My Research: Homotopy invariants & topological classification of quantum simulators

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My research is motivated by questions at the intersection of quantum information theory and quantum many-body physics. The goal is to derive rigorous and model independent results about the properties and dynamics of these systems. In particular, I am interested in identifying and characterizing so called quantum phases in the context of quantum lattice systems and quantum simulators, that is, quantum systems moving on a discrete lattice and in discrete time-steps. Such quantum phases have been at the forefront of quantum many-body physics over the last decades due to their surprising physical properties as well as their potential for new technological applications - at the same time this research field opens up a variety of connections to exciting mathematics in particular homotopy invariants and stability questions that I am going to describe in the following.

In the classical **Landau-theory** of phases, a phase transition is characterized by a broken symmetry and an order-parameter. Specific examples include the transition from water to ice, where the continuous translation symmetry of a fluid is broken to the discrete translation symmetry of the crystal structure of the solid or the formation of magnetic states in spin-systems at low enough temperature, which breaks the rotation symmetry of the unordered system. Even though there are few rigorous results, Landau-theory has been very successfully applied and even been considered to capture essentially all phase-transitions that can be observed in nature. However, over the last fifty years, it has become clear that the theory is not exhaustive but that for quantum systems at zero-temperature there exist exotic orders nowadays called **topological quantum phases**, where the ground state after the phase transition does not break any symmetry of the system Hamiltonian, i.e. the Hermitian operator characterizing the system's properties and dynamics. Instead, a stable ground state degeneracy depending only on the topology of the sample can be observed, which is robust with respect to any local perturbation. Examples of topologically ordered quantum systems include fractional quantum Hall-systems, quantum spin-liquids, string-net-models and quantum double models. From a quantum information point of view, topologically ordered quantum systems with long-range excitations and exotic statistics are considered candidates for both the implementation of quantum computers and quantum memories with very small error-rates.

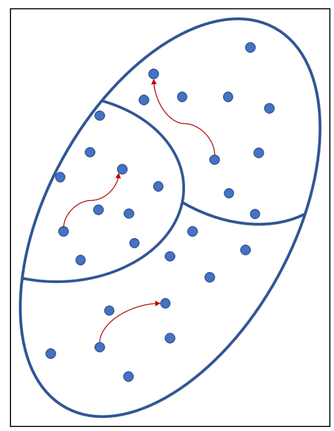

FIG. 1. Phase classification of a set of Hilbert space operators. Different phases are given by connected components under the allowed operations (red arrows).

In contrast to quantum systems exhibiting such an intrinsic topological order, **symmetry protected topological (SPT) phases** are only stable with respect to perturbations that do not violate the symmetries of the system. The main reason for this reduced robustness can be attributed to the fact that their ground states do not possess long-range entanglement but exhibit a local entanglement structure. Nevertheless, if only physical systems with certain symmetries, e.g. particle-hole, time-reversal or chiral symmetry, are considered, this leads to a non-trivial phase classification, even in one spatial dimension. Examples of these systems include the celebrated AKLT-model and topological insulators.

Mathematically, a quantum phase classification problem is given by

1. a set of Hilbert space operators to classify, e.g. Hamiltonians, unitary operators; possibly with some additional constraints, e.g. spectral gap condition, symmetry constraint
2. a set of allowed operations or deformation rules, e.g. norm-continuous deformations, local perturbations

and the goal is to determine the connected components of the set of operators with respect to the allowed operations. In particular, we would like to answer the following questions:

1. Is the classification non-trivial, i.e. does the set decompose into more than one connected component?
2. Which properties distinguish the phases/can we identify an index quantity?
3. Is the classification complete, i.e. are two systems in the same phase if and only if they can be deformed into each other by the allowed operations?
4. Is there a bulk-boundary principle, i.e. stable observable consequences at interfaces between different phases?

In quantum information theory, a well studied class of model systems, which also play an important role for quantum algorithms are so-called **quantum walks**. Quantum walks can be seen both as quantum mechanical generalizations of classical random walks or the time-discrete local evolution of a single quantum particle. In the last years, it has become clear that these systems also provide an excellent playground to study symmetry protected topological phases. In one spatial dimension, the Hilbert space describing those systems is given by

$$\mathcal{H} = \bigoplus_{x\in\mathbb{Z}} \mathcal{H}_x \ , \tag{1}$$

where each $\mathcal{H}_x$ is of finite dimension. We interpret the labels x as the particle's position and the local Hilbert spaces $\mathcal{H}_x$ as describing some internal degree of freedom. In the case, where all $\mathcal{H}_x$ have the same dimension, $\mathcal{H}$ is given by $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^k$. Let us denote by $P_{\geqslant a}$ the orthogonal projection onto the half-chain $\mathcal{H}_{\geqslant a} = \bigotimes_{x\geqslant a} \mathcal{H}_x$.

A unitary operator W on $\mathcal{H}$ is called a **quantum walk operator** if it is **essentially local**, i.e. $[W, P_{\geqslant a}]$, the commutator of W with any half-space projector $P_{\geqslant a}$, is a compact operator. Note, that if this condition is satisfied for any $a \in \mathbb{Z}$ then it is true for all $b \in \mathbb{Z}$, because the difference between any two commutators $[W, P_{\geqslant a}]$ and $[W, P_{\geqslant b}]$ is compact, too.

A canonical example of such an evolution are the so-called **shift-coin quantum walks**. Let us introduce the orthonormal basis $\{\delta_x \otimes e_i\}$ of $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ with the vectors $\delta_x(y) = \delta_{x,y}$ and $\{e_1, e_2\}$ the standard basis of $\mathbb{C}^2$. We can then define the conditional shift S acting in this basis by $S(\delta_x \otimes e_i) = \delta_{x+(-1)^i} \otimes e_i$. The overall timestep is then given by $W = S \cdot (\mathbb{1} \otimes C)$, with C a unitary 2×2-matrix, acting exclusively on the internal degree of freedom. In comparison to a classical random walk, were the particle moves with probability 1/2 to the left or the right, C is interpreted as a quantum coin toss, which in turn via S determines the coherent movement of the particle through the lattice. However, in contrast to classical diffusive dynamics, translation invariant quantum walks generically lead to ballistic dynamics of the particle.

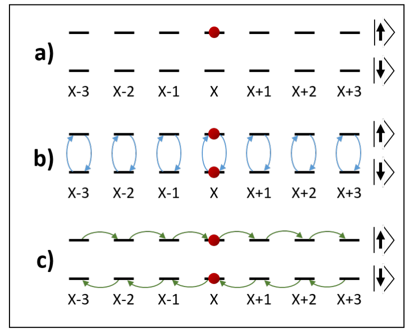

FIG. 2. Schematics of the evolution of a shift-coin quantum walk $W = S \cdot (\mathbb{1} \otimes C)$. a) initially, the particle is located at x in a specified internal state. b) The application of the coin-operation to the internal degree of freedom creates a superposition of the internal degree of freedom. c) The conditional shift transfers the particle coherently to the neighbouring lattice sites depending on the internal degree of freedom.

Without any additional assumptions, the class of one-dimensional quantum walks admits a phase classification under norm-continuous deformations, which was shown by Gross et al. in 2012 for strictly local walk operators. The index they identify to label the different phases is integer valued, additive under direct sums and concatenation of walk operators and basically measures the imbalance between shifts of the particle to the left versus right in a single timestep. Accordingly, the index of the shift-coin walks described before as an example would be zero, since the conditional shift S is balanced. The fact, that quantum walks without additional constraints, admit a non-trivial classification even in a single spatial dimension is a remarkable observation, because for continuous-time quantum processes, the corresponding classification has indeed been shown to be trivial.

We can now go one step further and introduce additional, physically motivated symmetry constraints to the system. These come in the form of involutive unitary or anti-unitary operators that satisfy certain commutation relations with W. A particular example of this is **chiral symmetry**, which is represented on $\mathcal{H}$ by a unitary operator γ with $\gamma^2 = \mathbb{1}$. A walk operator W is then called chiral symmetric if it satisfies $\gamma W = W^* \gamma$ and furthermore is essentially gapped at the symmetry-invariant spectral points ± 1, i.e. the spectrum of W at ± 1 is finitely degenerate and isolated. Two other symmetries, which come from physical consideration are particle-hole and time-reversible symmetry, which arise in a similar manner from an (anti-)unitary operator and certain commutation relations with W. The absence or presence and the relations amongst these symmetry operators lead to a total of 10 different symmetry classes, the so called ten-fold way.

We can now ask how the phase classification of quantum walks changes under these additional symmetry constraint and if we can find topological invariants labeling the different phases. Indeed, this is possible and the different phases under homotopic and symmetric deformations can be associated to a set of three independent symmetry indices

$$\big(\mathrm{si}(W), \overrightarrow{\mathrm{si}}(W), \mathrm{si}_+(W)\big). \tag{2}$$

We will not define those indices rigourously, but rather offer an intuitive interpretation and mention that they can be identified with Fredholm indices and trace formulas of certain restrictions of W . The index $\overrightarrow{\mathrm{si}}(W)$ describes the structure of the walk far to the right, and can be determined from the walk projected to any subspace $P_{\geqslant a}\mathcal{H}$. Similarly, $\overleftarrow{\mathrm{si}}(W) = \mathrm{si}(W) - \overrightarrow{\mathrm{si}}(W)$ relates to the far left. The indices $\mathrm{si}_\pm(W)$ classify the symmetry protected eigenspaces of W at ± 1. Finally, $\mathrm{si}(W)$ acts as the connecting bit between these two types of indices. In the translation invariant case the index triples are of the form $(0, n, 0)$, and each n is given by a winding number determined from the band structure. The possibility of having $\mathrm{si}(W) \neq 0$ is crucial for describing a crossover of two bulk systems with different topological characteristics. In such systems, topologically protected eigenvalues appear in the spectral gap at ± 1, which is known as bulk-boundary correspondence. It is expressed by the sum relation

$$\mathrm{si}(W) = \mathrm{si}_-(W) + \mathrm{si}_+(W) = \overrightarrow{\mathrm{si}}(W) + \overleftarrow{\mathrm{si}}(W). \tag{3}$$

Here $|\mathrm{si}_\pm(W)|$ is a lower bound to the dimension of the eigenspace of W at ± 1. Hence, if $\mathrm{si}(W) \neq 0$, (3) implies some topologically protected eigenvalues in the gap. Since $\mathrm{si}_\pm(W)$ are individually invariant under deformation, topological protection also holds for these eigenvalues separately. A new phenomenon in the classification of unitaries (as opposed to Hamiltonians, i.e., continuous time systems) is the possibility of "non-gentle" local perturbations: These are modifications of a walk, for example only on finitely many cells, which cannot be done along a continuous path (preserving all the defining conditions). Then the index $\mathrm{si}_+(W)$ exactly classifies this new possibility, i.e., the set of compact perturbations up to homotopy. In contrast, the right/left indices $\overrightarrow{\mathrm{si}}(W), \overleftarrow{\mathrm{si}}(W)$ are invariant also under compact perturbations.

Furthermore, we can also ensure the **completeness** of the classification of symmetric walks in terms of the symmetry indices introduced above. Not only is the classification stable against gentle or non-gentle perturbations, but also the converse is true: Whenever two symmetric walks share the values of the independent invariants $\mathrm{si}, \overrightarrow{\mathrm{si}}$ and si_+ they can be deformed into each other along an admissible path. Hence, we can summarize our discussion in the following theorem

Theorem 1. *Let W be a symmetric walk with essential gaps at ± 1. Then:*

(1) The indices $\mathrm{si}_\pm(W)$ are integer valued and invariant under gentle perturbations.

(2) The indices $\mathrm{si}(W)$, $\overrightarrow{\mathrm{si}}(W)$ and $\overleftarrow{\mathrm{si}}(W)$ are integer valued and invariant under gentle and compact perturbations.

(3) The index triple $(\mathrm{si}(W), \overrightarrow{\mathrm{si}}(W), \mathrm{si}_+(W))$ is independent and complete, and, moreover, every index combination in $\mathbb{Z}^3$ can be reached, i.e. for every element of $\mathbb{Z}^3$ there is a walk with corresponding index triple.

Going beyond the single particle and/or one-dimensional regime is one of my current research interests right now, which in particular leads to the study of quantum cellular automata and therefore to the study of topological phases of interacting quantum many-body systems in discrete time and space.