15.   Room: AKB Aud 3.
Speaker: **Riccardo Pengo** (MPI Bonn, Germany).
Title: *Standard conjectures in Arakelov geometry: from the projective space to Grassmannians.*
Abstract: The seminal work of Gross and Zagier, generalized by Zhang and encompassed in a celebrated program of Kudla, has shown how special values of L-functions at the center of the critical strip can be related to the Néron-Tate height of certain algebraic cycles on a given arithmetic variety. As Zhang has shown, this allows one to prove the positivity of these central values under the assumption that the arithmetic Chow groups of the aforementioned arithmetic variety satisfy a strong analogue of the standard conjectures in algebraic geometry, proposed by Soulé as a generalization of a weaker conjecture put forward by Gillet and Soulé. In this talk, based on joint work with Paolo Dolce and Roberto Gualdi, I will explain how to find an almost complete characterisation of the hermitian line bundles on the projective space which satisfy the arithmetic standard conjectures. Moreover, I will outline our strategy to give a similar characterisation for hermitian line bundles on general Grassmannians, based on a new presentation of their Arakelov Chow rings.

Time: Wednesday 30.08, 11:15-12:15.   Room: HCØ Aud 8.
Speaker: **Dennis Eriksson** (Chalmers University of Technology, and the University of Gothenburg, Sweden).
Title: *Eisenstein series and enumerative geometry.*
Abstract: Eisenstein series, and versions thereof, play an important part in arithmetic and algebraic geometry. In some lucrative situations, their q-expansions can be thought of as generating series of interesting invariants in enumerative geometry. I will depict this with two examples coming from the world of elliptic curves. One of the examples is related to the Gross-Kohnen-Zagier formula, which will be discussed in more detail in other lectures in the Masterclass. The other is an example coming from mirror symmetry and the Kronecker limit formula, which relates real-analytic Eisenstein series to generating series of coverings of elliptic curves.

Time: Thursday 31.08, 11:15-12:15.   Room: HCØ Aud 8.
Speaker: **Farbod Shokrieh** (Univ. Washingthon, Seattle, USA).
Title: *Arakelov invariants and tropical geometry.*
Abstract: I will describe some connections between arithmetic/Arakelov geometry and non-archimedean/tropical geometry. The interplay arises from the study of analytic invariants on degenerating families of curves and abelian varieties, as well as the theory of heights of abelian varieties. (Based on recent and ongoing projects with Robin de Jong and Robert Wilms.)

Conference dinner:

Tuesday, August 29, 2023, at 19:00 at
Food Club Nørrebro - Sortedam dossering 7C baghuset,
2200 København N

Excursion:

Wednesday, August 30, 2023, from 13:30, we will go to
Helsingør.

Helsingør is a very interesting town in the north of Sjælland, the drive should be around one hour. When we arrive, you will have the option of visiting:

either the famous Kronborg Castle (also known as the "Hamlet Castle", as it is an important location in Shakespeare's play): `kronborg.dk/om-kronborg`,

or the famous Maritime Museum of Denmark: `mfs.dk/en/exhibitions`.

The bus will bring us back to Copenhagen, departure 18:00.