EULER CHARACTERISTICS AND p-ELEMENTS

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I have been working on polynomial covering maps, mapping spaces, groups of self-homotopy equivalences, genus of spaces, spaces of the same N-type, colorings of simplicial complexes, p-compact groups and fusion systems. Recently, I have been fascinated by equivariant Euler characteristics and Euler characteristics of finite categories. But rather than give a full review I am going to tell a little anecdote from my research life about the fate of a small result that I stumbled over almost by accident. The anecdote is about applications of Euler characteristics to finite group theory.

It is well known that finite simplicial complexes have Euler characteristics. It is less widely known that we can also define the Euler characteristic of (some) square matrices and even (some) finite categories.

Let ζ be a square matrix with rational entries. A weighting for ζ is a column vector k such that all cooordinates in ζk equal 1 and a coweighting is a row vector k such that all coordinates in $k\zeta$ equal 1. If ζ admits both a weighting and a coweighting then the Euler characteristic of ζ is defined to be the the sum of the entries of a weighting or a coweighting, the two sums are identical. Not all square matrices have Euler characteristics.

If now $\mathcal{C}$ is a finite category, let $\zeta(\mathcal{C}) = (|\mathcal{C}(a,b)|)_{a,b\in\mathcal{C}}$, the ζ-matrix of $\mathcal{C}$, be the square matrix recording the sizes of all the morphism sets between objects of the category. We define the Euler characteristic of $\mathcal{C}$ to be the Euler characteristic of $\zeta(\mathcal{C})$ if it exists, $\chi(\mathcal{C}) = \chi(\zeta(\mathcal{C}))$. For instance, $\chi(G) = |G|^{-1}$ when we view the finite group G as a category with one object. Also, if K is a finite simplicial complex then the usual Euler characteristic of K is the Euler characteristic in this new sense of the poset of simplices of K. Therefore this new defintion agrees with the old definition of the Euler characteristic of a finite simplicial complex. Not all finite categories have Euler characteristics. The reduced Euler characteristic is defined to be simply $\widetilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1$.

Let now G be a finite group and p a prime number. An element of G is a p-element if its order is a power of p. The set of p-elements of G,

$$G_p = \{g \in G \mid g^{|G|_p} = e\} = \bigcup \mathrm{Syl}_p(G)$$

is the union of the Sylow p-subgroups of G. The number of p-elements is always a multiplum of $|G|_p$, the p-part of the group order.

Theorem 1 (Frobenius 1907). $|G|_p \mid |G_p|$

The Brown poset $\mathcal{S}_G^p$ is the poset of all p-subgroups of G ordered by inclusion. Also the reduced Euler characteristic $\widetilde{\chi}(\mathcal{S}_G^{p+*})$ is always a multiplum of $|G|_p$.

Theorem 2 (Brown 1975). $|G|_p \mid \widetilde{\chi}(\mathcal{S}_G^{p+*})$

Is there any relation between these two seemingly similar theorems?

The key to answer this question lies hidden in the orbit category $\mathcal{O}_G^p$ of G. The objects in $\mathcal{O}_G^p$ are again all p-subgroups of G and the morphisms from $H \le G$ to $K \le G$ are $\mathcal{O}_G^p(H,K) = \{g \in G \mid H^g \le K\}/K$, K-orbits of elements of G that conjugate H into K. Composition of morphisms is induced from group composition. In fact, it will be enough to consider the full subcategory $\mathcal{O}_G^{p+\mathrm{rad}}$ of the so-called p-radical subgroups. The ζ-matrix of $\mathcal{O}_G^{p+\mathrm{rad}}$ also goes under the name of the table of marks for G. When the subgroups are arranged in descending order, the table of marks becomes a lower triangular matrix. I will write $[\mathcal{O}_G^p]$ for the table of marks and $[[\mathcal{O}_G^{p+\mathrm{rad}}]]$ for the modified table of marks obtained by dividing each column in $[\mathcal{O}_G^{p+\mathrm{rad}}]$ by its diagonal element.

We can count the number of p-elements in G by computing the Euler characteristic of the orbit category $\mathcal{O}_G^{p+\mathrm{rad}}$ of G. The orbit category happens to have Euler characteristic, the Euler characteristic is the density

$$\chi(\mathcal{O}_G^{p+\mathrm{rad}}) = \frac{|G_p|}{|G|}$$

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/Users/jespermoller/projects/euler/sym/version1/myres.tex.

of the p-elements in G and by computing its weighting we get the basic relation

$$|G_p| = |G|\chi(\mathcal{O}_G^{p+\mathrm{rad}}) = |G|\sum_K -\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})|K| = |G|\sum_{[K]} \frac{-\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})}{|N_G(K):K|} \tag{3}$$

where the first sum runs over all p-radical subgroups of G and the second sum is over conjugacy classes of such subgroups. This expression involves the (reduced) Euler characteristic of the poset $\mathcal{S}^{p+*}_{N_G(K)/K}$ of nontrivial p-subgroups of the quotient $N_G(K)/K$.

I will present two applications of identity (3). First, isolating the contribution from the trivial subgroup gives

$$|G_p| + \widetilde{\chi}(\mathcal{S}^{p+*}_G) + \sum_{[K]\neq 1} \frac{\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})}{|N_G(K):K|_p}\frac{|G|}{|N_G(K):K|_{p'}} = 0$$

Observe that the sum is divisible by $|G|_p$ since we may inductively assume that the first factor under the sum, $-\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})/|N_G(K):K|_p$, is an integer and the second factor, $|G|/|N_G(K):K|_{p'}$, clearly is divisible by $|G|_p$. It follows that $|G_p|$ is divisible by $|G|_p$ if and only if $\widetilde{\chi}(\mathcal{S}^{p+*}_G)$ is. This means that the basic relation (3) shows that Frobenius' and Brown's theorems are in fact equivalent.

Here is another application of identity (3). We shall also use the general miracle that the vector $(-\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K}))_K$ is the weighting for the modified table of marks $[[\mathcal{O}_G^{p+\mathrm{rad}}]]$. Clearly, identity (3) is particularly useful when we know the p-radical subgroups as we do for characteristic p finite groups of Lie type in defining. Let's do an example. We take the group $\mathrm{GL}_3(\mathbf{F}_2)$ of order $168 = 8\cdot 21$ and let $p=2$. The modified table of marks $[[\mathcal{O}^2_{\mathrm{GL}_3(\mathbf{F}_2)}]]$ and its weighting are

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 3 & 1 & 0 & 0\\ 3 & 0 & 1 & 0\\ 21 & 7 & 7 & 1\end{pmatrix}\begin{pmatrix}1\\-2\\-2\\8\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

and the above formula (3) gives

$$|\,\mathrm{GL}_2(\mathbf{F}_3)_2| = 168\cdot\left(\frac{1}{1}+\frac{-2}{6}+\frac{-2}{6}+\frac{8}{168}\right) = 64 = 8^2 = |\,\mathrm{GL}_2(\mathbf{F}_3)|_2^2$$

We note that

- the Euler characteristics $\pm\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})$ for 2-radical subgroups K of $\mathrm{GL}_2(\mathbf{F}_3)$ are powers of 2
- the number of 2-elements in $\mathrm{GL}_3(\mathbf{F}_2)$ is the square of the 2-part of the group order

With a computer program it is possible to generate a lot of examples. In fact, the basic identity (3) implies the following theorem.

Theorem 4. *Let G be a finite group of Lie type in characteristic p.*

(1) For any p-radical subgroup K of G, the Euler characteristic $\pm\widetilde{\chi}(\mathcal{S}^{p+*}_{N_G(K)/K})$ equals p raised to the power of the number of positive roots of the Levi complement $N_G(K)/K$

(2) $|G_p| = |G|_p^2$

Both results are already known. The first item is a weak version of the Solomon–Tits theorem and the second one was proved by Steinberg in 1968. However, the proofs sketched here are completely different from the known proofs so they might nevertheless be of some interest.

What about the cross-characteritic case? The number of p-singular elements in finite groups of Lie type is unknown in the cross-characteristic case. Out of curiosity, I found that the number of p-classes in $\mathrm{GL}_n(\mathbf{F}_q)$, $p\nmid q$, is

$$|\,\mathrm{GL}_n(\mathbf{F}_q)_p/\,\mathrm{GL}_n(\mathbf{F}_q)| = \frac{1}{n!}\sum_{\lambda\vdash n} T(\lambda)\prod_{b\in\lambda}(q^b-1)_p$$

where λ ranges over all partitions of n and $T(\lambda)$ is the number of permutations of cycle type λ in Σ_n.

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