My research - in general terms

My life as a research mathematician began in Aarhus in 1972 where I wrote a small preprint with a vising professor from Japan, M. Goto; "On Intersecting Geodesics". In the end we never submitted it, but S. Helgason was nice enough to mention it in one of his books.

Around the same time, Niels Skovhuus Poulsen came back from MIT with a Ph.D. degree and a big typed manuscript by his advisor, Irving E. Segal, "The Chronometric Theory", as well as his own publications on representation theory, some still in draft form. We were three eager students, Palle Tikøb Jørgensen, Bent Ørsted, and myself, who got so inspired that we, in 1974, became part of what could be called the second wave of Aarhus students at MIT (though Palle went to Penn.). Of course, times were very favorable.

Segal had his "brown bag seminar" with many exciting people. After my Ph.D. in 1976 Segal facilitated that I got a job as assistant professor at Brandeis for three years.

At MIT I had the big privilege of working together with Michèle Vergne on 2 articles. Later I worked with Michael Harris at Brandeis.

Both MIT and Brandeis were extremely inspiring and friendly places.

After Brandeis I got a three year "Senior Stipend" at Copenhagen. Here, Mogens Flensted Jensen was instrumental. Towards the end of that period, times were suddenly not so good. The State of Denmark put a stop to hirings, but I was personally lucky. A group of people wanted to mark the centennial of Niels Bohr in 1985 in different ways. One was to create 15 3-year stipends, funded by Industry and private funds, in all of science, and they were meant to *terminate* in 1985. I got one of those!

Even before the stipend expired, things began to loosen up, and I became Lektor at Copenhagen in November 1984.

Though there have been several interesting deviations over the years - collaboration with Victor Kac, Hechun Zhang, and others - one recurring topic for my research has been representation theory. Specifically, unitarity and "covariant differential operators".

At MIT, right from day 1, Segal pointed to the importance of the Wave operator, given in its flat or, equivalent, curved versions as

$$\Box_F = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \qquad \text{and} \qquad \Box_C = \frac{\partial^2}{\partial \tau^2} - \nabla_{S^3} + 1,$$

respectively. Segal was only interested in the conformal group $SU(2,2) \simeq SO(4,2)$, but I became interested in more general groups connected with Hermitian symmetric spaces of the non-compact type, especially $SU(n,n)$ amd $Mp(n,\mathbb{R})$.

Let us say that a (matrix) differential operator $\mathcal{D}$ is covariant with respect to some group G if there are two representations U_1, U_2 of G is some nice spaces $\mathcal{H}_1^o, \mathcal{H}_2^o$ of functions with values in finite-dimensional complex vector spaces V_{τ_1}, V_{τ_2}, such that $\mathcal{D} : \mathcal{H}_1^o \to \mathcal{H}_2^o$, and such that

$$\forall g \in G, \forall f \in \mathcal{H}_1^o : \mathcal{D}(U_1(g)f) = U_2(g)(\mathcal{D}f).$$

It was, and still is, of interest to determine such covariant differential operators, but at the same time it was of interest to determine the range of unitarity for holomorphically induced representations. Here luck really strikes, because to determine the range of unitarity was tantamount to finding the most singular unitary representations, the singularity being measured by how many K-types that are missing in these representations compared to "generic" ones. The missing K-types can be seen as giving rise, by duality, to homomorphisms between generalized Verma modules

$$\phi_{\tau_2',\tau_1'} : \mathcal{M}(V_{\tau_2}') \to \mathcal{M}(V_{\tau_1}'),$$

where, quite generally,

$$\mathcal{M}(V_\Lambda) = \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathcal{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)} V_\Lambda,$$

and V' denotes the dual space. One uses here a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+$ of the complexification any simple Lie algebra $\mathfrak{g}$ connected with a Hermitian symmetric space of the non-compact type. In our case, $\mathfrak{g} = su(n,n)$.

The big point is that the transposed of such a ϕ is a covariant differential operator $\mathcal{D}_\phi$.

Actually, there are many more covariant differential operators than those that arise in connection with unitarity, but the covariant differential operators for which the (clearly invariant!) subspace $\mathcal{D}f = 0$ carries a unitary representation are believed to be much more fundamental.

As an example, consider $SU(n,n)$. Generalized Verma modules can here be realized as vector valued functions on the set $M(n,\mathbb{C})$ of $n\times n$ complex matrices W. For a specific set of 1-dimensional representations τ_1, τ_2 of $\mathfrak{k}^{\mathbb{C}}$, the *determinant*, $\det W$, yields a $\mathcal{U}(su(n,n)^{\mathbb{C}})$ homomorphism

$$\mathcal{M}(V'_{\tau_2}) \ni f(W) \to \det W f(W) \in \mathcal{M}(V'_{\tau_1}).$$

The corresponding covariant differential operator is then $\det(\frac{\partial}{\partial Z_{ij}})$. In the case of 2×2 matrices we thus get

$$\frac{\partial}{\partial Z_{11}}\frac{\partial}{\partial Z_{22}} - \frac{\partial}{\partial Z_{21}}\frac{\partial}{\partial Z_{12}} \simeq \Box_F,$$

that is, The Wave Operator.

Before looking at one more example, let us move to the q world, say $\mathcal{U}_q(su(n,n)^{\mathbb{C}})$: One big challenge is here that polynomial algebras are replaced by quadratic (non-commutative) algebras which means that there is a need for distinguishing between right and left operators. Leaving this aside, it is actually possible to generalize the above picture. Especially the generalized Verma modules are straightforward to "quantize". In the case of 2×2 matrices one gets two quadratic algebras, one of which is generated by w_1, w_2, w_3, w_4 and relations:

$$w_1 w_i = q w_i w_1 (i=2,3), w_i w_4 = q w_4 w_i (i=2,3), w_2 w_3 = w_3 w_2, \text{ and}$$
$$w_1 w_4 - w_4 w_1 = (q - q^{-1}) w_2 w_3.$$

The q-analogue of $\det W$ is $\det_q W = w_1 w_4 - q w_2 w_3$. The analogue of the Dirac operator (in its guise as a homomorphism) is given as follows: Set

$$\mathbb{D} = \begin{pmatrix} w_1 & q w_2 \\ w_3 & q w_4 \end{pmatrix}, c(\mathbb{D}) = \begin{pmatrix} w_4 & -q^{-1} w_2 \\ -w_3 & q^{-1} w_4 \end{pmatrix}, \text{ and } \not{\nabla}_q = \begin{pmatrix} 0 & \mathbb{D} \\ c(\mathbb{D}) & 0 \end{pmatrix}.$$

Then

$$\not{\nabla}_q^2 = \det_q W \cdot I_4.$$

There is another theme, which is related, namely how one can get very interesting representations of eg $SU(n,n)$ from the canonical commutation relations. Actually, the original proof of the Kashiwara-Vergne Conjecture, as furnished by myself, tells much of the story. But not all: In connection with the study of a natural setting for quantized differential operators, including both left and right multiplication operators as well as duals of such, I have recently been led naturally to a quantum analogue $\mathcal{W}_q(n,n)$ of the Weyl algebra which is a collection of n^2 commuting algebras, each equal to $\mathcal{W}_q(1,1)$, and where the latter is generated by two operators D, M and relations

$$DM - qMD = H^{-1} \qquad DM - q^{-1}MD = H$$
$$HD = q^{-1}DH \qquad HM = qMH.$$

This is a very good candidate for a quantum Heisenberg algebra (studied by T. Hayashi), but the big point is that this appears at the output level! Much work still needs to be done on this, work that will also include my other recent interest; quantum cluster algebras.