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My Research

Algebraic Geometry

My field of research is algebraic geometry, which is the study of (algebraic) varieties. Roughly, a variety is the set of solutions to a system $S = \{f_i\}_{i \in I}$ of polynomials

$$f_i = f_i(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n],$$

where K is a field. A solution is a tuple $(a_1, \ldots, a_n) \in K^n$ for which *all* the f_i-s vanish. It is costumary to equip K^n with the *Zariski topology*, in which the closed sets are precisely the solution sets; this is called the affine space, denoted $\mathbb{A}^n_K$. So every system S of polynomials as above yields a variety (a closed subset) $X = X(S) \subset \mathbb{A}^n_K$.

Varieties are among the first geometrical objects one encounters as a student in mathematics; consider for example the *parabola* $C \subset \mathbb{A}^2_{\mathbb{R}}$ given as the solution set of the polynomial $y - x^2 \in \mathbb{R}[x, y]$. If we move in positive direction along the y-axis, we find *two* distinct branches of the parabola tending to infinity. In fact, the parabola is not compact, but it can be compactified by adding points "at infinity" to $\mathbb{A}^2_{\mathbb{R}}$, which, as it turns out, results in the two branches joining in a unique point at infinity.

In algebraic geometry, one usually prefers to work with compact objects. In order to do this, and generalizing our elementary example of the parabola, one replaces affine space by the so called *projective space* $\mathbb{P}^n_K$. Points can now be represented by $n+1$-tuples $(a_0 : \ldots : a_n)$ of elements in K, modulo simultaneous scaling by non-zero scalars $\lambda \in K$. One can then speak of sets of solutions to systems of *homogeneous* polynomials, these are called projective varieties, which turns out to always be compact. As an example, consider the degree 2 polynomial

$$x_0 x_2 - x_1^2 \in \mathbb{R}[x_0, x_1, x_2].$$

Its set of solutions $\overline{C} \subset \mathbb{P}^2_{\mathbb{R}}$ compactifies the affine parabola C. Indeed, in the open part where $x_0 \neq 0$, we can, using the scaling, assume that the first coordinate equals 1. Then our solution set is precisely the (affine) solutions of $y - x^2 = 0$, where $y = x_2/x_0$ and $x = x_1/x_0$. However, we have added a point at infinity, namely $P_\infty = (0 : 1 : 0)$, thus $\overline{C} = C \cup P_\infty$.

Projective varieties in mathematics

Projective varieties appear in many parts of mathematics. For instance, a (smooth) variety in $\mathbb{P}^n_{\mathbb{C}}$ is also a compact complex manifold. In the other direction, any compact Riemann surface X can be embedded as a subvariety of some projective space. If X has genus 1, i.e., if X is topologically the surface of a donut, a classical result asserts that X can be embedded in $\mathbb{P}^2_{\mathbb{C}}$ as the zero locus of a cubic polynomial $F \in \mathbb{C}[x, y, z]$. These varieties are commonly referred to as *elliptic curves*, and are of great importance and interest in both number theory and geometry. Because algebraic geometry allows one to work over any field K, it is often possible to translate questions in number theory into algebraic geometry. The interplay between arithmetic and geometry has in fact been a central theme in modern mathematics; stunning results are, to name a few, Deligne's proof of the Weil Conjectures (we will return to this below) and Wiles' proof of Fermat's Last Theorem (which deeply uses the theory of elliptic curves).

Weil Zeta Functions

Let us consider, as an illustration, the Fermat cubic $F = x^3 + y^3 + z^3 \in \mathbb{Z}[x, y, z]$. It yields an elliptic curve $X \subset \mathbb{P}^2_{\mathbb{Q}}$, whose set of points $X(\mathbb{C})$ with coefficients in $\mathbb{C}$ forms a compact Riemann surface. However, F also makes sense as cubic in $\mathbb{F}_p[x, y, z]$, for any prime p, by "reduction modulo p", thus we also obtain a cubic curve $X_p \subset \mathbb{P}^2_{\mathbb{F}_p}$ for every p. The curve X_p is smooth unless $p = 3$, due to the fact that $F = (x + y + z)^3$ in characteristic 3; one says that X *degenerates* at $p = 3$.

The objects $X(\mathbb{C})$ and X_p (for general p) are quite different in nature, but in fact related in a beautiful way through the Weil Zeta Function $Z_p(X, T)$. To define this function, let us fix an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. For every integer $r > 0$ we have a unique degree r field extension

$$\mathbb{F}_p \subset \mathbb{F}_{p^r} \subset \overline{\mathbb{F}_p},$$

and we denote by N_r the (finite) number of points $X_p(\mathbb{F}_{p^r})$ with coefficients in the field $\mathbb{F}_{p^r}$. Then

$$Z_p(X, T) = \exp(\sum_{r=1}^{\infty} N_r \cdot \frac{T^r}{r}) \in \mathbb{Q}[[T]].$$

It turns out that $Z_p(X, T) = \frac{P_1(T)}{P_0(T) \cdot P_2(T)}$, where each P_i is a polynomial with integer coefficients, of degree equal to the i-th Betti number of $X(\mathbb{C})$! These properties are (part of) the statement of the Weil Conjectures, formulated by A. Weil in 1949 for smooth varieties over finite fields, and finally proved by P. Deligne in 1973.

Motivic Zeta Functions

To conclude, I will discuss some features of my own research. We saw above that, for a variety defined over a finite field, the Weil Zeta Function encodes the asymptotic behaviour of the sets of points of X with coefficients in finite extension fields. In joint work with J. Nicaise (Imperial College), we investigate the properties of an analogous series attached to varieties defined over discretely valued fields. A typical example of such a field is $K = \mathbb{C}((t))$, the field of formal Laurent series with complex coefficients.

A useful way to think of a variety defined over K is as a 1-parameter family varying continuously with the (non zero) parameter t. Let us, as an example, consider the curve $C \subset \mathbb{P}^2_K$ defined as the zero locus of the cubic polynomial

$$F = t \cdot (x^3 + y^3 + z^3) + xyz.$$

Since F actually has coefficients in the power series ring $\mathbb{C}[[t]]$, we can reduce modulo t, which yields the cubic $F_0 = xyz \in \mathbb{C}[x, y, z]$. Its zero locus $C_0 \subset \mathbb{P}^2_{\mathbb{C}}$ is a degenerate cubic, consisting of a triangle of lines.

For any integer $d > 0$, extracting a d-th root t_d of t yields a degree d field extension

$$K \subset K(d) := \mathbb{C}((t_d)),$$

where $t \mapsto (t_d)^d$. Thus, it makes sense to ask, for any variety X defined over K, for a generating series encoding the growth of the sets of points $X(K(d))$ with coordinates in $K(d)$, as d tends to infinity. But now we encounter a problem that did not occur over finite fields; usually the sets $X(K(d))$ are infinite! Thus, one can not simply work with the cardinality. This can be seen already in the elementary example above: any smooth point on C_0 are of the form $(a : b : c)$ with a, b and c complex numbers where exactly one of them is zero. Any of these (infinitely many) points can be lifted (by Hensel's Lemma) to a K-point of C.

To deal with this issue, it is necessary to restrict attention to varieties that allow a differential form ω_X of top degree without zeroes or poles. This holds in many cases of importance, for instance for abelian varieties (elliptic curves are of this type) and Calabi-Yau varieties. In this situation, one can measure the "size" of $X(K)$ by a motivic integral $\int_X \omega_X$ taking values in the so-called Grothendieck ring of $\mathbb{C}$-varieties. Replacing K by the field extensions $K(d)$ for arbitrary d, one can define the "motivic" zeta function $Z_X^{\mathrm{mot}}(T)$ of X as the generating series for the motivic integrals measuring the sets $X(K(d))$.

Working over $K = \mathbb{C}((t))$, the series $Z_X^{\mathrm{mot}}(T)$ turns out to be a rational function in T (this is not known in general in positive characteristic). It encodes, for instance, geometric information related to the behaviour of X under degeneration, often in quite subtle ways. By now, we have a rather complete understanding of these motivic zeta functions for abelian varieties, whereas in the general case there are still many intriguing open questions!