MANIFOLDS AND MORE

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I have worked on several topics in algebra and topology. More recently I have also been dabbling in parts of number theory.

Manifolds

The topic I've spent the most time thinking about is *manifolds*. A subset $M \subset \mathbb{R}^n$ defined by a set of equations

$$\begin{gathered} f_1(x_1, \ldots, x_n) = 0 \\ \cdots \\ f_k(x_1, \ldots, x_n) = 0 \end{gathered}$$

is a smooth submanifold of dimension $d = n - k$, provided that the functions f_i are defined on an open subset $U \subset \mathbb{R}^n$, that their partial derivatives of all orders exist and are continuous, and that the gradient of f is non-zero at all points of M. More generally, $M \subset \mathbb{R}^n$ is a smooth submanifold if it is locally of this form. For instance, with $n = 2$ and $k = 1$ the equation $x_1^2 + x_2^2 - 1 = 0$ defines the unit circle, which is a smooth manifold of dimension 1.

The equations

$$\begin{gathered} |z_0|^2 + |z_1|^2 + |z_3|^2 + |z_4^2| + |z_4|^2 = 1 \\ z_0^5 + z_1^3 + z_2^2 + z_3^2 + z_4^2 + z_4^2 = 0 \end{gathered}$$

in five complex variables can be rewritten as three equations in ten real variables, and defines a smooth seven-dimensional submanifold $\Sigma \subset \mathbb{R}^{10}$ called a *Brieskorn sphere*. It gives an example of "exotic spheres", a phenomenon first discovered by John Milnor, for which he was awarded the Fields medal in 1962 and many subsequent prizes.

My own work on manifold theory, in recent years mostly joint with Oscar Randal-Williams, investigates the *collection* of all d-dimensional submanifolds $M \subset \mathbb{R}^n$ as a mathematical object in its own right, a "moduli spaces of manifolds". For $d = 0$, such moduli spaces are traditionally called *configuration spaces* and have been much studied and used in algebraic topology and elsewhere. For $d = 2$ an influential development happened in the early 2000s, when I was still a PhD student in Århus, with a result by Ib Madsen and Michael Weiss. My work with Randal-Williams started as a quest to understand the case $d > 2$ from a similar point of view. In three long papers [GRW14, GRW18, GRW17] we obtained satisfactory answers when d is an even number larger than 4.

Moduli spaces of manifolds of odd dimensions remain mysterious, although progress has been made by others recently.

Free groups and their automorphisms

This topic has Danish footprints going back to the work of Jakob Nielsen a century ago. Nielsen was plenary speaker at the International Congress of Mathematicians in 1936, while he worked at the Technical University of Denmark.

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The free group F_n on n letters $x_1, \dots, x_n$ is a non-commutative analogue of $\mathbb{Z}^n$, and its automorphism group $\mathrm{Aut}(F_n)$ is a non-commutative analogue of the general linear group $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$. Comparing sets, groups, and $\mathbb{Z}$-modules leads to interesting group homomorphisms

$$S_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}), \tag{0.1}$$

between the symmetric groups, the automorphism group of free groups, and the general linear groups of $\mathbb{Z}$. The middle group $\mathrm{Aut}(F_n)$ is probably less familiar than S_n and $\mathrm{GL}_n(\mathbb{Z})$, and one may wonder whether it is more similar to one or the other.

The homomorphisms (0.1) are of course far from being isomorphisms, but one may nevertheless wonder whether $\mathrm{Aut}(F_n)$ is more similar to S_n or to $\mathrm{GL}_n(\mathbb{Z})$, in terms of its qualitative properties. In some regards it is more similar to $\mathrm{GL}_n(\mathbb{Z})$, for example both are infinite which S_n is not, but in some other ways it turns out to behave more like S_n. An example of this is given in my paper [Gal11], where I show that their *group homology* agrees, $H_i(\mathrm{Aut}(F_n)) \cong H_i(S_n)$ at least for $i < n/2$.

Actions of Galois groups

More recently I have been interested in actions of Galois groups of number fields $K \subset \mathbb{C}$. In a joint paper with Venkatesh [GV18], we investigate representations of such groups and how they may be deformed over simplicial rings.

A question I'm currently very interested in concerns a fourth group which may be added to the end of (0.1), namely the *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{Z})$. This group has important relations to both the moduli spaces of manifolds mentioned earlier, as well as to objects of interest in algebraic geometry and number theory. In particular, the relationship to *abelian varieties* can be used to define an action of $\mathrm{Aut}(\mathbb{C})$ on the K-theory of the symplectic groups, at least after completing at a prime p. In ongoing joint work with Feng and Venkatesh, we attempt to describe this action by a universal property.

For odd p we find a short exact sequence

$$0 \to K_{4k-2}(\mathbb{Z})^\wedge_p \to \mathrm{KSp}_{4k-2}(\mathbb{Z})^\wedge_p \to \mathbb{Z}_p(2k-1) \to 0, \tag{0.2}$$

which is equivariant for $\mathrm{Aut}(\mathbb{C})$ when $K_{4k-2}(\mathbb{Z})^\wedge_p$ is given trivial action, and

$$\mathbb{Z}_p(2k-1) = \varprojlim_n \mu_{p^n}(\mathbb{C})^{\otimes(2k-1)}$$

where $\mu_{p^n}(\mathbb{C}) \subset \mathbb{C}$ is the group of p^n-th roots of unity with the canonical action of $\mathrm{Aut}(\mathbb{C})$. We show that the action factors through a continuous action of the profinite group $\Gamma = \mathrm{Gal}(H_{p^\infty}/\mathbb{Q})$, where H_{p^∞} is the union of the fields $H_{p^n} \subset \mathbb{C}$, defined as the maximal everywhere-unramified p-extension of $\mathbb{Q}(e^{2\pi i/p^n})$. We want to characterize (0.2) as the universal extension of p-adic Γ-representations with all these properties.

References

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