“Homotopy combinatorics”

Joachim Kock (28/02/2022)

Most of my recent research lies at the interface between higher category theory and homotopy theory on one side, and enumerative and algebraic combinatorics on the other side. The word ‘homotopy combinatorics’ is in quotes because it it not a well-established notion. I hope it will have a much broader meaning than the topics I am involved in so far. I’ll first say something general about ∞-groupoids, and then try to be a bit more concrete concerning applications to combinatorial bialgebras.

The homotopy type of a topological space is essentially the algebraic data of all its homotopy groups (and the way these groups interact). Originally this was considered an algebraic invariant of the space, which deliberately throws away a lot of data. But gradually, through the insights of Grothendieck, Kan, Quillen, Joyal, Lurie, Voevodsky, and many others, it has become clear that homotopy types are a much more fundamental notion than topological spaces, and possibly also more fundamental than sets. In fact, people are beginning to rethink the foundations of mathematics, using homotopy types instead of sets as fundamental building blocks. The idea is that the sets we see in everyday mathematics are then considered to be just π_0, the set of ‘connected components’, of fancier objects. And that a lot of trouble in mathematics is caused by neglecting the higher structure, and that things become clearer in a natively homotopical language.

Homotopy types do not form an ordinary category, but rather an *∞-category*, the formalisation of the idea of ‘weak category’, or ‘category up to homotopy’. In this context homotopy types are presented as *∞-groupoids*. In both words, the ∞ refers to the fact that there are (potentially) infinite levels of homotopy involved. ∞-groupoids relate to ∞-categories just as sets relate to ordinary categories. The hierarchy starts with 0-groupoids, which are just sets. But most set-based structures have symmetries, and these are handled with groups and groupoids, so it is fruitful to work with groupoids instead of sets. This first step up the ladder is already very important, both historically and in daily practice. For example, it is the step that allowed Grothendieck and his school in algebraic geometry to build moduli spaces as stacks instead of schemes — stacks are essentially schemes in groupoids instead of sets. But groupoids are examples of categories, and they in turn have higher homotopies (natural transformations), which should be handled by 2-groupoids, and so on, *ad infinitum*. It quickly becomes quite complicated. However, somehow at ∞, things come together in their natural harmony, and it is actually much easier to work with ∞-groupoids than with 2-groupoids or 3-groupoids. Even to deal with ordinary 1-groupoids, often the ∞-language is the most efficient!

While the theory of ∞-categories is highly technical to bootstrap, experts at the forefront of research in algebra and geometry nowadays handle the language quite comfortably, giving the hope that there should be a more synthetic and elementary foundation. The remarkable thing is that once the technical complications have been subsumed in a sufficiently synthetic language where all concepts are replaced by their homotopy analogue, ∞-groupoids behave very much like sets! Furthermore, in various ways, the ∞-category of ∞-groupoids is actually nicer than the category of sets. First of all, it has better colimits. The trouble with colimits of sets is felt in many situations all over mathematics, often in connection with symmetries. For example, when a group acts on a set, the naive quotient *set* (an example of an old-fashioned colimit) does not have good formal properties (for example it does not interact well with pullbacks or cardinality), unless the action happens to be free. Instead the good notion is the *homotopy quotient*, where one sews in paths instead of collapsing points, as a way of keeping track of symmetries. The homotopy quotient is no longer just a set but rather a groupoid (whose π_0 is the naive set quotient). In the ∞-world, it is as if all group actions were free! This example is one illustration of the phenomenon that when the burden of symmetries is loaded off to the native homotopy formalism, things begin to look *more* discrete and more combinatorial than they do in the set-based setting. I will try to give another illustration with some more details about one particular topic: *combinatorial bialgebras*.

Combinatorial co-, bi-, and Hopf algebras serve — in many areas of mathematics — as a way to encode recursive structure. The comultiplication is generally given by splitting objects into smaller ones. A very simple example is the *chromatic Hopf algebra*: let H be the vector space spanned by the set of (iso-classes of) simple graphs, and define a comultiplication by the assignment

$$\begin{array}{rcl} \Delta : H & \longrightarrow & H \otimes H \\ G & \longmapsto & \displaystyle\sum_{A+B=V} G|A \otimes G|B. \end{array}$$

Here G is a graph with vertex set V, and the sum is over all ways of splitting V into two disjoint subsets, to which the graph structure is then restricted.

Möbius inversion is a powerful counting device in this general context, to count objects in terms of alternating sums of smaller objects; a basic example is the inclusion-exclusion principle. Algebraic renormalisation of perturbative quantum field theories is a more elaborate example closely related to Möbius inversion. The theory of incidence algebras of posets, developed by Rota in the 1960s, is a fundamental toolbox, long considered the canonical setting for Möbius inversion. Now, posets can be viewed as a special case of categories, namely those where there are no non-identity invertible arrows and at most one arrow between any two objects. Leroux (1975) generalised the fundamentals of Rota's theory from posets to categories, but it only works for a very restrictive class of categories: in particular, they are still not allowed to contain non-identity invertible arrows. The basic comultiplication law is defined on the vector space C spanned by the arrows of the category by

$$\begin{array}{rcl} \Delta : C & \longrightarrow & C \otimes C \\ f & \longmapsto & \displaystyle\sum_{f = b \circ a} a \otimes b, \end{array}$$

decomposing an arrow in all ways, and returning the two factors. Möbius inversion is given in terms of chains of non-identity arrows. This is where invertible arrows screw things up, as they allow for arbitrarily long chains.

Motivated by problems in perturbative quantum field theory, where invertible arrows abound and carry essential information, Imma Gálvez, Andy Tonks, and myself, in a series of seven papers so far, worked out a far-reaching abstraction of incidence algebras and Möbius inversion, generalising Rota–Leroux theory in three directions. First of all we upgrade from ordinary categories to ∞-categories, in the form of certain simplicial ∞-groupoids called *Rezk-complete Segal spaces.* This is already very useful, even in classical combinatorics: from an ∞-viewpoint, every category behaves as if it had no invertible arrows other than the identities, and in this way Möbius inversion suddenly applies to a much wider class of categories. For example, the category of finite sets and surjections cannot have Möbius inversion in the Leroux sense, but considered as an ∞-category it *does* have Möbius inversion. The incidence bialgebra in this case is the ubiquitous Faà di Bruno bialgebra, which is dual to composition of formal power series. In this way, ∞-categories are more poset-like than ordinary categories are!

The second direction of generalisation is the discovery that certain simplicial ∞-groupoids more general than ∞-categories admit incidence algebras and Möbius inversion. These we call *decomposition spaces.* Where category structure expresses the ability to compose, here we are concerned instead with the ability to *decompose* in a certain controlled way, tailor-made for comultiplications. There are plenty of combinatorial bialgebras in combinatorics (virtually all?) that can be realised as the incidence bialgebra of a decomposition space, but not of a category.[1] An easy example is the chromatic Hopf algebra above, which is the incidence Hopf algebra of a certain decomposition space of coloured graphs (but not of any category or ∞-category). Decomposition spaces turn out to be the same thing (but very differently formulated) as the *2-Segal spaces* of Dyckerhoff and Kapranov, discovered around the same time in homological algebra, in connection with Waldhausen's S-construction in K-theory and Hall algebras in representation theory.

The third generalisation is to work at the *objective* level instead of working with vector spaces and numbers. Everybody knows that natural numbers can be seen as cardinalities of finite sets, and that multiplication then corresponds to the cartesian product of sets, and so on. This analogy can be taken quite far, so that slice ∞-categories play the role of vector spaces and homotopy-colimit-preserving functors play the role of linear maps. The link back to numbers is given by the notion of homotopy cardinality of (sufficiently homotopy-finite) ∞-groupoids. It can be quite tricky to work at this level, but the stronger the combinatorial content of the algebra in use, the easier. The benefit is that one can actually give up finiteness conditions altogether, when required, to handle infinite quantities and divergent series quite safely. While the numerical identity $\infty = \infty$ is generally quite useless, an explicit bijection between infinite sets is often useful, and allows tranfer of structure.

The main theorems of the theory so far belong to the objective level: we establish the up-to-coherent-homotopy coassociativity of the incidence coalgebra for any decomposition space, and a general Möbius inversion principle for a wide class of decomposition spaces (including all Rezk-complete Segal spaces). These results can be seen as providing the elbow room required to use classical combinatorial tools in a wider context. Now that the basic theory has been set up, I have recently been interested in applications to various areas of mathematics where Möbius inversion and combinatorial bialgebras are used. After roaming in quantum field theory for some years, I recently ventured into free probability and into rewrite systems. Currently I am trying to apply the machinery in the theory of finite groups. . .

[1] Classical combinatorics often gets around the problem by realising these coalgebras as quotients of incidence coalgebras of auxiliary posets.