My research

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A common thread running through my research has been *algebraic K-theory*. At a coarse level, algebraic K-theory is a machine for extracting interesting invariants from certain algebraic or geometric structures. For an algebraic example, every ring R has an associated algebraic K-theory $K(R)$; for a geometric example, every variety or scheme X has an associated algebraic K-theory $K(X)$. Rings and schemes are very rigid objects, but some of my more current research also involves defining and studying algebraic K-theory of looser geometric objects such as analytic spaces or manifolds. I'll talk a bit about that at the end of this article.

To define algebraic K-theory one first extracts some notion of *linear algebra* from the object in question, be it a ring, scheme, manifold, topological space, or what have you. Then the algebraic K-theory is, by definition, the solution to a certain classification problem of these linear-algebraic objects. For example, the most basic calculation in algebraic K-theory is the following: for a field F, we have

$$K_0(F) = \mathbb{Z}.$$

In words, the so-called 0^{th} algebraic K-group of a field identifies with the group of integers under addition. This comes from the fact that finite-dimensional vector spaces over a field are classified by their dimensions. In general, the appropriate linear algebraic objects, generalizing finite-dimensional vector spaces over a field, are usually some kind of *vector bundles*.

Despite the nature of the definition, the reason I study algebraic K-theory is *not* because I care so much about classifying linear-algebraic objects like vector bundles. Rather, it turns out that algebraic K-theory has very close ties to all sorts of other invariants of the geometric objects in question, even when these other invariants, on the face of it, have nothing to do with the ingredients involved in defining algebraic K-theory. Then algebraic K-theory is uniquely positioned to throw a different light on these other invariants, because the other ones usually involve something much more complicated than just linear algebra. In this way, it's reasonable to say that I'm not interested in algebraic K-theory because of what it is; I'm interested in it because of what it reveals.

One example of this is in *intersection theory*, where a basic problem is how to count the multiplicity of intersection of two closed subschemes of a given scheme. One wants to do this in such a way that one obtains appropriate generalizations of Bézout's theorem on intersections of plane curves. The most general solution to this problem, following Grothendieck and Gillet-Soulé, goes through algebraic K-theory, where intersection corresponds to the simple linear-algebraic operation of tensor product, and is therefore much more manageable.

An aspect of algebraic K-theory which I haven't touched on yet is that, perhaps surprisingly given its algebraic nature, it is actually an object of homotopy theory. Why is that? The K_0 group is gotten by taking all isomorphism classes of vector bundles, and imposing the relation that the middle term of an extension should be equal to the sum of the outer terms. But it is known that the K_0 group alone does not contain enough data to make algebraic K-theory a fully workable and informative invariant. To remedy this, Quillen introduced the full algebraic K-theory $K(-)$. Instead of identifying isomorphic vector bundles, one remembers the isomorphisms as higher data. And instead of just imposing additivity in extensions, one adds higher data coming from iterated extensions. In the end one gets a well-defined homotopy type $K(-)$ whose π_0 is the K_0-group, but whose higher homotopy groups, and other algebraic topology invariants, are just as interesting and informative. My papers often (always?) make use of homotopy-theoretic techniques, for this and other reasons.

The paper I'm most proud of is my first paper, *p-adic J-homomorphisms and a product formula*, which combines homotopy theory and algebraic K-theory in a way which I believe has real significance in number theory. In it, I managed to show that a simple geometric construction of spheres from finite-dimensional real vector spaces — just take the one-point compactification — in some sense has a hidden p-adic analog, and that these real and p-adic constructions fit together into a global picture. One expresses this using K-theory; then, surprisingly, when one looks on second homotopy groups π_2 at the end, a highly nontrivial theorem from number theory pops out: the quadratic reciprocity law. Most of my other papers were in some sense about exploring the ramifications of this idea.

More recently, with Peter Scholze I have been working on new foundations for all sorts of basic objects in mathematics. Starting with what we call *condensed sets*, a new replacement for topological spaces, we define new replacements for the basic objects in functional analysis, continuous cohomology, algebraic geometry, and analytic geometry — and quite likely, other subjects as well. The defining guide for these new definitions is that they should behave very nicely algebraically, while still capturing the usual topological phenomena, which have traditionally been annoyingly difficult to mix with algebra. One of our motivations was precisely the desire to understand algebraic K-theory in less rigid/algebraic contexts, and it seems we've come a long way in this understanding.