My Research

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My research is in large deviation theory. The subject began in Scandinavia with the work of H. Cramér in the 1930s. While an analyst by training, he also worked in insurance and later probability theory and gave a formal solution to the *ruin problem*, which studies the probability that an insurance company will become bankrupt at some future time. Mathematically, this is achieved by introducing a stochastic process with positive linear drift, where the insurance company gains capital from a deterministic flow of premiums income, and loses capital from random claim payments occurring sporadically over time. Under some natural assumptions on the process, it was shown by Cramér that, as the initial capital u tends to infinity, the probability of ruin decays at an exponential rate; that is, if Y_t denotes the company's capital at time t, then

$$\mathbf{P}\{Y_t < 0, \text{ for some } t \geq 0\} \sim Ce^{-Ru} \quad \text{as} \quad u \to \infty. \tag{1}$$

In establishing this result, Cramér made use of techniques from complex analysis; in particular, he showed that the solution to this problem satisfies the Wiener-Hopf equation.

The modern theory of large deviations was developed beginning in the 1960s in a famous series of papers by Donsker and Varadhan. Motivated by theoretical questions as well as applied problems arising in statistical physics, information theory and other areas, they introduced a theory describing, roughly, the path properties of certain stochastic processes, where the probability that the process follows a "rare" path is quantified. In complex problems, we typically have that under appropriate scalings, these rare-event probabilities will decay at explicit exponential rates (as seen in (1)), and these rates can be characterized. For example, in my Ph.D. dissertation, a multivariate version of (1) was considered, where $\{Y_t\} \subset \mathbb{R}^d$ and $Y_0 = 0$, and one considers $\mathbf{P}\{Y_t \in uA, \text{ for some } t \geq 0\}$ for a general open set A in Euclidean space. This yields an estimate of the form

$$\lim_{u\to\infty} \frac{1}{u} \log \mathbf{P}\{Y_t \in uA, \text{ for some } t \geq 0\} = -\inf_{x\in A} I(x),$$

where the exponential decay of these probabilities is captured by the convex "rate function" I, which is minimized over the given set $A \subset \mathbb{R}^d$.

More recently, I have been interested in the large deviation behavior of iterated random systems, namely sequences of the form

$$V_n = F_n(V_{n-1}), \quad n = 1, 2, \ldots, \tag{2}$$

where $\{F_n\}$ is an i.i.d. sequence of random functions. In various incarnations, such recursive sequences have been the focus of much recent interest in pure and applied probability. An example is the matrix-driven recursion

$$V_n = M_n V_{n-1} + R_n, \quad n = 1, 2, \ldots, \tag{3}$$

where $\{M_n\}$ is an i.i.d. sequence of $d \times d$ random matrices and $\{R_n\}$ is an i.i.d. sequence of random vectors. Under the rough assumption that M_1 is contracting, then $\lim_{n\to\infty} V_n \equiv V$ will exist as a proper random vector, and it is of interest to study the probability that V takes on large values (that is, to find the tail probabilities of the limiting distribution). A well-known estimate for the matrix recursion (3) is due to H. Kesten (*Acta Math.*, 1973), where it was shown that

$$\mathbf{P}\{V \in uA\} \sim Du^{-\gamma} \quad \text{as} \quad u \to \infty, \tag{4}$$

for any given halfspace $A \subset \mathbb{R}^d$. Kesten's work was motivated by branching processes in random environments. Here, the limiting size of a population is characterized assuming, roughly, a random offspring *distribution* which is contracting on average. However, the recursion (3) also arises in other settings, including the study of the popular ARCH and GARCH financial time series models, which provides some motivation for my work. For example, determining the tail distribution of $\lim_{n\to\infty} V_n$ in (3) is tantamount to characterizing the probability that a large financial fluctuation occurs for a GARCH process in steady state.

In separate collaborations with S. Mentemeier and A. Vidyashankar, we have described certain path properties for the matrix-driven recursive sequence defined in (3). These estimates yield refinements and extensions of Kesten's classical theorem. A simple estimate of this type involves the time of large exceedance, $T_u(A) := \inf\{n : V_n \in uA\}$, where A is any open subset of $\mathbb{R}^d$. It can be shown that this time grows roughly at the rate u^γ for γ given as in (4); more precisely,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{T_u(A)}{u^\gamma} \le t\right\} = 1 - e^{-Kt},$$

for a constant $K > 0$ which can be explicitly specified. We have also developed, and are in the process of further developing, other path estimates which describe, for example, the behavior of the process conditional on the rare event $\{V_n \in uA\}$. There is a nice analogy between these estimates and the earlier work in large deviations developed in the simpler setting of i.i.d. random walks. We intend to extend these estimates to a wider class of recursions, as described in (2), and to relate these results to some more complex problems arising in insurance and finance, statistics, and Monte Carlo simulation.