My Research

My research is centered in the fields of orthogonal polynomials and special functions and focuses on application of methods from complex analysis. Here I describe some characteristics of the areas, how they are related to other fields, and some of my contributions.

What is a special function?

On one hand it must be more complicated than the elementary functions such as the trigonometric functions. On the other hand there should be something *special* about it; it could e.g. appear in various mathematical areas so often that it deserves a name!

Euler's gamma function is a classical function, introduced hundreds of years ago. It appears almost everywhere, and it has a name. It is a natural extension to the positive real line of the factorial function. It is usually defined via Euler's first integral for positive real values of the variable and it has a meromorphic extension to the entire complex plane with poles at the non-positive integers.

What is an orthogonal polynomial?

This is not the right question, since orthogonal polynomials never come alone; but as a sequence. Given a positive measure on the real line with respect to which any polynomial is integrable then we perform Gram-Schmidt orthogonalization of $\{1, x, x^2, \ldots\}$ in the L^2-space and obtain the corresponding sequence of orthogonal polynomials. The theory of orthogonal polynomials deals very roughly with abstract results and more concrete results. How are orthogonal polynomials related to special functions? Let us consider a concrete measure on the real line, perhaps of compact support. Compute the orthogonal polynomials! In this way the classical Hermite, Laguerre and Chebychev polynomials appear. This gives a link between orthogonal polynomials and special functions.

Think positive!

Positive definite matrices and operators, positive measures and ordinary positive functions. Positive harmonic functions play a prominent role in potential theory and complex analysis. A certain class of holomorphic functions, the so-called Nevanlinna-Pick functions and Stieltjes functions, enters in the study of Jacobi matrices, orthogonal polynomials and special functions. A Nevanlinna-Pick function is simply a holomorphic function defined in the upper half plane and mapping it into itself. (Its imaginary part is thus a positive harmonic function.)

Examples

Let me conclude by giving a few examples indicating how complex methods, and in particular the Nevanlinna-Pick functions and special functions come together. The median in the gamma distribution is an implicitly defined function $m(x)$ of the shape parameter x:

$$\frac{1}{\Gamma(x)}\int_0^{m(x)} e^{-t}t^{x-1}\,dt = \frac{1}{2}.$$

The problem of finding properties of $m(x)$ is a real valued one, but it can be attacked by complex methods. ("Real problems call for complex solutions".) One can indeed show that the median is a convex function, and hence resolve the so-called Chen-Rubin conjecture, using Nevanlinna-Pick functions and conformal mapping in a crucial place of the proof!

A subclass of the Nevanlinna-Pick functions enters in the theory of operator monotone functions and this has lead to the study of certain inverse functions of the gamma function. But there are more gamma functions, despite the Bohr-Mollerup theorem. Some are called multiple gamma functions, and they were introduced by Barnes more than a century ago. They constitute a kind of hierarchy $\{\Gamma_n\}$ satisfying recurrence relations of the form

$$\Gamma_{n+1}(x+1) = \Gamma_n(x)\Gamma_{n+1}(x).$$

(Putting $\Gamma_0(x) = x$ and $\Gamma_1(x) = \Gamma(x)$ we revisit the familiar functional equation for Euler's gamma function.) The so-called double gamma function, Γ_2 can be studied using conformal mapping, and one of its inverses can be extended to a Nevanlinna-Pick function.

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