My research

The key words should be: C*-algebras, K-theory (originally topological but now also algebraic), cyclic homology, deformation theory, index theory, quantum groups...

The "index theory" part involves most of these terms, so this note will be devoted to:

Index theorems

The term *index theorems* is usually used to describe the equality of, on one hand, analytic invariants of certain operators on smooth manifolds and, on the other hand, topological/geometric invariants associated to their “symbols". A convenient way of thinking about this kind of results is as follows.

One starts with a C^*-algebra of operators A associated to some geometric situation and a K-homology cycle (A, π, H, D), where $\pi\colon A \to B(H)$ is a $*$-representation of A on a Hilbert space H and D is a Fredholm operator on H commuting with the image of π modulo compact operators $\mathcal{K}$. The explicit choice of the operator D typically has some geometric/analytic flavour, and, depending on the parity of the K-homology class, H can have a $\mathbb{Z}/2\mathbb{Z}$ grading such that π is even and D is odd.

Given such a (say even) cycle, an index of a reduction of D by an idempotent in $A \otimes \mathcal{K}$ is the integer

$$Ind(eD \otimes 1e) = dim\,\mathrm{Ker}(e(D \otimes 1)e) - dim\,\mathrm{Coker}(e(D \otimes 1)e).$$

This define pairing of K-homology and K-theory, i. e. the group homomorphism

$$KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \longrightarrow \mathbb{Z}. \tag{1}$$

One can think of this as a Chern character of D defining a map

$$K_0(A) \longrightarrow \mathbb{Z},$$

and the goal is to compute it explicitly in terms of some topological data extracted from the construction of D.

Basic example - the theorem of Atiyah and Singer

$A = C(X)$, where X is a compact manifold and D is an elliptic pseudodifferential operator acting between spaces of smooth sections of a pair of vector bundles on X. The number $< ch(D), [1] >$ is the Fredholm index of D, i. e. the integer

$$Ind(D) = \dim(Ker(D)) - \dim(Coker(D))$$

and the Atiyah–Singer index theorem identifies it with the evaluation of the $\hat{A}$-genus of T^*X on the Chern character of the principal symbol of D.

The functional $ch(D)$ on K-theory of A descends to a cyclic periodic cocycle and the pairing of cyclic periodic homology with cyclic periodic cohomology. Cyclic periodic (co-)homology is a version of the de Rham cohomology adopted to a not necessarily commutative algebra A.

Suppose again that X is a smooth manifold. A more general class of representatives of K-homology classes of $C(X)$ is given by operators on $L^2(X)$ of the form

$$D = \sum_{\gamma \in \Gamma} P_\gamma \pi(\gamma).$$

Here Γ is a discrete group with a representation π on $L^2(X)$ by Fourier integral operators of order zero and P_γ is a collection of pseudodifferential operators on X, all of them of the same (non-negative) order.

As in the case of (pseudo-) differential operators, there is again a notion of the principal symbol,

$$\sigma_\Gamma(D) = \sum_{\gamma \in \Gamma} \sigma_p(P_\gamma)\gamma$$

This time as an element of the C^*-algebra $C(S^*X) \rtimes_{max} \Gamma$. Here S^*X is the cosphere bundle of X and the subscript "max" $in \rtimes_{max}$ means that we use the largest possible C*-norm. Invertibility of $\sigma_\Gamma(D)$ implies that D is Fredholm and the index theorem in this case would express the integer $Ind_\Gamma(D)$ in terms of some equivariant cohomology classes of X and an appropriate equivariant Chern character of $\sigma_\Gamma(D)$.

A typical computation of index proceeds via a reduction of the algebra of operators D under consideration to an algebra of (complete) symbols,. In the case considered in the theorem of Atiyah and Singer this is just the quotient of pseudodifferential operators by the smoothing operators. This algebra which can be thought of as a "formal deformation" $\mathcal{A}^\hbar$ of $C^\infty(T^*X)$. What this means is the following.

A formal deformation quantization of a symplectic manifold (M, ω) *is an associative* $\mathbb{C}[[\hbar]]$*-linear product* $\star$ *on* $C^\infty(M)[[\hbar]]$ *of the form*

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \sum_{k \geq 2} \hbar^k P_k(f, g);$$

where $\{f,g\}$ is the canonical Poisson bracket induced by the symplectic structure and P_k are given by bidifferential operators.

We will use $\mathcal{A}^\hbar(M)$ to denote the algebra $(C^\infty(M)[[\hbar]],\star)$. The ideal $\mathcal{A}^\hbar_c(M)$ in $\mathcal{A}^\hbar(M)$, consisting of formal power series of the form $\sum_k \hbar^k f_k$, where f_k are compactly supported, has a unique (up to a normalization) trace Tr with values in $\mathbb{C}[\hbar^{-1},\hbar]]$.

The product in $\mathcal{A}^\hbar_c(M)$ is local and the computation of the pairing of K-theory and cyclic cohomology of $\mathcal{A}^\hbar_c(M)$ reduces to a differential-geometric problem and the result of the resulting computation is the "algebraic index theorem". The computation of the index of the operator of the type $\sum_{\gamma\in\Gamma} P_\gamma \pi(\gamma)$ as above reduces to a computation of the pairing of K-theory and cyclic cohomology for the algebra $\mathcal{A}^\hbar_c(M) \rtimes \Gamma$, where $M = T^*X$.

Cyclic periodic homology (just like de Rham homology) is homotopy invariant and i, the result of the pairing depends only on the $\hbar = 0$ part of the K-theory of $\mathcal{A}^\hbar_c(M)\rtimes\Gamma$. The $\hbar = 0$ part of the full symbol algebra $\mathcal{A}^\hbar_c(M)\rtimes\Gamma$ is just $C^\infty_c(M) \rtimes \Gamma$, hence the Chern character of D, originally an element of K-homology of the $C(M)$, enters into the final result only through a class in the equivariant cohomology $H^*_\Gamma(M)$.

Let us describe the result. Given a group cocycle $\xi \in H^k(\Gamma,\mathbb{C})$,

$$\begin{aligned}&Tr_\xi(a_0\gamma_0 \otimes \ldots \otimes a_k\gamma_k) =\\ &\delta_{e,\gamma_0\gamma_1\ldots\gamma_k}\xi(\gamma_1,\ldots,\gamma_k)Tr(a_0\gamma_0(a_1)\ldots(\gamma_0\gamma_1\ldots\gamma_{k-1})(a_k)).\end{aligned}$$

defines a cyclic cocycle on $\mathcal{A}^\hbar_c(M) \rtimes \Gamma$. The corresponding formula produces a cyclic cocycle $\Phi(\xi)$ on the algebra $C^\infty_c(M) \rtimes \Gamma$ is given by

$$\begin{aligned}&\Phi(\xi)(f_0\gamma_0 \otimes \ldots \otimes f_k\gamma_k) =\\ &\delta_{e,\gamma_0\gamma_1\ldots\gamma_k})\textstyle\int_M(\alpha_\xi a_0 d\gamma_0(a_1)\ldots(d\gamma_0\gamma_1\ldots\gamma_{k-1})(a_k))\end{aligned}$$

where α_ξ is a certain closed differential form on M associated to ξ.

Γ-equivariant index theorem
Let e, $f \in M_N\left(\mathcal{A}^\hbar(M)\right)$ be a couple of idempotents such that the difference $e-f \in M_N\left(\mathcal{A}^\hbar_c(M) \rtimes \Gamma\right)$ is compactly supported. Then $[e]-[f]$ is an element of $K_0(\mathcal{A}^\hbar_c(M) \rtimes \Gamma)$ and its pairing with the cyclic cocycle Tr_ξ is given by

$$< Tr_\xi, [e]-[f] >= \left\langle \Phi\left(\hat{A}_\Gamma e^{\theta_\Gamma}[\xi]\right), ch([\sigma(e)]-[\sigma(f)])\right\rangle . \qquad (2)$$

Here $\hat{A}_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant $\hat{A}$-genus of M, $\theta_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant characteristic class of the deformation $\mathcal{A}^\hbar(M)$.

This particular result is joint work with Alexander Gorokhovsky and Niek de Klejin.