Term Structure Models

by David Skovmand

My research has mainly involved term-structure problems in mathematical finance. A term structure model is a mathematical description of the risk in a market where you can buy (or sell) the right (or obligation) to receive (or pay) the same quantity of some asset, unit of currency, interest rate or other financial variable, at different time points in the future. The simplest example being a market for a series of bonds that expire at different time points in the future. My research has focused on building mathematical models that enable the users to quantify and manage this risk when investing in these markets. Below is a description of one approach to modelling a term structure and some of the challenges involved.

Assessing whether a model for a financial market is correctly specified is inherently an empirical question. However, the model should not only be consistent with empirical observation (say having maximum likelihood) but should also be free of *arbitrage* - meaning the proverbial 'free lunch' must not exist within the market generated by the model. We can formalize the no-arbitrage restriction by putting requirements on the so called **pricing kernel**. The pricing kernel is a stochastic process $\{\pi_t\}_{\{0\leq t\}}$ that determines both the time-value of money and the value of risk in a market. If we let $\{\mathcal{F}_t\}_{0\leq t}$ be a filtration measuring the information flow in the market, then for a generic financial asset with an $\{\mathcal{F}_t\}_{0\leq t}$-adapted price process $\{S_{tT}\}_{0\leq t\leq T}$, which is characterised by a cash flow S_{TT} at the fixed date T the no-arbitrage pricing formula at time t is:

$$S_{tT} = \frac{1}{\pi_t}\mathbb{E}[\pi_T S_{TT} \,|\, \mathcal{F}_t].$$

We can see that the fair price at time t is not merely the expectation of the future cash flow S_{TT} but also involves adjustment for risk and time value of money via the pricing kernel $\{\pi_t\}_{0\leq t}$. A particularly simple asset is the *zero coupon bond*, which has a certain payoff at time T of just one. We can write the no-arbitrage price of the zero coupon bond as:

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}[\pi_T \,|\, \mathcal{F}_t]. \tag{1}$$

P_{tT} thus corresponds to the value you must invest at time t to receive one unit of currency at time T. If we (for simplicity) assume that interest rates are positive and cash is available in the economy it is clear that that we must have, $0 < P_{tT} \leq 1, \forall t \in (0,T)$. For example if $P_{tT} > 1$, one could just sell the bond at time t, place the money in cash, and a earn a risk free profit of $P_{tT} - 1 > 0$, at time T without any initial investment – in other words an *arbitrage*. Thus it follows from (1) above that under these assumptions a necessary condition to preclude arbitrage is that $\{\pi_t\}_{0\leq t}$ must be a non-negative supermartingale.

In recent research projects we have considered a specific class of models for the pricing kernel where :

$$\pi_t = \pi_0(P_{0t} + y(t)X_t), \quad X_0 = 0.$$

$\{X_t\}_{0\leq t}$ is assumed to be a martingale and the function y is assumed deterministic and non-decreasing. We can think of X_t as determining the state of the economy at time t and the market filtration as being generated by $\{X_t\}_{0\leq t}$. From the martingality of $\{X_t\}_{0\leq t}$ we can write the zero coupon bond prices as rational functions of this stochastic process:

$$P_{tT} = \frac{P_{0T} + y(T)X_t}{P_{0t} + y(t)X_t}.$$

We can consider the above model as one that takes an initial observable set of bond prices $\{P_{0t}\}_{0\leq t}$ and sets them in motion via the stochastic process $\{X_t\}_{0\leq t}$.

My recent research projects has dealt with various ways of selecting the proper model for the $\{X_t\}_{0\leq t}$ process. Pinning down this process can be done by looking at historical prices of bonds, but also more structural approaches like including economic indicators such as GDP and inflation can be used. But more often than not the users primarily demand that the model matches a set of observed prices of interest rate *derivatives*. An example would be an interest rate *caplet*. Investors buy this instrument to hedge against increases in interest rates at some future point in time. Let $L(T, T+\delta)$ be a $\mathcal{F}_T$-measurable random variable denoting the annual rate you have to pay on a loan that starts at time T and has to be paid back at time $T+\delta$. The caplet contract pays the following at time $T+\delta$:

$$h_{T+\delta} = N \times \max\left[L(T, T+\delta) - K, 0\right].$$

In this case the contract pays out a positive amount $L(T, T+\delta) - K$, (scaled by the notional value N) if the interest rate rises above K. If the interest rate falls below K it pays zero. We can price this contract by noting first that the rate can expressed in terms of zero coupon bond prices using the no-arbitrage principle:

$$L(T, T+\delta) = \frac{1 - P_{T,T+\delta}}{\delta P_{T,T+\delta}}.$$

We can then price the caplet contract at $t = 0$ by solving the following expectation, which turns out to be particularly simple given the above model structure:

$$h_0 = \frac{1}{\pi_0}\mathbb{E}[\pi_{T+\delta}h_{T+\delta}|\mathcal{F}_0] = N\mathbb{E}[\max(a + bX_T, 0)|\mathcal{F}_0], \tag{2}$$

where a and b are known constants ($a = 1/\delta P_{0,T} - \tilde{K}P_{0,T+\delta}$, $b = y(T)/\delta - \tilde{K}y(T+\delta)$, $\tilde{K} = 1/\delta + K$).

The market for interest rate derivatives is by far the largest derivative market in the world measured in market value. On any given day thousands of different contracts like the one above are traded, both for different values of K and T, but also other and much more complicated contract specifications exist. By observing their prices you may hope to infer certain properties of the marginal distributions of the process $\{X_t\}_{0\leq t}$ (say scaling behavior of moments) as well as infer the function $y(T)$. This procedure is broadly known as *calibration* and typically involves solving an optimization problem, using numerical methods, to determine the parameters of the process $\{X_t\}_{0\leq t}$, such that model prices are as close as possible to the observed market prices. This can be quite computationally intensive so the space of potential models for $\{X_t\}_{0\leq t}$ is often restricted to those that make calibration computationally feasible. In particular, a process $\{X_t\}_{0\leq t}$ that allows for an explicit solution or a fast numerical approximation to the pricing equation in (2) is desirable.

From a modelling perspective the problem is further complicated by *frictions* such as counterparty credit risk - the risk that the counterparty you are buying the derivative from may go bankrupt before the contract has expired. Modelling this phenomena can be quite challenging as the value of the derivative can be linked to the probability of bankruptcy in many subtle ways. Other complications occur when the payments involve multiple currencies or multiple different interest rates or both. All of these things also have to be included in the pricing kernel $\{\pi_t\}_{0\leq t}$ in an arbitrage free manner, and in a manner that still that makes it feasible to perform calculations with the model within a reasonable time-frame.