Tensor surgery

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In the 1940s, Shannon had the ambition to create a theory of information that is abstracted from all possible physical realisations. Think bit (0/1) versus either lightswitch (on/off) or harddrive (magnetisation up/down). He succeeded in capturing all classical theories of physics, but missed out on quantum mechanics, which was developed around the same time. Information-theoretic puzzles about quantum mechanics arose right from the start (think Schrödinger's cat or Einstein's spooky action at a distance) and marked the beginning of the development of a full theory of quantum information with key contributions by Holevo (channel coding 1970s) and Bennett et al. (teleportation 1990s). *Quantum information theory* has by now matured as a research field and is situated at the intersection of mathematics, physics and computer science, yet it still offers a wealth of fundamental open puzzles and questions that one day might be realised quantum-physically in e.g. a fridge in HCØ building 3, 4th floor. As the last runner in this year's research relay race, I would like to tell you about a question that has kept me awake this past year.

In quantum information theory, the state of k quantum bits (qubits) is a vector in $\psi \in (\mathbf{C}^2)^{\otimes k}$ (we will also look at higher dimensions; for the purpose of this note, you might replace $\mathbf{C}$ by your favourite number field). Let us write ψ as an array of numbers $\psi_{i_1,\ldots,i_k}$, a tensor:

$$\psi = \sum_{i_1,\ldots,i_k \in \{0,1\}} \psi_{i_1,\ldots,i_n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k},$$

where $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. When created in the lab, some of such states might be more useful than others. The two-qubit state $e_0 \otimes e_0 + e_1 \otimes e_1$ (an entangled bit), which corresponds to the unit matrix δ_{i_1,i_2} will allow you to quantum teleport; the 3-qubit state $e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$, which corresponds to a $2 \times 2 \times 2$ cube with 1's on the diagonal, can be used to win the Greenberger-Horne-Zeilinger game better than any classical strategy will.

Let us now fix k and let us think of the k-particle versions of the above tensors in dimension d: $T_d[k] := \sum_{i=1}^{d} \underbrace{e_i \otimes \cdots \otimes e_i}_{k}$ as a precious resource that costs $\log_2 d$ kroner each (we take the $\log_2$ so that teleporting n qubits from Alice to Bob costs n kroner, and for Alice, Bob and Charlie to win n games they need to invest n kroner). We can now ask:

How much does a given k-particle quantum state ψ cost to produce?

Formally, the cost is given by the $\log_2$ of the minimal d, s.th. there are matrices $A_1, A_2, \ldots, A_k$ that turn $T_d[k]$ into ψ:

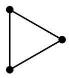
$$(A_1 \otimes A_2 \otimes \cdots \otimes A_k) T_d[k] = \psi \quad ?$$

The minimal d turns out to equal the *tensor rank*

$$R(\psi) = \min\{d : \psi = \sum_{i=1}^{d} v_{i_1}^{(1)} \otimes v_{i_2}^{(2)} \otimes \cdots \otimes v_{i_k}^{(k)}\}$$

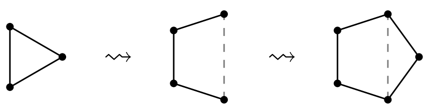
which is a generalisation of matrix rank (the $k = 2$ case). Computing tensor rank for $k \geq 3$ is computationally hard (NP-hard). We will focus on upper bounds for a set of combinatorially defined tensors.

For a given graph, consider the tensor, where every edge in the graph corresponds to an entangled pair connecting the two vertices (you need such a quantum state if you want to perform teleportation among the pairs connected by an edge). Famously, Strassen showed in 1969 that the tensor rank of the 3-tensor given by the triangle graph

T() $= \sum_{i \in \{0,1\}^3} (e_{i_1} \otimes e_{i_2}) \otimes (e_{i_2} \otimes e_{i_3}) \otimes (e_{i_3} \otimes e_{i_1}) \ \in \ (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes 3},$

is only seven (and not eight).[1] No nontrivial upper bounds on other graphs (triangle-free) were known and with Jeroen Zuiddam from Amsterdam, I started wondering at the beginning of the year about how to do the 5-cycle.

After some initial numerics, we came up with a method we call *tensor surgery* (`https://arxiv.org/abs/1606.04085`) that allows us to transform a good decomposition of a well-chosen starting tensor into a good decomposition of a goal tensor. If we want to end up with the 5-cycle, we start with Strassen's decomposition of the 3-cycle and split, in a first step, the vertex into two, keeping track of this cut with a dotted edge. We then insert two new edges in a second step:



Note that the dotted edge and the two new edges form a triangle! If we now carefully keep track of the increase in rank that the components in Strassen's original decomposition experience under this surgery procedure and in particular realise that the new triangle only has rank 7 (again by Strassen's result!), we can shave off 1 from the trivial rank of 32 and obtain a rank upper bound of 31 for the five cycle.

[1] Thereby he showed that one can multiply $N \times N$ matrices with $O(N^{\log_2 7})$ operations, thus better than the $O(N^3) = O(N^{\log_2 8})$ that your high school row-times-column algorithm would need.

$31 < 32$ doesn't sound spectacular, but tensor surgery works surprisingly well when considering asymptotics (give each edge the same large multiplicity) and sparse graphs (such as the k-cycle) yielding often optimal results. Tensor rank also shows up in many other areas of maths, physics, computer science and engineering. Get in touch, if some surgery is needed! We could also try lasering if surgery doesn't give the desired results.