MODULI OF CURVES

DAN PETERSEN

Much of my research has involved *moduli spaces* in algebraic geometry. One of the simplest moduli problems in algebraic geometry is to parametrize *one-dimensional algebraic varieties*, i.e. curves. It seems fair to say that understanding this particular example has been a driving force behind much of the development of the general theory of 'moduli', going back to the work of Riemann and Hurwitz. The modest goal of this note is to explain what the moduli space of curves is. The starting point for the moduli theory of curves is thinking about n-pointed algebraic curves and how they vary in families.

By an *n-pointed algebraic curve*, we mean a projective nodal geometrically connected curve $X \to S$ equipped with n ordered disjoint sections $\sigma_i \colon S \to X$ contained in the smooth locus of X. Over the complex numbers, an equivalent definition is that we are considering compact connected Riemann surfaces X which are allowed to have simple nodes, and which are equipped with an embedding

$$[n] = \{1, 2, \ldots, n\} \hookrightarrow X \setminus X_{\mathrm{sing}}.$$

We say that an n-pointed algebraic curve X is *stable* if its automorphism group is finite. If X is smooth, i.e. $X_{\mathrm{sing}} = \varnothing$, then X is stable if and only if $2g - 2 + n > 0$, or equivalently, the Euler characteristic of the complement of the markings is negative.

Let $2g-2+n > 0$. We denote by $\mathcal{M}_{g,n}$ the 'moduli space' which parametrizes smooth n-pointed algebraic curves of genus g. Over $\mathbf{C}$, $\mathcal{M}_{g,n}$ can be thought of as the space parametrizing isomorphism classes of complex structures on a compact oriented surface S with an embedding $[n] \hookrightarrow S$. One can also think of $\mathcal{M}_{g,n}$ in terms of hyperbolic geometry: since the Euler characteristic of the surface minus the punctures is negative, by uniformization it is a quotient $\Gamma \setminus \mathfrak{H}$ of the upper half plane by a discrete group of orientation-preserving isometries. This gives a hyperbolic metric on $S \setminus [n]$, and one can thus think of $\mathcal{M}_{g,n}$ as the space parametrizing finite area hyperbolic metrics on $S \setminus [n]$. In this picture, the n marked points are actually cusps of the hyperbolic surface and more naturally thought of as punctures.

The remarkable fact is then that $\mathcal{M}_{g,n}$ is in a canonical way itself an algebraic variety. Informally, the set of all one-dimensional varieties is itself a variety! Certainly this does not happen in other areas of geometry: the set of all Riemannian manifolds is not in a natural way a Riemannian manifold, and so on. This is a peculiar phenomenon that occurs in algebraic geometry, illustrating in a sense how 'rigid' algebraic varieties are.

So what do we mean when we say that $\mathcal{M}_{g,n}$ is naturally a variety? A priori it is not even clear that $\mathcal{M}_{g,n}$ should be a 'space', and not just a set (or groupoid); that is, why should there even be a topology on the set of points of the moduli space,

$$\{\text{compact Riemann surfaces of genus } g \text{ with } n \text{ markings}\}/\text{isomorphisms},$$

in a natural way? One needs to start thinking of curves varying in families. One can for instance consider the family of curves

$$y^2 = x(x-1)(x-t)$$

as t varies in $\mathbf{C}$; when $t \notin \{0,1\}$ we get an elliptic curve, i.e. a smooth curve of genus 1 with a marked point. We would like to say that the topology on $\mathcal{M}_{1,1}(\mathbf{C})$ should be such that elliptic curves defined by nearby values of t should be close to each other in moduli space. If one contemplates how to formalize this vague idea, one thinks perhaps after a while of the condition that the map

$$\mathbf{C} \setminus \{0,1\} \to \mathcal{M}_{1,1}$$

that assigns to a number t the isomorphism class of the curve $y^2 = x(x-1)(x-t)$ is continuous. In fact, this map will be a *morphism* of algebraic varieties; moreover, if we impose the stronger requirement that morphisms $S \to \mathcal{M}_{1,1}$ should be in *bijective correspondence* with isomorphism classes of families of elliptic curves over S, for *any* base space S, then this determines the structure of algebraic variety on $\mathcal{M}_{1,1}$ completely (via the Yoneda lemma). This is essentially[1] the definition of the spaces $\mathcal{M}_{g,n}$.

Yet another way to think about the moduli space of curves, which is perhaps more appealing to topologists, is via its fundamental group. Let us consider a loop γ in $\mathcal{M}_{g,n}(\mathbf{C})$, and let us for simplicity assume that it does not meet the locus of curves with nontrivial automorphism group. Fix an n-pointed smooth surface S of genus g. We identify the starting point of γ with a choice of a complex structure on S. We can represent moving along γ by a continuous deformation of the chosen complex structure. As we have reached the end of the loop, we have a *new* complex structure on the *same* underlying surface S. But we know moreover that the resulting two Riemann surfaces are uniquely isomorphic, and this isomorphism is realized by a diffeomorphism $\phi_\gamma \colon S \to S$ (which fixes the marked points). Now (contrary to what the notation suggests) ϕ_γ does not only depend on the loop γ, since we needed a choice of a lift of γ to the space of all complex structures on S. But it is true that ϕ_γ is uniquely determined up to isotopy, and if we modify γ by a homotopy, then the isotopy class remains invariant. Hence we obtain a map from $\pi_1(\mathcal{M}_{g,n}(\mathbf{C}))$ to the *mapping class group* $\Gamma_{g,n}$ of S, i.e. the group of all oriented diffeomorphisms of S moduli isotopy. This group is of central importance in low-dimensional topology and geometric group theory.

In the preceding paragraph we assumed for simplicity that γ does not meet the locus of curves with automorphisms, so that there was a single well defined isomorphism between the two complex structures on S. All we said remains true without this assumption if we take $\pi_1(\mathcal{M}_{g,n}(\mathbf{C}))$ to be the *orbifold* fundamental group, i.e. when we think of $\mathcal{M}_{g,n}(\mathbf{C})$ as a topological stack rather than a topological space; for this one needs to be a bit more careful in defining precisely what it means to have a loop in an orbifold.

In fact, the map $\pi_1(\mathcal{M}_{g,n}(\mathbf{C})) \to \Gamma_{g,n}$ is an *isomorphism*. This amounts to saying that the space of complex structures on S, considered up to diffeomorphisms isotopic to the identity, is simply connected. The latter space is called the *Teichmüller space* $\mathscr{T}_{g,n}$ of the pointed surface S, and in fact more is true: $\mathscr{T}_{g,n}$ is a complex manifold which is real-analytically equivalent to a ball in $\mathbf{R}^{6g-6+2n}$, so it is in fact contractible. This shows that $\mathcal{M}_{g,n}(\mathbf{C})$ is homotopy equivalent[2] to the

[1]The above discussion sweeps lots of 'stacky' issues under the rug. In particular, according to the definition above, spaces $\mathcal{M}_{g,n}$ do not even exist. To get a correct definition, one needs to replace isomorphism classes with groupoids everywhere, in which case the moduli spaces are not varieties or schemes but rather *stacks*. In fact, the stability condition implies that these stacks are even Deligne–Mumford, which is the kind of stack closest to an ordinary algebraic variety. Over the complex numbers, a Deligne–Mumford stack corresponds to a not necessarily effective orbifold.

[2]One must be slightly careful here. Either one should work with coarse moduli spaces, in which case the correct statement is that $\mathcal{M}_{g,n}(\mathbf{C})$ is only rationally homotopic to $B\Gamma_{g,n}$, as $\Gamma_{g,n}$ acts on Teichmüller space with finite stabilizers. Alternatively, one may work with fine moduli spaces and say that $\mathcal{M}_{g,n}(\mathbf{C})$ and $B\Gamma_{g,n}$ are homotopy equivalent as topological stacks.

classifying space $B\Gamma_{g,n} = B\mathrm{Diff}^+(S,n)$. Thus all homotopic information of $\mathcal{M}_{g,n}$ is contained in the mapping class group.

A basic fact is that the space $\mathcal{M}_{g,n}$ is not compact. What this means concretely is that one can find 1-parameter families of Riemann surfaces which 'diverge'; one cannot assign to them a limiting complex structure. In the hyperbolic picture, one can visualize this phenomenon e.g. by considering the family obtained by letting the length of a bounding curve shrink to zero. In the limit, the surface acquires a cusp, as in Figure 1.

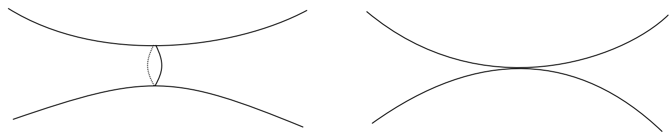

FIGURE 1. The length of a curve shrinks to zero and a cusp appears in the limit.

In the algebraic picture, this means that one needs to allow singular algebraic curves in order to assign a limit to this family. By now, there is a well established practice that whenever a moduli space is not compact, one should try to construct a compactification by enlarging the scope of the moduli problem to encompass also some appropriately 'mild' degenerations. In general, allowing 'too few' kinds of degenerations will result in a moduli space which is not compact, whereas allowing 'too many' results in a space which is not Hausdorff. A miracle is that the stability condition introduced in the beginning of this note gives the nicest possible compactification of $\mathcal{M}_{g,n}$.

If X is a stable n-pointed curve, we define its genus as $\dim H^1(X, \mathcal{O}_X)$; when X is smooth, this coincides with the usual definition of the genus of a surface. Over $\mathbf{C}$, there is a topological definition of the genus of a stable curve: if one removes a neighborhood of each node and instead glues in a cylinder, then the genus of the nodal curve can be defined as the genus of the resulting smooth surface. Pictorially, this definition is what you get when you run the process of Figure 1 backwards.

Let $\overline{\mathcal{M}}_{g,n}$ be defined as the moduli space parametrizing *stable* n-pointed curves of genus g. Then clearly $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$, and the fact is that $\overline{\mathcal{M}}_{g,n}$ is smooth, projective, irreducible, and the complement $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a normal crossing divisor. In the world of algebraic geometry, this means that the compactification $\overline{\mathcal{M}}_{g,n}$ is the nicest possible you could have hoped for. Moreover, there is also a rich combinatorial structure in the structure of the boundary; this combinatorial structure was formalized by Getzler and Kapranov via the introduction of the notion of a *modular operad*.

My own research has to a large extent been concerned with topological properties of the spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, in particular questions about its cohomology ring, or Chow ring, or tautological ring. The notion of tautological ring goes back to work of Mumford in the early '80's. One can think of the tautological ring as a more easily understood approximation to the cohomology ring/Chow ring; a precise statement in this direction was proved by Kawazumi and Morita. One of the landmark results in this area is due to Ib Madsen and Michael Weiss, a small part of which says that as $g \to \infty$, this approximation of the cohomology ring becomes better and better, so that in the large genus limit, the cohomology ring and the tautological ring coincide.

The curious reader who wants to learn more about moduli spaces of curves is referred to office 04.1.16.