My Research

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I am mainly interested in various control problems held by an individual or a household of (probabilistically and/or economically dependent) individual members. The starting point is a classical Merton's consumption-investment problem dating back to 1969, which goes like this: Assume a wealth dynamics in the following form

$$dX(t) = X(t)\,dG^{\pi}(t) - c(t)\,dt,$$

where $X(t)\,dG^{\pi}(t)$ forms the capital gains arising from some investment strategy π and $c(t)\,dt$ represents the consumption withdrawn (over dt) from the saved capital. Now the control problem is to maximize expected utility gained over a finite time horizon n from deciding the investment strategy π and the consumption rate c, i.e.

$$\max_{c,\pi} E\left[\int_0^n u(c(t))\,dt\right], \tag{1}$$

where u is some utility function (essentially a concave function) of the consumption rate and the expectation operates over risky gains from investment opportunities.

Shortly after Merton introduced the problem (and its solution for u being a power function), the problem with uncertain lifetime and access to life insurance was solved in 1975. Here we say that the individual, in addition to investment and consumption, decides on an insurance cover $b(t)$, which is the insurance sum paid out if he dies, and the insurance company prices this cover at $b(t)\,p(t)\,dt$ where $p(t)\,dt$ relates to the probability of death (over dt). As long as the individual survives, he has then the wealth dynamics (capital gains added and consumption and insurance premium withdrawn),

$$dX(t) = X(t)\,dG^{\pi}(t) - c(t)\,dt - p(t)\,b(t)\,dt.$$

He then seeks to maximize utility of consumption while being alive plus utility upon death of the remaining capital plus the insurance sum paid. The latter can be thought of as utility on behalf of his heirs. The maximum is taken over investment, consumption and the insurance strategy. Introducing $I(t)$ as the indicator of survival, the optimization problem reads

$$\max_{c,\pi,b} E\left[\int_0^n I(t)\,u(c(t))\,dt + u(X(t)+b(t))\,d(1-I(t))\right], \tag{2}$$

where it is noted that the term $d(1-I(t))$ triggers a lump sum utility upon death. Now, the expectation operates over both the risky capital gains and the uncertain lifetime. Labor income while being alive can easily be incorporated in the setup.

During the last decade I have contributed to the research area by generalizing the problem and solving it in various directions of both mathematical-technical and practical interest. Some years ago I worked with extending the problem in the direction of the dimension of the insurance decision problem. In life insurance mathematics, generalizations to multi-state models beyond a life-death model is natural in order to cope with more involved economic entities and more involved risks during the course of life of the entity. E.g., during periods of disability, a disability insurance should provide benefits to consume. Another example is the case where whole household is considered and its members are connected through a household utility function and perhaps even dependent risks across individuals. A key result of mine (appeared first in 2008 but has been repeated later in various connections by myself and others) is to characterize the proportion f^{ij} of the wealth that an insured should optimally hold after entering state j at time/age t in his course of life (e.g. after getting disabled or after the death of his spouse), which we denote by $X^j(t)$, from state i having there the wealth $X^i(t)$,

$$X^j(t) = f^{ij}(t) X^i(t). \tag{3}$$

The objective is to maximize a function similar to (2) but extended to the multi-state case. The function f can be found by solution of a system of non-linear ordinary differential equations.

In more recent years I worked on generalizations where one leaves the assumption of so-called time-additive utility where we in the objective, like in (1) and (2), just integrate over utility of consumption earned at different points in time. Already around 1990 it was pointed out that time-additive utility assumes a particular homogeneity between an individual's preferences towards risk (consumption in different states of the world) and time (consumption at different time points). Generalizations serve, among other things, to disentangle preferences with respect to risk and time. Consider e.g. generalization of Merton's problem to

$$\max_{c,\pi} \left\{ \int_0^n g\left(E\left[u\left(c\left(t\right)\right)\right]\right) dt \right\},$$

for functions u and g relevant to express, jointly, preferences with respect to risk and time. In the case of uncertain life time, apart from the '+' hidden in the integration, there is even also the '+' between utility while alive and utility upon death. Then the generalized object becomes

$$\max_{c,\pi,b} \left\{ \int_0^n g\left(h\left(E\left[I\left(t\right) u\left(c\left(t\right)\right)\right]\right) + h\left(E\left[u\left(X\left(t\right) + b\left(t\right)\right) d\left(1 - I\left(t\right)\right)\right] \frac{1}{dt} \right) \right) dt \right\},$$

for functions u, g, and h relevant to express, jointly, preferences with respect to risk, time and consuming as 'dead or alive'. Such problems become non-trivial as optimization problems, if g and h and non-linear, because of time-inconsistency issues: *What you optimally plan to do at a future time point in a given state of the world differs from what you optimally do when you actually get there!* We search for solutions in terms of a so-called sub-game perfect equilibrium. Now, for specific functions u, g, and h, what is the optimal process of consumption, investment and insurance? Is the structure (3) preserved? And if so, how does the non-linear system of ordinary differential equations characterizing f generalize.

Ultimately the research seeks a better understanding of individual preferences towards key economic risk factors in life and translate this understanding into improved product design and advice to policy holders. In that respect my research enters into the most important strategic and risk management decision made by the financial business, namely how to design the financial contracts such that they meet the demand of individuals in the first place. Thereafter comes all subsequent but also important decision making while the contracts are in force.