MY RESEARCH: GEOMETRIC GROUP THEORY

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My research lies mainly within the area of Geometric Group Theory. Though, many motivations, applications, and tools originate in other branches, e.g. in metric and algorithmic graph theory, algebraic topology, geometric topology, 3–manifold topology, combinatorial optimisation, algebraic geometry, ergodic theory, operator algebras, coarse geometry, mathematical programming.

The main idea of *Geometric Group Theory* is studying (usually infinite) groups through their actions on geometric, topological, or combinatorial objects. As an independent research area this one is relatively new, has developed mainly from works in combinatorial group theory and 3–manifold topology in late 1980's.

Recall, that an *action* of a group G on a space X is a homomorphism $G \to \mathrm{Aut(X)}$ into the group of automorphisms of (a structure on) X. Here, I usually work with the space X being a metric space (then Aut(X) is the group of isometries of X) or a simplicial complex (then Aut(X) is the group of simplicial automorphisms of X).
Of course, some actions might be not so interesting, for example, the *trivial* action $X \to \{\mathrm{Id_X}\}$ always exists. Such an action has a lot of fixed-points, and consequently, we are rather interested in actions without them. Already existence of such actions is an intriguing question. For example, the celebrated Kazhdan's property (T) for discrete groups is equivalent to the non-existence of fixed-point-free actions on Hilbert spaces. This property has been introduced by Kazhdan in 1967 as a tool for proving an important result about finite generation of some lattices in Lie groups. Since then Kazhdan's property (T) has been studied thoroughly in the frame of operator algebras and in the mainstream of Geometric Group Theory. Non-existence of fixed-point-free actions on Hilbert spaces is an example of *rigidity* phenomena.
Such rigidity provides information on "representability" of a given group. If we further restrict to "nicer" actions we can observe "closer" relations between a group and a space it acts on. For example, the fundamental group G of a compact manifold acts *freely* (no non-trivial element fixes a point) and *cocompactly* (the quotient is compact) on the universal cover X of the manifold. This is an example of a *geometric* action. One of the main paradigms of Geometric Group Theory is that in such a situation the geometry of the group "resembles" the geometry of the space. For a finitely generated group G its "geometry" is given by a *word metric* — we define a distance between two group elements g and h as the length of a shortest word (in a given finite generating set) representing gh^{-1}. Then "resembles" means e.g. that the two spaces are *quasi-isometric* — there exists a map $G \to X$ (e.g. the orbit map) such that distances between images of points are controlled in a uniform way by distances between the points themselves. In such situation, many (geometric) properties of the space X translate to (geometric, algebraic, algorithmic) properties of the group G. For example, if we assume further that the space X is contractible, then the solvability of the word problem for G can be deduced from the features of the isoperimetric function for X.

A recurring theme of my research is finding a "nice" space on which a given group acts geometrically (or, sometimes, with a slightly "uglier" action). Various features of the group are then concluded immediately.

Already at the birth of Geometric Group Theory various notions of *non-positive curvature* played a central role. Trees (graphs without cycles) and universal covers of closed Riemannian manifolds of negative sectional curvature are main examples of "negatively curved" spaces. Gromov introduced the notion of *hyperbolicity* unifying the two examples and allowing a metric approach to many other spaces and groups acting on them geometrically. He also popularized another metric notion of non-positive curvature: the *CAT(0) property*. Roughly, both notions are based on the idea that in the presence of non-positive curvature geodesic triangles should be not "fatter" than in the Euclidean plane (which is considered to be of curvature zero). Knowing that a group admits a geometric action on a non-positively curved space provides o lot of information about the group. For example, non-positive curvature usually implies quadratic isoperimetric inequality hence, as aforementioned, the word problem is solvable for groups acting geometrically on such spaces.

My research within Geometric Group Theory usually goes along one of the two following questions:
i) for a given group or family of (classical) groups find non-positively curved spaces they act "nicely" on;
ii) construct new examples of groups with "exotic" properties using non-positive curvature.

For i) an integral part of the question is finding a suitable notion of non-positive curvature. Except of the hyperbolicity and CAT(0) property mentioned above, myself, I have been working mostly with various notions of "combinatorial" non-positive curvature. Applying concepts originating from e.g. metric graph theory (such as Helly graphs) or introducing new versions of non-positive curvature (weak systolicity) together with collaborators we were able to equip some classical groups (e.g. some Artin group) with geometric actions on non-positively curved spaces, hence proving interesting features of the groups.

For ii) one should mention that non-positive curvature can be usually expressed in "local" terms, meaning that obeying some local rules when constructing a space leads to a space whose universal cover is non-positively curved. For example, if such space is a model for $K(G, 1)$ then the group G acts geometrically on a non-positively curved space, hence G is usually infinite. *Small cancellation* is a classical and powerful combinatorial version of non-positive curvature, and using it I was able to construct the first examples of finitely generated groups containing interesting expanding families of graphs.