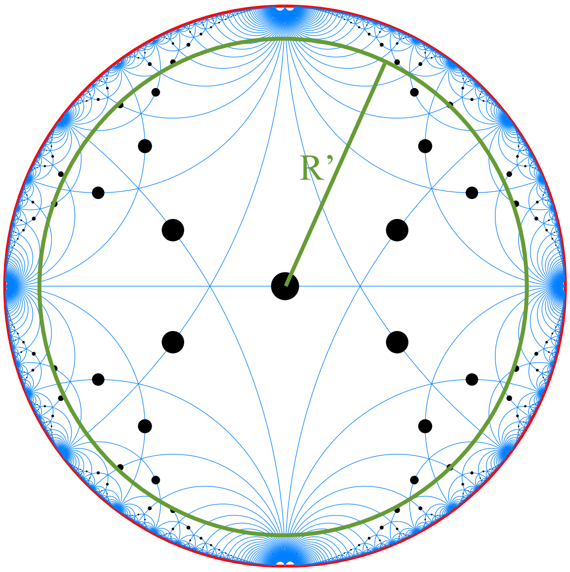
My research

My research interests include (among many other things) analytic number theory and automorphic functions. In number theory it is not uncommon that interesting problems can be easily stated, but in order to make progress towards their resolution we use advanced tools and methods from several areas of mathematics. These methods often comes from spectral theory, harmonic and complex analysis, algebraic geometry, group theory, and other sources. Let me try to illustrate this point by describing the problem I am working on right now:

A counting problem

Consider the group Γ of 2×2 matrices with integer coefficients and determinant 1. This group acts on the complex upper half-plane $\mathbb{H}$ by linear fractional transformations $z \mapsto \frac{az+b}{cz+d}$ and given a fixed point z the orbit Γz is a set of isolated points in the upper half-plane. Consider now how many of these points have hyperbolic distance less than R from another fixed point w. We want to estimate this count as R goes to infinity.

On the right we have illustrated this count in the so-called Poincaré disc model for hyperbolic geometry: If we choose $z = w = i$ the relevant count equals the number of black point within the green circle with radius R'. Here R' is a specific function of R which tends to 1 as R tends to infinity.



Moving to spectral theory

If we denote the relevant count by $N(R, z, w)$ it turns out that this has a spectral expansion in terms of so-called cusp forms (and other spectral data). Cusp forms are functions φ on the upper half plane characterised by

- being invariant under the Γ action, i.e. $\varphi(\gamma z) = \varphi(z)$ for $z \in \mathbb{H}$ and $\gamma \in \Gamma$,
- being eigenfunctions of the hyperbolic Laplace operator $-y^2(\partial_x^2 + \partial_y^2)$, and
- being square integrable with respect to a certain hyperbolic measure.

These are truly fascinating yet mysterious objects. In some sense we know very little about them: In fact we cannot construct explicitly even a single example. It took the full power of Selberg's trace formula to prove that cusp form exist. But they do. In fact there are infinitely many of them, and we have good asymptotic estimates for how many up to a given eigenvalue. Using this (and a few other ingredients) Selberg prove in the 1960'ies that

$$N(R, z, w) = 6e^R + O(e^{\frac{2}{3}R}), \tag{1}$$

which has never been improved. It is conjectured that the exponent $\frac{2}{3}$ can be replaced by $\frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. I am currently attempting to make progress towards this conjecture, i.e. I am trying to prove that there exists an $\alpha < 2/3$ such that $N(R, z, w) = 6e^R + O(e^{\alpha R})$.

Moving to L-functions

In order move the problem in a direction where techniques from number theory can be used we set $z = w = i$ (or more generally a Heegner points of a fixed discriminant). There is a deep formula due to Waldspurger and several others that cusp forms evaluated at points like $z = i$ equals a product of specific L-functions $L(\varphi, s)$ evaluated at special points $s = 1/2$ (together with their twists by Dirichlet characters).

To get a sense of what an L-function is recall the Riemann zeta function ζ. We know quite a bit about this function: It can be defined as meromorphic functions on the entire complex plane and it satisfies a functional equation relating s to $1 - s$. There are of course also many great unsolved conjectures about it: Most notably the Riemann hypothesis which predicts that all non-trivial zeroes lie on the line $\Re(s) = 1/2$.

L-functions are generalisation of the Riemann zeta function sharing many of its (provable and conjectural) properties, and there is a wide range of techniques to study them. Using Waldspurger's formula we can express $N(R, i, i)$ in terms of the infinitely many values $L(\varphi, 1/2)$.

Moving to Kloosterman sums

In order to move further we use yet another trace formula: This time the so-called Kuznetsov trace formula which translates the spectral data encoded in the infinitely many L-functions to an infinite sum whose terms involve Kloosterman sums. Kloosterman sums are sums defined as

$$S(m, n, l) = \sum_{a \in (\mathbb{Z}/l\mathbb{Z})^\times} \exp\left(2\pi i \frac{ma + na^{-1}}{l}\right).$$

Algebraic geometry provides very good bounds on Kloosterman sums. These bounds comes from the Riemann hypothesis for Weil zeta functions for curves over finite fields. This conjecture was proved by Deligne. His result implies that

$$|S(m, n, p)| \leq 2\sqrt{p}$$

when m, n are coprime with the prime p. For large primes this is a massive improvement over the trivial bound $|S(m, n, p)| \leq p - 1$.

Do we need to move further?

Does this all lead to an improvement of Selberg's bound (1). This is not so clear. YET. It seems that maybe we need to apply yet another summation formula called the Voronoï summation formula. This is a formula – related to the transformation properties of L-functions – which allows one estimate to certain averages of additive twists of Fourier coefficients of automorphic forms like the cusp forms described above. I am optimistic that this will eventually lead to an improvement on Selberg's bound.

Anyway, I hope I made my point: That understanding relatively simple counting functions in number theory involves using techniques from many different non-trivial sources.