MY RESEARCH: THE DILUTE BOSE GAS

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We claim that we know many of the fundamental equations of Physics. More precisely, we claim that these equations describe the world around us. But if we cannot solve the equations except in very special examples, then how can we be sure? My research focuses on extracting qualitative and quantitative results about systems in quantum mechanics, often when there are many particles involved. In this case, the system becomes high-dimensional, and therefore it becomes impossible to make accurate, direct numerical simulation (the so-called *curse of dimension*). Some very interesting and unsolved questions concern phase transitions. Why does water freeze to ice? Why do magnets exist? A similar phase transition - admittedly a bit further away from everyday life - is the transition to a Bose-Einstein Condensate (BEC) that an atomic gas undergoes at low temperature. In the last approx 5 years - and if all goes according to plan this will continue for the next years as well - I have dedicated my main research effort to understanding and improving the mathematics of this physical system.

To set up the description, let $N \in \mathbb{N}$ be the number of particles and let $L > 0$. We want to model N particles in a box of side-length L. Furthermore, we let the particles interact through a (radial, compactly supported) potential $v(x) \geq 0$. The quantum mechanical energy is then given through the Hamiltonian

$$H = H(N,L) = \sum_{j=1}^{N} -\Delta_j + \sum_{j<k} v(x_j - x_k),$$

acting as an unbounded operator on $L^2(\Lambda^N)$, with $\Lambda = [-\frac{L}{2}, \frac{L}{2}]^3$. Here we use the notation that $x_j = (x_{j,1}, x_{j,2}, x_{j,3}) \in \Lambda$ is the position of the j'th particle and $\Delta_j = \sum_{s=1}^{3} \frac{\partial^2}{\partial x_{j,s}^2}$. In general, we use the convention that an index j on an operator denotes that this operator only acts on the j'th particle. The term $-\Delta_j$ represents the quantum mechanical kinetic energy of the j'th particle and $v(x_j - x_k)$ is the potential energy of the interaction between the j'th and the k'th particles in the box. The allowed quantum mechanical energies are given by the eigenvalues of H, i.e. the solutions (Ψ, E) to the equation

$$H(N,L)\Psi = E\Psi,$$

with $\Psi \in L^2(\Lambda^N)$ (and where we have to impose boundary conditions on Ψ to have a self-adjoint operator H - for simplicity assume Dirichlet boundary conditions below). It follows from standard results that the spectrum consists of a sequence of eigenvalues $0 < E_0(N,L) < E_1(N,L) \leq \cdots$ converging to $+\infty$, In particular, we want to understand $E_0(N,L)$ and the associated (normalized) eigenfunction $\Psi_0 \in L^2(\Lambda^N)$.

The eigenvalue equation is of course an elliptic partial differential equation, so we are clearly here in the mathematical areas of partial differential equations as well as functional analysis and spectral theory.

However, we want to study what happens when N goes to infinity, thereby making the configuration space Λ^N high-dimensional. More precisely, the relevant situation is to fix a density $\rho > 0$ and study the behaviour in the *thermodynamic limit*, i.e. when $N, L \to \infty$ in such a way that the particle density $N/L^3 \to \rho$. In this way one gets, for example, the thermodynamic energy per particle, $e(\rho) = \lim N^{-1} E_0(N, L)$.

To define BEC, let P be the orthogonal projection on the constant function (describing a condensed particle) in $L^2(\Lambda)$ and define

$$N_0 := \sum_{j=1}^{N} P_j.$$

To have (complete) BEC in the thermodynamic limit is the statement that

$$N^{-1}\langle \Psi_0, N_0 \Psi_0 \rangle \to 1,$$

along the sequence(s) considered above.

With our current techniques it is not possible to prove such a result in the thermodynamic limit. However, together with Jan Philip Solovej, we have made important progress on the understanding of the energy in recent years and this also leads us to have the presently best available results on BEC for dilute systems at shorter length scales than thermodynamic. The result for the energy is the formula

$$e(\rho) = 4\pi\rho a \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3}(1 + o(1))\right),$$

in the dilute limit $\rho a^3 \to 0$, where $a = a(v)$ is a characteristic length scale associated with v called its scattering length. This is the famous Lee-Huang-Yang formula for the energy of the dilute Bose gas.

Together with my postdocs Léo Morin, Marco Olivieri and Théotime Girardot as well as my phd-student Lukas Junge, we have also recently extended these results to 2 dimensions. It is very interesting to compare 2 and 3 dimensions, because BEC is supposed to behave very different in these 2 geometries. Our 2 dimensional result corresponding to the Lee-Huang-Yang formula is,

$$e^{2\mathrm{D}}(\rho) = 4\pi\rho Y \Big(1 - Y|\log Y| + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)Y\Big) + o(\rho Y^2),$$

in the dilute limit $\rho a^2 \to 0$, where $Y = |\log(\rho a^2)|^{-1}$ and $\Gamma = 0.577\ldots$ is the Euler-Mascheroni constant.