MINIMAL SURFACES AND GEOMETRIC FLOW SOLITONS

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1. Minimal Surfaces Everywhere

Minimizing the area of an interface between two regions in a geometric space is a fundamental problem which arises naturally in many mathematical contexts, pure or applied, regardless of scale or finer properties of the system. Thus in our own physical universe it is a common feature shared between such different objects as the apparent horizons of black holes, certain (hypothetical) structures in neutron stars, the colorful wings of birds and butterflies, and inside the human cell as well the organelles called endoplasmic reticula as the tiny DNA molecules (see Figure 1). All of these have in common that they are area-minimizing structures[1]: *Minimal surfaces.* It means by definition being a critical point of the area functional under any local variation, and is equivalent to, for an oriented, smooth hypersurface $\Sigma^n \subseteq (M^{n+1}, g)$ in an ambient Riemannian manifold, the local curvature condition:

$$H_\Sigma = 0, \quad \text{for} \quad H_\Sigma = \sum_{i=1}^{n} \kappa_i.$$

Here the κ_i denote the principal curvatures of the embedding (or immersion) $\Sigma^n \to M^{n+1}$. The appearance of minimal surfaces in so many places is quite similar to the way that the harmonic oscillator potential $V(x) = c|x|^2$ is ubiquitous in (low energy) physics, as a lowest order approximation near equilibria of more complicated true energy functionals that one might consider.

2. Spaces in Motion: Geometric Flows

The notion of minimality is a static one. To consider a dynamical process which potentially can create such surfaces, the canonical candidate is the mean curvature flow, being defined as the fastest, namely *steepest descent*, way to locally decrease the area of a given hypersurface in space.

Before going into more detail, let us digress to explain how a deeper reason for the usefulness of such geometric flows is that the flowed objects can have much improved properties. In a best case scenario, the process can be continued all the way to a *canonical* situation, where one can then relate backwards to obtain highly nontrivial information about the original state. One basic example where this works out is the flow proof of the *isoperimetric inequality*, which states that among all curves of unit area, round circles attain the least circumference possible[2]:

$$(\text{Length of } \gamma)^2 \geq 4\pi(\text{Area enclosed by } \gamma), \quad \text{for any closed planar curve.} \tag{2.1}$$

Flow Proof: Flow to a round circle (of some radius R), our canonical object, where the (in-)equality is very easy to prove, namely $(2\pi R)^2 = 4\pi(\pi R^2)$. (NB: The inequality behaves well along the flow).

Going beyond this simple example, geometric flows involving curvature have already had many much more striking applications in finding out the possible shapes and features of space, as understood both in the most general mathematical sense of topology (geometrization of 3-manifolds, Poincaré's conjecture) and in the manifestly physical sense of space in the theory of general relativity (Positive mass theorems/Penrose's inequality, entropy of black holes), whereby these ideas and techniques have already very much proven their worth. Where the Ricci flow works well for the intrinsic geometric setting, mean curvature flow is a natural candidate for questions involving submanifolds.

To get back to specifics, the mean curvature flow is formulated as a nonlinear partial differential equation, which turns out to be very similar in nature to (in fact is a nonlinear version of) that which governs the flow of heat in a material, the parabolic *heat equation*:

$$\frac{\partial f}{\partial t} = \Delta f,$$

[1] In case you wonder, the names of examples of "nature's own" minimal surfaces here are, respectively: Spheres (in a positively curved background), helicoids, gyroids, helicoids and helicoids. (See also Figure 1.)

[2] However, in the opposite direction, recall that by taking e.g. $\gamma_L = \partial([0, L] \times [0, L^{-1}])$, and letting $L \to +\infty$, the RHS in (2.1) can stay constant while the LHS goes to infinity.

where $f = f(x,t)$ and Δ is the Laplace-Beltrami operator in the x-variables (on a Riemannian manifold). For yet more details, the mean curvature flow means to consider a family of immersions $X : M^n \times [0,T) \to N^{n+1}$, between Riemannian (oriented) manifolds M^n and N^{n+1} solving (we write $\Sigma_t := X(M^n,t)$ and think of the right hand side as a nonlinear version of the Laplacian):

$$\frac{\partial X}{\partial t} = -H_{\Sigma_t}\nu_{\Sigma_t}, \tag{2.2}$$

Here H again denotes the mean curvature, X the position vector and ν_{Σ_t} a normal unit length vector field. That is: We "push" points at (signed) velocities faster or slower proportionally to their curvature.

The flow of heat is a classical subject with a rich and well-developed theory. Drawing in large part on this, the key foundational results concerning mean curvature flow are now known, and powerful applications have started appearing, yet many basic questions remain unanswered: Namely, many very complicated behaviors can arise as we try to flow our way towards a final minimal hypersurface.

3. Singularities and Solitons in Geometric Curvature Flows

The main topics that I'm currently interested in thus concern two central notions for geometric flows: *long-time behavior* and *singularity formation*. The notion of *solitons* of the flow, which means the surprising emergence of special evolutions which retain a fixed profile for all time, is intimately connected with the set of possible singularities.

Historically, such soliton behavior for a nonlinear partial differential equation was first observed by Scottish naval engineer John Scott Russell in 1834, for solitary wave solitons in a shallow water canal: This "translating wave" of water was so stable in shape that he could follow it by horseback long enough to determine its speed to be approximately 13 km/hour.

The idea that singularities correspond to solitons has the following precise mathematical content: Singularities for mean curvature flow will, under appropriate (quite general) geometric circumstances that allow for parabolic flow rescalings, lead to a hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ satisfying the self-shrinking soliton equation, where ν_Σ is a unit field normal to Σ^n (after moving the shrinking center to the origin and scaling the time it takes to go singular to $T = 1$):

$$H_\Sigma = -\frac{\langle X, \nu_\Sigma\rangle}{2}, \quad X \in \Sigma^n \subseteq \mathbb{R}^{n+1}. \tag{3.1}$$

This is a time-independent, nonlinear (in fact: quasi-linear) elliptic partial differential equation involving the mean curvature. (See Figure 2, and check the equation for the round spheres of appropriate radii, if you wish). The geometric solitons here turn out to be minimal hypersurfaces, but in a conformally changed background (via Gaussian densities that come from the heat kernel).

Despite the role as "atoms" for the singularity theory, the exhaustive list of the rigorously known examples of self-shrinking solitons in $\mathbb{R}^{n+1}$ is very short, the simplest (and only stable) ones being "round" cylinders $\mathbb{R}^{n-k} \times \mathbb{S}^k \subseteq \mathbb{R}^{n+1}$. In my own work, I have constructed new examples that have arbitrarily high genus and still satisfy all the reasonable and desirable geometric properties one imposes, such as completeness and embeddedness (see Figure 2), by using gluing techniques for minimal surfaces to "fuse" known examples. Another project, using similar techniques, is to give constructive answers to some old questions on classical minimal surfaces in $\mathbb{R}^3$ (see Figure 3).

To study the geometric solitons one needs to develop the corresponding exact geometric analysis tools applicable to such problems, including for stability operators (i.e. the Hessians of the functionals). F.ex. one is lead to many hard questions in the detailed theory of Schrödinger operators (as in Quantum Mechanics) with geometric content, including those of Ornstein-Uhlenbeck-Witten type, meaning with drift terms, on noncompact manifolds (often with prescribed asymptotical ends):

$$\mathcal{L} = \Delta_g + Y \cdot \nabla_g + \mathcal{K}. \tag{3.2}$$

Here Y is a vector field and $\mathcal{K}$ a potential term involving curvatures (the second fundamental form and the Ricci curvature). Both Y and $\mathcal{K}$ can be quite sizeable and hence exert a nontrivial influence on the analysis. This, (3.2), is closely related to Riemannian manifolds with density (M^n, g, φ).

The other side of the coin is to understand which geometric flow singularity shapes are actually "forbidden", e.g.: Which are the computable, *a priori* criteria which obstruct their existence? Thus, we try painting the "*obstructions versus constructions*" roadmap for navigating the intricate landscape of mean curvature singularities. But that's a story for another (singular?) time!

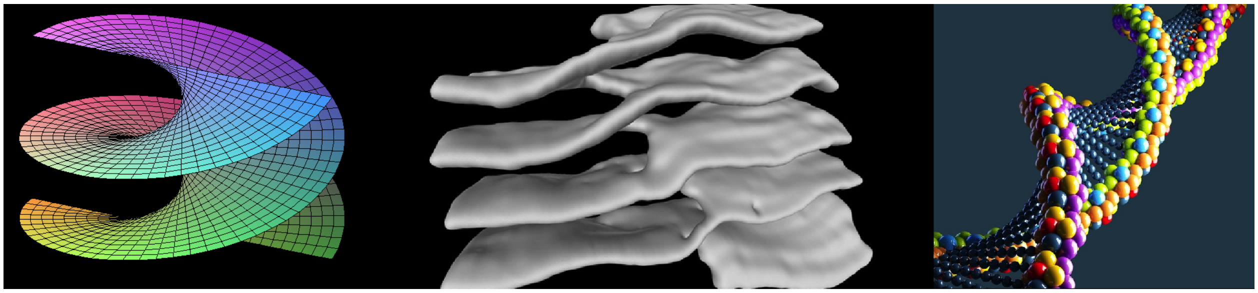

FIGURE 1. Minimal surfaces in theory and in nature: The helicoid, the endoplasmic reticulum and the DNA molecule.

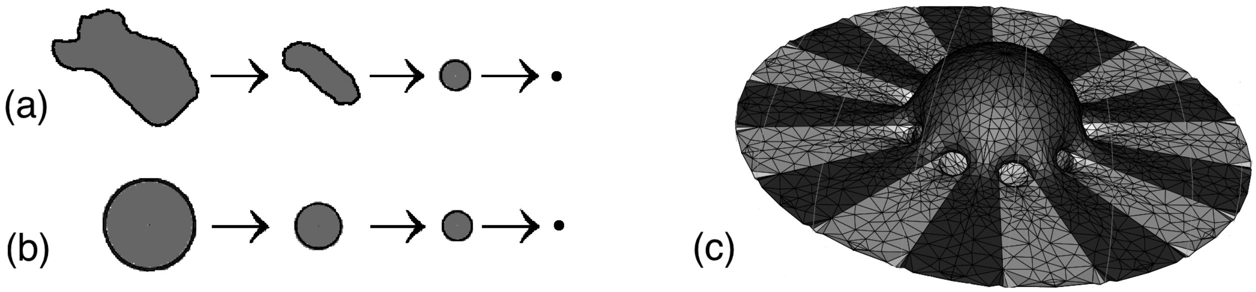

FIGURE 2. (a) Surface becoming increasingly round under mean curvature flow, while shrinking in size. (b) Round spheres are simple "self-shrinker" solitons: Retaining shape, shrinking by homothety until disappearing in a point singularity. (c) Genus g self-shrinking solitons Σ_g like the ones constructed in my work (found numerically by Tom Ilmanen in 1994).

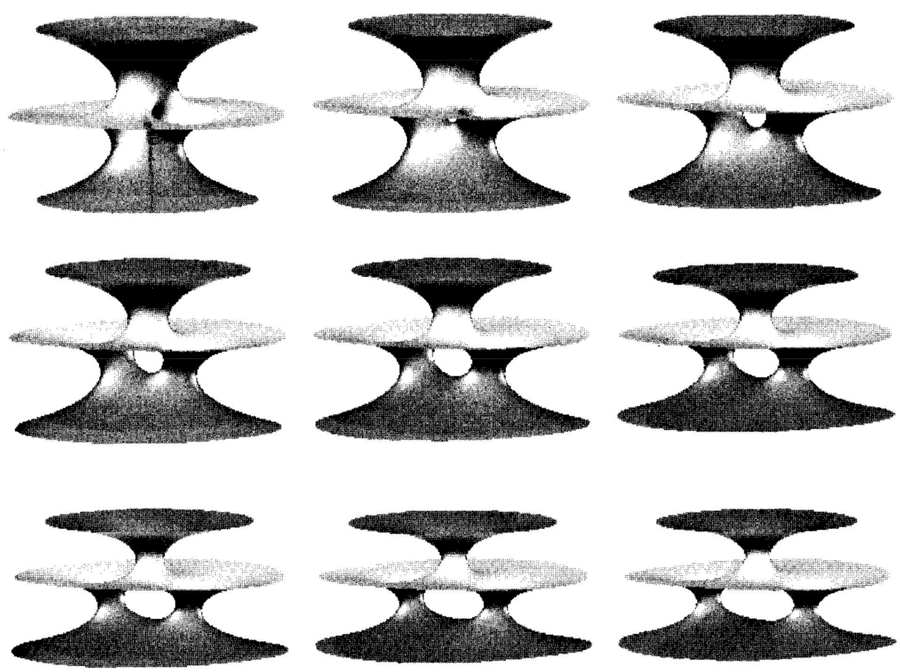

FIGURE 3. Family of surfaces in $\mathbb{R}^3$ exhibiting noncompactness of the moduli space of complete embedded minimal surfaces with genus g, with r ends and of finite total Gauß curvature. Are there such examples for all values of (g, r)?