My research: spectral universality

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In the 1950s Eugene Wigner faced a problem that is often encountered when analysing high dimensional disordered systems. He wanted to understand the energy spectra of heavy nuclei. However, the physics are too complicated to be analysed numerically to the necessary accuracy even with modern computational methods. Furthermore, at the time the relevant interactions were not sufficiently well understood. His bold solution to this problem was to replace the detailed microspcopic interactions by random ones and to determine the spectrum of the resulting random matrix model instead. Mathematically this is a much more approachable problem. Behind his idea lies the deep insight that across a vast range of different systems certain statistical properties of spectral information are universal, i.e. they do not depend on model details. This line of thinking has lead to numerous successful applications of random matrix theory (RMT) in fields as diverse as communication and number theory, condensed matter physics, neural networks and statistics. My research is focussed on developing robust mathematical tools for identifying and establishing spectral universality phenomena for a wide range of models that are often motivated by applications from physics or engineering.

The simplest random matrix models have entries that all follow the standard normal distribution and are otherwise independent up to specific symmetry constraints. For example, the self-adjoint $n\times n$-random matrix $H=(h_{ij})$ with $h_{ij}=\overline{h}_{ij}$ is said to belong to the Gaussian orthogonal ensemble (GOE) if its entries are real valued normal random variables and to the Gaussian unitary ensemble (GUE) if they are complex valued. In both cases the joint distribution $p_n^{(n)}(x_1,\dots,x_n)$ of all n eigenvalues and their k-point marginals or k-point correlation functions

$$p_k^{(n)}(x_1,\dots,x_k) := \int \mathrm{d}x_{k+1}\dots\mathrm{d}x_n\, p_n^{(n)}(x_1,\dots,x_n)$$

are explicitly computable. In particular, the 1-point function $\rho(x) := \lim_{n\to\infty} p_1^{(n)}(x)$ follows the celebrated semicircular distribution when the matrix is scaled so that $\mathbb{E}|h_{ij}|^2=\frac{1}{n}$. Although the eigenvalue density ρ depends on the model under consideration, the **local spectral statistics**

$$\rho_k(y_1,\dots,y_k) := \lim_{n\to\infty}\frac{1}{\rho(x)^k}\, p_k^{(n)}\left(x_1+\frac{y_1}{n\rho(x)},\dots,x_k+\frac{y_k}{n\rho(x)}\right), \qquad (1)$$

rescaled to the typical eigenvalue spacing scale $\frac{1}{n\rho(x)}$, are universal across a wide range of models. The functions ρ_k are conjectured to reflect the joint statistical behaviour of k eigenvalues around a fixed point x in the bulk of the spectrum for systems as diverse as chaotic quantum billiards, the hypothetical Hilbert-Pólya operator that determines the non-trivial zeros of the Riemann zeta functions, as well as the Anderson Hamiltonian, describing the motion of a quantum particle in a random potential.

An overarching goal of my research is to identify the main mechanisms that lead to spectral universality and to use such insights to prove (1) for models that go far beyond the exactly solvable GOE and GUE. The first step towards this goal is to control the eigenvalues $\lambda_1,\dots,\lambda_n$ of H on mesoscopic scales just above their typical spacing distance. More precisely, the **local law**

$$\frac{1}{n}\#\{i:\lambda_i\in[a_n,b_n]\}\approx\int_{a_n}^{b_n}\rho(x)\,\mathrm{d}x\,, \qquad (2)$$

asserts that with high probability the deterministic density $\rho(x)$ correctly predicts the number of eigenvalues in any interval $[a_n,b_n]$ as long as $b_n-a_n\gg\frac{1}{n}$. Such statement also indicates the strong correlation among the eigenvalues since we expect (2) only in the regime $b_n-a_n\gg\frac{1}{\sqrt{n}}$ if λ_i are assumed to be independent. Due to the relation

$$\frac{1}{n}\operatorname{Tr}G(z)=\int\frac{\mu_n(\mathrm{d}x)}{x-z}\,, \qquad z\in\mathbb{C}\setminus\operatorname{Spec}(H)\,,$$

between the resolvent $G(z):=(H-z)^{-1}$ and the Stieltjes transform of the empirical spectral distribution $\mu_n=\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i}$, proving a local law is equivalent to controlling $\frac{1}{n}\operatorname{Tr}G(z)$ in the limit $n\to\infty$ in the regime $\operatorname{Im}z\gg\frac{1}{n}$. In fact, for a large class of Hermitian matrices the fluctuation of every entry of the resolvent vanishes as $n \to \infty$, showing that the resolvent $G = (g_{ij})$ is well approximated by a deterministic matrix M. The following informal theorem identifies the matrix M as the solution to a self-consistent equation, the **matrix Dyson equation** (3). The main assumption on H is that the correlations among its entries decay sufficiently rapidly to ensure that not too many entries are determined by the same source of randomness. Otherwise the entry distributions are very general. In particular, they do not have to be Gaussian.

THEOREM (Resolvent of random matrices with correlated entries). *Let H be a random matrix with decaying correlations among its entries and $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Then all entries of the resolvent satisfy $g_{ij}(z) - m_{ij}(z) \to 0$ with high probability as $n \to \infty$, where $M = (m_{ij})$ solves the matrix equation*

$$M(z) = \frac{1}{A - z - \mathcal{S}M(z)} . \tag{3}$$

Here, $A := \mathbb{E}H$ is the expectation of H and the linear operator $\mathcal{S} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is determined by the covariances of the entries of H through

$$\mathcal{S}X := \mathbb{E}(H - \mathbb{E}H)X(H - \mathbb{E}H) .$$

The non-linear high dimensional matrix equation (3) is used to infer properties about the spectrum of H. It is obtained via a renormalisation procedure that remains valid when the spectral parameter z is chosen n-dependent, as long as $\operatorname{Im} z \gg \frac{1}{n}$. As a consequence, the local law (2) holds with the spectral density ρ implicitly defined in terms of the solution M through the relation

$$\frac{1}{n} \operatorname{Tr} M(z) = \int_{\mathbb{R}} \frac{\rho(x)}{x - z} \, \mathrm{d}x .$$

Despite its high dimensionality the algebraic equation (3) enforces surprisingly strong regularity properties on ρ. Under some additional assumptions on the data A and $\mathcal{S}$ in (3), the bulk regime $\{x : \rho(x) > 0\}$ consists of finitely many disjoint open intervals, the spectral bands. Inside each band ρ is real analytic. The boundaries of these intervals are called spectral edges and ρ has a one sided square root growth behaviour at these points. Only when two bands touch a cusp singularity emerges with a cubic root growth on both sides. Put differently, the singularities of the spectral density ρ are all algebraic of degree two or three (see Figure 1).

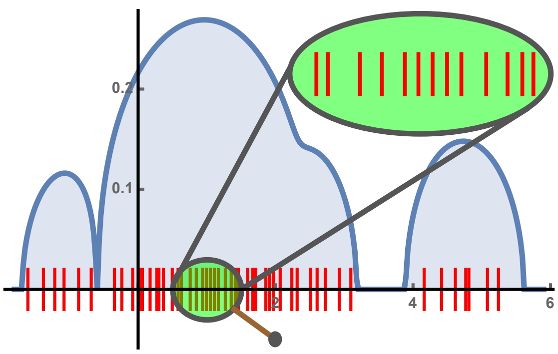

Figure 1: Eigenvalues & Spectral density

The local law for random matrices with general decaying correlations among their entires provides enough control on the eigenvalues to establish local spectral universality in the bulk. Furthermore, the two singularity types of the spectral density ρ are accompanied by their own local universality class.

THEOREM (Local spectral universality). *1) For a large class of random matrices with decaying correlations among their entires the local k-point correlation functions* (1) *coincide with the ones from the GOE or GUE, depending on whether the matrix entries are real or complex valued.*
2) Across the entire model class the distribution of all edge eigenvalues also coincides with the GOE/GUE edge distribution (Tracy-Widom).
3) For random matrices with independent entries in the complex Hermitian symmetry class the local eigenvalue statistics around any cusp singularity is given by the Pearcey process.

Despite such detailed understanding of the universality classes for a large range of random matrix models, it remains a mystery why spectral universality extends to systems with much lower degrees of disorder. For the above theorem to be applicable to an $n \times n$-matrix H, the randomness that generates H has to span all its degrees of freedom, i.e. it has to span an n^2-dimensional space. On the other hand, the universal GOE eigenvalue statistics are observed also e.g. for the 3-dimensional Anderson model whose randomness is restricted to the diagonal, i.e. it spans an n-dimensional space. Reducing the gap of mathematical understanding between these models is a major motivation for my research.