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The work of Stanislav Smirnov

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Stanislav (Stas for short) Smirnov is receiving a Fields medal for his ingenious and astonishing work on the existence and conformal invariance of scaling limits or continuum limits of lattice models in statistical physics.

Like many Fields medalists, Stas demonstrated his mathematical skills at an early age. According to Wikipedia he was born on Sept 3,1970 and was ranked first in the 1986 and 1987 International Mathematical Olympiads. He was an undergraduate at Saint Petersburg State University and obtained his Ph.D. at Caltech in 1996 with Nikolai Makarov as his thesis advisor. Stas has also worked on complex analysis and dynamical systems, but in these notes we shall only discuss his work on limits of lattice models. This work should make statistical physicists happy because it confirms rigorously what so far was only accepted on heuristic grounds. The success of Stas in analyzing lattice models in statistical physics will undoubtedly be a stimulus for further work.

Before I start on the work for which Stas is best known, let me mention a wonderful result of his (together with Hugo Duminil-Copin, [21]) which he announced only two months ago. They succeeded in rigorously verifying that the connective constant of the planar hexagonal lattice is $\sqrt{2+\sqrt{2}}$. The *connective constant* μ of a lattice $\mathcal{L}$ is defined as $\lim_{n\to\infty}[c_n]^{1/n}$, where c_n is the number of self-avoiding paths on $\mathcal{L}$ of length n which start at a fixed vertex v. It is usually easy to show by subadditivity (or better submultiplicativity; $c_{n+m} \leq c_n c_m$) that this limit exists and is independent of the choice of v. However, the value of μ is unknown for most $\mathcal{L}$. Thus this result of Stas is another major success in Statistical Physics.

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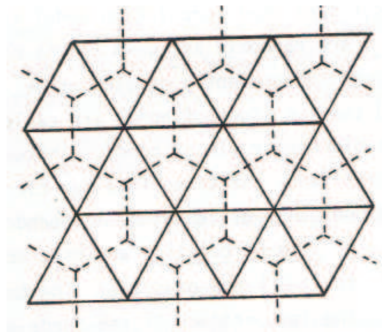
1. Percolation

Since the result for which Stas is best known deals with percolation, it is appropriate to describe this model first. The first percolation problem appeared in Amer. Math Monthly, vol. 1 (1894), proposed by M.A.C.E. De Volson Wood ([9]). He proposes the following problem: "An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end? "As a special example, suppose there are 30 balls in the length of the box, 10 in the width, and 5 (or 10) layers deep." Apart from an incorrect solution by one person who misunderstands the problem, there is no reaction and we still have no answer. Next there is a hiatus of almost 60 years to 1954 when Broadbent ([1]) asks Hammersley at a symposium on Monte-Carlo methods a question which I interpret as follows: Think of the edges of $\mathbb{Z}^d$ as tubes through which fluid can flow with probability p and are blocked with probability $1-p$. Alternatively we assign the color blue or yellow to the edges or call the edges occupied or vacant.) p is the same for all edges, and the edges are independent of each other. If fluid is pumped in at the origin, how far can it spread? Can it reach infinity? Physicists are interested in the model since it seems to be one of the simplest models which has a phase transition. In fact Broadbent and Hammersley ([1, 2]) proved that there exists a value p_c, strictly between 0 and 1, such that ∞ is reached with probability 0 when $p < p_c$, but can be reached with strictly positive probability for $p > p_c$. p_c is called the *critical probability.* The *percolation probability* $\theta(p)$ is defined as the probability that infinity is reached from the origin (or from any other fixed vertex).

Let E be a set of edges. Say that a point a is *connected* (in E) to a point b if there is an open path (in E) from a to b. One can then define the *open clusters* as maximal connected components of open edges in E. By translation invariance, the Broadbent and Hammersley result shows that on $\mathbb{Z}^d$, for $p < p_c$, with probability 1 all open clusters are finite, while it can be shown for $p > p_c$, that with probability 1 there exists a unique infinite open cluster (see [3] for uniqueness). We can do the same thing when we replace $\mathbb{Z}^d$ by another lattice. We can also have all edges open, but the vertices open with probability p and closed with probability $1-p$. In obvious terminology we talk about bond and site percolation. Site percolation is more general than bond percolation, in the sense that any bond percolation model is equivalent to a site percolation model on another graph, but not vice versa. For Stas' brilliant result we shall consider exclusively site percolation on the 2-dimensional triangular lattice. See Figure 1.

We would like to have a global (as opposed to microscopic) description of such systems. Can we tell what $\theta(p, \mathcal{L})$ is? And similarly, what is the behavior of the "average cluster size" and some other functions. We have a fair understanding of the system for $p \neq p_c$ fixed. For instance, if $p < p_c$, then (with probability 1) there is a translation invariant system of finite clusters, and the probability that the volume of the cluster of a fixed site exceeds n decreases



—— $= \mathcal{G}$, the triangular lattice,
$- - - = \mathcal{G}_d$, the hexagonal lattice.

Figure 1.

exponentially in n (see [10], Theorem 6.75). If $p > p_c$, then there is exactly one infinite open cluster. Also, if $\mathcal{C}$ denotes the open cluster of the origin, then for some constants $0 < c_1(p) \le c_2(p) < \infty$,

$$c_1 n^{(d-1)/d} \le -\log\big[P_p\{|\mathcal{C}| = n\}\big] \le c_2 n^{(d-1)/d}.$$

For $d = 2$ we even know that

$$0 < -\lim_{n\to\infty} n^{-(d-1)/d} \log\big[P_p\{|\mathcal{C}| = n\}\big] < \infty,$$

i.e., for some $0 < c(p) < \infty$,

$$P_p\{|\mathcal{C}| = n\} = \exp\big[-(c + o(1))n^{(d-1)/d}\big]$$

(see [10], Section 8.6). For these reasons the most interesting behavior can be expected to be for p equal or close to p_c. We have here a system with a function $\theta(p,\mathcal{L})$, which has a phase transition, but, at least in dimension 2, is continuous. I am told that physicists have been successful in analyzing such systems by making an extra assumption, the so-called *scaling hypothesis*: for $p \ne p_c$ there is a single length scale $\xi(p)$, called the *correlation length*, such that for p close to p_c, at distance n the picture of the system looks like a single function of $n/\xi(p)$. More explicitly, it is assumed that many quantities behave like $T\big(n/\xi(p)\big)$ for some function T which is the same for a class of lattices $\mathcal{L}$. What happens when $p = p_c$ where there is no special length scale singled out (other than the lattice spacing)? The correlation length is assumed to go to ∞ as $p \to p_c$. Therefore, investigating what happens as $p \to p_c$ automatically entails looking at a piece of our system which is many lattice spacings large. For convenience we shall think of looking at our system in a fixed piece of space, but letting the lattice spacing go to 0. We shall call this "taking the scaling limit" or "taking the continuum limit." We shall try to explain Stas' result that this limit exists and is conformally invariant if we consider critical site percolation on the triangular lattice in the plane.

2. The Scaling Limit

What do we expect or hope for? One hopes that at least the cluster distribution and the distribution of the curves separating two adjacent clusters converge in some sense in the scaling limit. Since there is no special scale, one expects scale invariance of the limit. If $\mathcal{L}$ has enough symmetry you can also hope for rotational symmetry of the scaling limit. In dimension two, scale and rotation invariance together should give invariance under holomorphic transformations. If one believes in scale invariance, then one can expect power laws, i.e., that certain functions behave like a power of n or $|p - p_c|$ for n large or p close to p_c. E.g., if we set $R = R(p) =$ the radius of the open cluster of the origin, then scale invariance at $p = p_c$ would give that

$$\frac{P_{p_c}\{R \ge xy\}}{P_{p_c}\{R \ge y\}} \to g(x) \tag{2.1}$$

for some function $g(x)$, as $y \to \infty$ and $x \ge 1$ fixed. This, in turn, would imply $g(xy) = g(x)g(y)$ and $g(x) = x^\lambda$ for some constant λ. Necessarily $\lambda \le 0$, since (2.1) is less than or equal to 1 for $x \ge 1$. Now let $\varepsilon > 0$ and $(1+\varepsilon)^k \le t \le (1+\varepsilon)^{k+1}$. Then

$$P_{p_c}\{R \ge t\} \le P_{p_c}\{R \ge (1+\varepsilon)^k\} = P_{p_c}\{R \ge 1\} \prod_{j=1}^{k} \frac{P_{p_c}\{R \ge (1+\varepsilon)^j\}}{P_{p_c}\{R \ge (1+\varepsilon)^{(j-1)}\}}. \tag{2.2}$$

Since

$$\frac{P_{p_c}\{R \ge (1+\varepsilon)^j\}}{P_{p_c}\{R \ge (1+\varepsilon)^{(j-1)}\}} \to g(1+\varepsilon) = (1+\varepsilon)^\lambda \text{ as } j \to \infty,$$

we obtain

$$P_{p_c}\{R \ge t\} \le t^{\lambda + o(1)} \text{ as } t \to \infty.$$

By replacing k by $k+1$ and reversing the inequality in the lines following (2.2) we see that

$$P_{p_c}\{R \ge t\} = t^{\lambda + o(1)} \text{ as } t \to \infty \text{ or } \lim_{t\to\infty} \frac{\log P_{p_c}\{R \ge t\}}{\log t} = \lambda. \tag{2.3}$$

Of course we did not prove (2.1) here, nor did we obtain information about λ. The complete proof of (2.3) and evaluation of λ in [15] is much more intricate.

An example of a different but related kind of power law which one may expect says

$$\frac{\log\big[\theta(p)\big]}{\log(p - p_c)} \to \beta \text{ as } p \downarrow p_c.$$

Exponents such as λ and β are called *critical exponents*. It is believed that all these exponents can be obtained as algebraic functions of only a small number of independent exponents. Physicists have indeed found (non-rigorously) that various quantities behave as powers. Still on a heuristic basis, they believe that these exponents are *universal*, in the sense that they depend basically on the dimension of the lattice only. In particular they should exist and be the same for the bond and site version on $\mathbb{Z}^2$ and the bond and site version on the triangular lattice. For the planar lattices physicists even predicted values for these exponents.

The pathbreaking work of Stas and Lawler, Schramm, Werner has made it possible to prove some power laws for various processes such as site percolation on the triangular lattice, loop erased random walk, or processes related to the uniform spanning tree. Nevertheless, there still is no proof of universality for percolation, because the percolation results so far are for one lattice only, namely site percolation on the triangular lattice. As stated by Stas in his lecture at the last ICM ([20],p. 1421), "The point which is perhaps still less understood both from mathematics and physics points of view is why there exists a universal conformally equivalent scaling limit." *From now on, all further results tacitly assume that we are dealing with site percolation on the triangular lattice.* As far as I know no other two dimensional percolation results have been proven. For this lattice p_c equals $1/2$.

Somehow, the knowledge and guesses about other similar systems convinced people that it would be helpful to prove that the scaling limit for percolation at p_c in two dimensions exists and is conformallly invariant. This is still vague since we did not specify what it means that the scaling limit exists and is conformally invariant. It seems that M. Aizenman (see [13], bottom of p. 556) was the first to express this as a requirement about the scaling limit of *crossing probabilities*.

A crossing probability of a Jordan domain $\mathcal{D}$ with boundary the Jordan curve $\partial\mathcal{D}$ is a probability of the form

$$P\{\exists \text{ an occupied path in } \overline{\mathcal{D}} \text{ from the arc } [a,b] \text{ to the arc } [c,d]\},$$

where $\overline{\mathcal{D}} =$ closure of $\mathcal{D}$, and a, b, c, d are four points on $\partial\mathcal{D}$ such that one successively meets these points as one traverses $\partial\mathcal{D}$ counterclockwise, and the interiors of the four arcs $[a, b], [b, c], [c, d]$ and $[d, a]$ are disjoint. We may also replace "occupied path" by "vacant path" in this definition. It seems reasonable to require that each crossing probability converges to some limit if our percolation configuration converges. As we shall see soon that this is indeed the case in the Stas' development. However, see [6] and [7] for a stricter sense of convergence.

To be more specific, let $\mathcal{D}$ be a Jordan domain in $\mathbb{R}^2$ with a smooth boundary $\partial\mathcal{D}$. Also let $\tau = \exp(2\pi/3)$ and consider three points of $\partial\mathcal{D}$ and label these $A(1), A(\tau), A(\tau^2)$ as one traverses $\partial\mathcal{D}$ counterclockwise. (More general $\mathcal{D}$ should be allowed, but we don't want to discuss technicalities here.) As shown by Stas, there then exist three functions

$$h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z), \quad \alpha \in \{1, \tau, \tau^2\},$$

which are the unique harmonic solutions of the mixed Dirichlet-Neumann problem

$$\begin{aligned}
&h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = 1 \text{ at } A(\alpha),\\
&h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = 0 \text{ on the arc } A(\tau\alpha), A(\tau^2\alpha),\\
&\frac{\partial}{\partial(\tau\nu)} h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = 0 \text{ on the arc } A(\alpha), A(\tau\alpha),\\
&\frac{\partial}{\partial(-\tau^2\nu)} h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = 0 \text{ on the arc } A(\tau^2\alpha), A(\alpha),
\end{aligned} \tag{2.4}$$

where these functions are regarded as functions of z, and ν is the counterclockwise pointing unit tangent to $\partial\mathcal{D}$. The harmonic solution to these boundary conditions (2.4) is unique, and hence its determination is a conformally invariant problem. More specifically, let Φ be a conformal equivalence between $\mathcal{D}$ and a domain $\widetilde{\mathcal{D}}$, and for simplicity assume that the equivalence extends to $\partial\mathcal{D}$. Let $\widetilde{h}$ be the harmonic solution of the boundary problem (2.4) with A replaced by $\widetilde{A} = \Phi(A)$. Then the uniqueness of the solution implies that

$$h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = \widetilde{h}\Big(\Phi(A(\alpha), \Phi(A(\tau\alpha), \Phi(A(\tau^2\alpha), \Phi(z)\Big).$$

In shorter notation,

$$h = \widetilde{h} \circ \Phi. \tag{2.5}$$

Thus, the solution of (2.4) is a conformal invariant of the points $A(1), A(\tau), A(\tau^2), z$ and the domain $\mathcal{D}$. By the Riemann mapping theorem we may choose Φ such that $\widetilde{\mathcal{D}}$ has a simple form and then use (2.5) to obtain h on $\mathcal{D}$. Carleson observed that if we take $\widetilde{\mathcal{D}}$ to be an equilateral triangle, then the solution $\widetilde{h}(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z)$ is just a linear function which is 1 at the vertex $A(\alpha)$, and 0 on the opposite side $(A(\tau\alpha), A(\tau^2\alpha))$, and similarly when α is replaced by $\tau\alpha$ or $\tau^2\alpha$. For Stas this elegant form made the problem that much more attractive to work on.

Stas achieves his main result by making the following choices: On the triangular lattice, let $A(1) = (2/\sqrt{3}, 0)$, $A(\tau) = (1/\sqrt{3}, 1)$, $A(\tau^2) =$ the origin. These are the vertices of an equilateral triangle $\mathcal{D}$ of height 1 and one vertex at the origin. One further takes z on the arc $[A(\tau^2), A(1)] = [(0,0), A(1)]$. Actually we are cheating a bit because the points $A(1), A(\tau), A(\tau^2)$ and z may not lie in $\delta\mathcal{L}$, but we shall ignore this difficulty here and on several places below. For $\alpha \in \{1, \tau, \tau^2\}$, and $z \in [(0,0), A(1)]$, define

$$\begin{aligned}
Q^\delta_\alpha(z) = {}&\text{there exists in } \mathcal{D} \text{ a simple, occupied path, from the}\\
&\text{arc } [A(\alpha), A(\tau\alpha)] \text{ to the arc } [A(\tau^2\alpha), A(\alpha)], \text{ and this}\\
&\text{path separates } z \text{ from the arc } [A(\tau\alpha), A(\tau^2\alpha)],
\end{aligned} \tag{2.6}$$

and

$$H^\delta(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) = P\{Q^\delta_\alpha(z)\}. \tag{2.7}$$

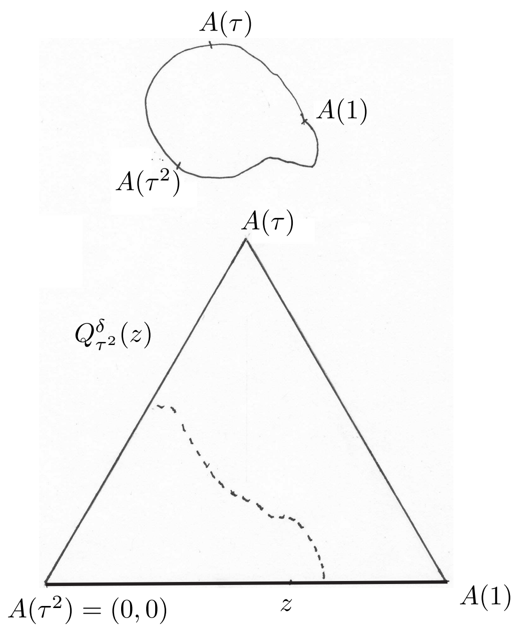


Figure 2.

Stas then formulates his main result as follows: For percolation on $\delta\mathcal{L}$, with $\mathcal{L}$ the triangular lattice, as $\delta \to 0$,

$$H^\delta(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z) \to h(A(\alpha), A(\tau\alpha), A(\tau^2\alpha), z),\ \text{uniformly on } \mathcal{D}. \tag{2.8}$$

The basic structure of the argument is now well known. It is shown that the H^δ are Hölder continuous, so that every sequence $\delta_n \to 0$ has a further subsequence δ_n^* along which the functions $H^{\delta_n^*}$ converge. Moreover the limit along this subsequence has to be harmonic and to satisfy the boundary conditions (2.4). The limit is therefore unique and independent of the choice of the subsequence δ_n^*. Thus $\lim_{\delta\to 0} H^\delta$ exists and is harmonic and conformally invariant (because the solution h to the problem (2.4) is conformally invariant). Note that this proof also yields the convergence of crossing probabilities to a computable limit. Indeed, it follows directly from the definitions that $Q^\delta_{\tau^2}(z)$ is just the event that there exists a crossing in $\mathcal{D}$ from the arc $[A(\tau), A(\tau^2)]$ to the arc $[z, A(1)]$. It then follows from (2.8) that the probability of the existence of such a crossing converges, (as $\delta \to 0$) to $h(A(\tau^2), A(1), A(\tau), z_1) = 1 - z_1\sqrt{3}/2$, where $\|z - (z_1, 0)\| \to 0$. The value $1 - z_1\sqrt{3}/2$ comes from the fact that $h(z)$ is linear on the segment from $A(\tau^2)$ to $A(1)$ and that $z \to (z_1, 0)$ as $\delta \to 0$.

Stas' proof is quite ingenious. Quite apart from the clever introduction of the variable z, there are steps which one would never expect to work. It uses estimates which rely on quite unexpected cancellations. The principal part of the argument is to show that any subsequential limit (as $\delta \to 0$ through some subsequence δ_n) of the H^δ is harmonic. In turn, this relies on the H^δ being approximations to discrete harmonic functions. Rather than trying to prove harmonicity locally from properties of a second derivative, Stas shows that certain contour integrals of H^δ tend to zero as $\delta \downarrow 0$ and applies Morera's theorem.

Thus these crossing probabilities have limits, which can be computed explicitly. These limits agree with Cardy's formula ([8]). This shows that certain finite collections of crossings of (suitably oriented equilateral) triangles converge weakly and that their probabilities behave as expected, or desired. But much more can be said. [6, 7], and later [4, 23], show that in "the full scaling limit" there is also weak convergence of the occurrence of loops, and loops inside loops or touching other loops, etc. As stated in the abstract of [5]: "These loops do not cross but do touch each other-indeed, any two loops are connected by a finite 'path' of touching loops."

3. Schramm-Loewner Evolutions (SLE)

A short time before Smirnov's paper, Schramm had tried to find out how conformal invariance could be used (if shown to apply) to study also other models than percolation. Loewner introduced his evolutions when he tried to prove Bieberbach's conjecture. Roughly speaking, Loewner represented a family of curves (one for each $z \in \mathbb{H}$) by means of a single function U_t. Here $\mathbb{H}$ is the open upper halfplane, U_t is a given function, and after a reparametrization, g_t is a solution of the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{3.1}$$

Let

$$T_z = \sup\{s : \text{ solution is well defined for } t \in [0, s) \text{ with } g_s(z) \in \mathbb{H}\}$$

and $H_t := \{z : T_z > t\}$. Then g_t is the unique conformal transformation from H_t onto $\mathbb{H}$ for which $g_t(z) - z \to 0$ as $z \to \infty$ (see [14], Theorem 4.6). The g_t arising in this way are called *Loewner chains* and $\{U_t\}$ the *driving function*. See [14], Theorem 4.6. The original Loewner chains were defined without any probability concepts. In particular the driving function $\{U_t\}$ was deterministic. [16] raised the question whether a random driving function could produce some of the known random curves as Loewner chain $\{g_t\}$. Schramm showed in [16] that if the process $\{g_t\}$ has certain Markov properties, then one can obtain this process as Loewner chain only if the driving function is $\sqrt{\kappa} \times$ Brownian motion, for some $\kappa \geq 0$. The processes which have such a driving function are called SLE's (originally this stood for "stochastic Loewner Evolution", but is now commonly read as Schramm-Loewner evolution). When a chain is an SLE_κ (in obvious notation) new computations become possible or much simplified. In particular, the existence and explicit values of most of the critical exponents have now been rigorously established (but see questions Q2 and Q4 below). Stas has made major contributions to these determinations in [15, 22].In particular he provided essential steps for showing that a certain interface between occupied and vacant sites in percolation is an SLE_6 curve.

The SLE calculations confirm predictions of physicists, as well as a conjecture of Mandelbrot. As a result, the literature on SLE_κ has grown by leaps and bounds in the last few years, and the study of properties of SLE is becoming a subfield by itself. SLE_κ processes with different κ can have quite different behavior. A good survey of percolation and SLE is in [17], and [14] is a full length treatment of SLE.

4. Generalization and Some Open Problems

I don't know of any lattice model in physics which has as much independence built in as percolation. It is therefore of great significance that Stas has a way to attack problems concerning the existence and conformal invariance of a scaling limit for some models with dependence between sites, and in particular for the two-dimensional Ising model. This is perhaps the oldest lattice model, and the literature on it is enormous. I am largely ignorant of this literature and have not worked my way through Stas' papers on these models. Nevertheless I am excited by the fact that Stas is seriously attacking such models.

For the people who are new to this, the Ising model again assigns a random variable (usually called a spin) to each site of a lattice $\mathcal{L}$. Denote the spin at a site v by $\sigma(v)$. Again $\sigma(v)$ can take only two values, which are usually taken to be ± 1. The interaction between two sites u and v is $J(u,v)\sigma(u)\sigma(v)$ and in the simplest case

$$J(u,v) = \begin{cases} J \text{ if } u \text{ and } v \text{ are neighbors} \\ 0 \text{ otherwise.} \end{cases}$$

We restrict ourselves to this simplest case, which takes $J \geq 0$ constant. However, in order to discuss boundary conditions we also need another constant, $\widetilde{J}$ say. For any *finite* set $\Lambda \subset \mathcal{L}$ we consider the probability distribution of the spin configuration on Λ. This configuration is of course the vector $\{\sigma(v)\}_{v\in\Lambda}$, and so can also be viewed as a point in $\{-1,1\}^\Lambda$. For any fixed $\widetilde{\sigma}$ and Λ we define

$$H(\sigma,\widetilde{\sigma}) = H_\Lambda(\sigma,\widetilde{\sigma}) = -\sum_{u,v\in\Lambda} J\sigma(u)\sigma(v) - \sum_{u\in\Lambda, v\notin\Lambda} \widetilde{J}\sigma(u)\widetilde{\sigma}(v), \tag{4.1}$$

and the normalizing constant (also called partition function)

$$Z = Z(\Lambda,\beta,\widetilde{\sigma}) = \sum_\sigma \exp\Big[-\beta H_\Lambda(\sigma,\widetilde{\sigma})\Big].$$

Here the sum over σ runs over $\{-1,1\}^\Lambda$, the collection of possible spin configurations on Λ. $\beta \geq 0$ is a parameter, which is usually called the "inverse temperature."

Let $\widetilde{\sigma}$ be fixed outside Λ. Then, given the boundary condition $\sigma(v) = \widetilde{\sigma}(v)$ for $v \notin \Lambda$, the distribution of the spins in Λ is given by

$$P\{\sigma(u) = \tau(u) \text{ for } u \in \Lambda \mid \sigma(v) = \widetilde{\sigma}(v), v \notin \Lambda\} = [Z(\Lambda, \beta, \widetilde{\sigma})]^{-1} \exp\Big[-\beta H_\Lambda(\tau, \widetilde{\sigma})\Big].$$

This defines a probability measure for the spins in a finite Λ. A probability distribution for all spins simultaneously has to be obtained by taking a limit as $\Lambda \uparrow \mathcal{L}$. The second sum in the right hand side of (4.1) shows the influence of boundary conditions. At sufficiently low temperature there can be two extremal states, obtained by taking $\Lambda \uparrow \mathcal{L}$ under different boundary conditions. It now becomes unclear how to deal with boundary conditions when one wants to take a continuum limit.

To conclude, here are some problems on percolation. These also have appeared in other lists, (see in particular [17]),but you may like to be challenged again.

Q1 Prove the existence and find the value of critical exponents of percolation on other two-dimensional lattices than the triangular one and establish universality in two dimensions.

This seems to be quite beyond our reach at this time. Probably even more so is the same question in dimension > 2.

Q2 Prove a power law and find a critical exponent for the probability that there are j disjoint occupied paths from the disc $\{z : |z| \leq r\}$ to $\{z : |z| > R\}$. For $j = 1$ this is the one-arm problem of [15]. For $j \geq 2$, the problem is solved, at least for the triangular lattice, if some of the arms are occupied and some are vacant (see Theorem 4 in [22]), but it seems that there is not even a conjectured exponent for the case when all arms are to be occupied or all vacant.

More specific questions are

Q3 Is the percolation probability (right) continuous at p_c? Equivalently, is there percolation at p_c? This is only a problem for $d > 2$. The answer in $d = 2$ is that there is no percolation at p_c;

Q4 Establish the existence and find the value of a critical exponent for the expected number of clusters per site. This quantity is denoted by

$$\kappa(p) = \sum_{n=1}^{\infty} \frac{1}{n} P_p\{|C| = n\}$$

in [10], p. 23. The answer is still unknown, even for critical percolation on the two-dimensional triangular lattice. It is known that $\kappa(p)$ is twice differentiable on $[0,1]$, but it is believed that the third derivative at p_c fails to exist; see [12], Chapter 9. This problem is mainly of historical interest, because there was an attempt to prove that p_c for bond percolation on $\mathbb{Z}^2$ equals $1/2$, by showing that $\kappa(p)$ has only one singularity in $(0,1)$.

5. Conclusion

I have been amazed and greatly pleased by the progress which Stas Smirnov and coworkers have made in a decade. They have totally changed the fields of random planar curves and of two dimensional lattice models. Stas has shown that he has the talent and insight to produce surprising results, and his work has been a major stimulus for the explosion in the last 15 years or so of probabilistic results about random planar curves.

As some of the listed problems here show, there still are fundamental, and probably difficult, issues to be settled. I wish Stas a long and creative career, and that we all may enjoy his mathematics.

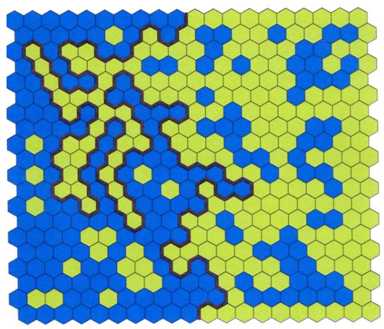


Figure 3.

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