C^*-algebras, classification and dynamical systems

After having done my Master's thesis in Copenhagen supervised by Erik Christensen in the mid 1980's on a new paper by Vaughan Jones: "Index for subfactors", I went to Philadelphia to do a PhD with Dick Kadison. I expected to continue working on von Neumann algebras, but was instead given a thesis problem in pure C^*-algebra theory about what one could call a quantitative version of the "Russo-Dye" theorem, which turned out to relate to the notion of *stable rank* introduced earlier by Marc Rieffel. I was furtunate that my thesis problem had a very nice answer, which got me started as a mathematician, and landed me my first job in Odense with Uffe Haagerup.

After finishing my PhD, and before going to Odense, I went on a mini postdoc stay in Toronto, where I got to know George Elliott. At that time George divided his time between Toronto and Copenhagen. He got me interested in K-theory and the classification of C^*-algebras. In the early days of C^*-algebras, classification results were quite sporadic, beginning with Glimm's classification of UHF-algebras in 1959, followed by Bratteli and Elliott's classification of AF-algebras in the 1960's and 1970's. Several other classes of C^*-algebras, including the irrational rotation C^*-algebras and the Cuntz algebras were also studied and understood quite well, but not in the unified way of today.

The kick-start of Elliott's classification program for C^*-algebras was his classification in the late 1980's of a certain class of inductive limit C^*-algebras arising from circles and intervals. Elliott suggested in his paper that, although the class of C^*-algebras he could handle seemed to be quite special, maybe they were not so special after all and could comprise a much larger class of C^*-algebras. The rest is history. Today, in fact very recently, we have a complete classification in terms of K-theory and related invariants for a specific natural class of simple C^*-algebras, the scope of which is way beyond what anyone would have expected 30 years earlier.

My own contributions to the classification program in the early 1990's were mostly in the socalled purely infinte case, which also was the first case to be completely understood. My research on the purely infinite C^*-algebras led to the solution of some problems left open by Joachim Cuntz, whom I got to know during my PhD studies in Philadelphia, relating both to the Cuntz algebras and to the Cuntz-Krieger algebras. I also got interested in a question raised by Dixmier and promoted further by Cuntz if the classification of von Neumann algebra factors into type I, II and III would carry over to C^*-algebras. Or could there exist a simple, preferably also nuclear, C^*-algebra that has both type II and type III properties at the same time. I thought about this problem for several years and finally solved it during a sabbatical in 2000-2001 at MSRI and UC Santa Barbera, by providing an example of a nuclear simple C^*-algebra with a finite and an infinite projection. This was also the first concrete counterexample to Elliott's classification conjecture. (As a result, one must specify and restrict the class of simple C^*-algebras that admit a classification.) Jesper Villadsen, a student of Klaus Thomsen, with whom I had many discussions during another sabbatical to the Fields Institute in Waterloo (later Toronto) in 1994, was the first to exhibit C^*-algebras with exotic high-dimensional behaviour. Villadsen's methods were crucial for my example and also later counterexamples to the Elliott conjecture by Toms.

The Elliott program had — and continues to have — several interesting byproducts and applications. One of them, that I was involved in, was the positive solution to a question promoted by Halmos if "almost commuting" self-adjoint matrices must be "close to exactly commuting" self-adjoint matrices. More formally: For each $\varepsilon > 0$ there is $\delta > 0$ such that for all integers $n \geq 1$ and all self-adjoint (contractive) $n \times n$ matrices A and B satisfying $\|AB - BA\| < \delta$ there exist *commuting* self-adjoint $n \times n$ matrices A_0 and B_0 such that $\|A - A_0\| + \|B - B_0\| < \varepsilon$. It is crucial that δ does not depend on n.

Lately, I have been interested in dynamical systems (groups acting on spaces or on C^*-algebras) and the interplay between groups and C^*-algebras. In the mid 1990's it was observed by several authors, including Anatharaman-Delaroche, that one can obtain simple purely infinite C^*-algebras from certain hyperbolic groups acting on their Gromov boundary. The "pure infiniteness" arises from the paradoxical property of the group, in the way that is also well-known from the famous Banach–Tarski theorem. I asked myself if one can construct an action of a non-amenable group on the Cantor set (or perhaps a more complicated space) such that the associated C^*-algebra is simple with a finite and an infinite projection. I still don't know, and there are indications that this may not be possible. Some of my recent research has directly or indirectly been concerned with this question, and has involved actions of non-amenable as well as non-supramenable groups on compact, respectively, locally compact spaces.

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