December 25th, 2023

MY RESEARCH Algebraic-Arithmetic Geometry: Cohomology Theories and Crystals

The areas of research in which I have been active during my mathematical life include complex algebraic geometry, Hodge theory, algebraic cycles, algebraic geometry over characteristic p-fields, in unequal characteristic, cohomology theories, and their coefficients, relation between arithmetic and geometric properties.

In the sequel I select three specific directions in which I initiated a program (for a large part together with co-authors) and could realize some steps of it.

1) a) The Lang-Manin conjecture predicts that Fano varieties (more generally (separably) rationally connected varieties) over a $C1$ field have a rational point. A field k is $C1$ is any hypersurface $X \subset \mathbb{P}^n_k$ of degree $d \leq n$ possesses a k-rational point. The most prominent fields which are $C1$ are the finite fields (Chevalley-Warning). I proved the conjecture (*Inventiones* 2002) by establishing an analogy between the notion of Hodge level ≥ 1 of the complex cohomology $H^i(X(\mathbb{C}), \mathbb{C})$ of a smooth complex projective variety $X/\mathbb{C}$ on one hand and the level in the sense of q -divisibility of the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ of a smooth projective variety $X/\mathbb{F}_q$ defined over the finite field $\mathbb{F}_q$ on the other hand.

The analogy relies on the following parallel. If for $X/\mathbb{C}$ as above, $H^i(X(\mathbb{C}), \mathbb{C})$ is supported in codimension ≥ 1, that is if there is a divisor $Z \subset X$ such that the restriction homomorphism $H^i(X(\mathbb{C}), \mathbb{C}) \to H^i((X \setminus Z)(\mathbb{C}), \mathbb{C})$ vanishes, then the Hodge level of $H^i(X(\mathbb{C}), \mathbb{C})$ is ≥ 1, that is $H^i(X, \mathcal{O}) = 0$ (Deligne, Hodge II). If for $X/\mathbb{F}_q$ as above, $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1, that is if there is a divisor $Z \subset X$ such that the restriction homomorphism $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell) \to H^i(((X \setminus Z)_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ vanishes, then the eigenvalues of the geometric Frobenius acting on $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, which are algebraic integers (Deligne), are divisible by q as such (a consequence of Gabber's purity and localization).

1) b) On the complex side, Grothendieck's generalized Hodge conjecture predicts the converse: if $H^i(X, \mathcal{O}) = 0$, that is if $H^i(X(\mathbb{C}), \mathbb{C})$ has Hodge level ≥ 1, it should imply that $H^i(X(\mathbb{C}), \mathbb{C})$ is supported in codimension ≥ 1. Except for $i = 1$, where it follows from Hodge duality, and for $i = 2$, where it follows from the existence of the exponential sequence, both ingredients being complex analytic (and not algebraic), we do not know the conjecture. One may hope that prismatic cohomology (Bhatt-Scholze) yields a non-trivial information on this profound question (current discussions with Mark Kisin).

2) Simpson's conjecture, which is anchored in complex geometry, predicts that rigid local systems are of geometric origin. By the work of Schloze, Liu-Zhu and Petrov (all of them p-adic), this conjecture is the consequence of the so-called relative Fontaine-Mazur conjecture (for geometry over number fields). If true, they are in particular integral. We proved with Groechenig integrality for cohomologically rigid local systems using the arithmetic Langlands program (Drinfeld, L. Lafforgue) and the resulting companions on smooth quasi-projective varieties over $\mathbb{F}_q$, the existence of which had been predicted by Deligne in Weil II.

If true, some crystalline properties should be true as well. In particular the p-adic completion of the local system, viewed on the p-adic variety, should descend to the variety over a non-ramified p-adic field and there be crystalline (in the sense of Fontaine) for p large. With Groechenig we proved this for X projective (*Acta Mathematica* 2020), also for Shimura varieties of real rank ≥ 2 for which Margulis super-rigidity holds. This latter result is a building block of the recent proof by Shankar-Pila-Tsimerman of the André-Oort conjecture in this case. A very recent proof of those results uses, aside of Faltings' functor from Fontaine-Lafaille modules to crystalline local systems, the classical strong p-adic topology on W-points of a scheme defined over W, where W is a non-ramified p-adic ring.

3) The Gieseker conjecture predicts that the étale fundamental group in characteristic $p > 0$ controls crystals on the infinitesimal site in the sense that if the étale fundamental group is trivial, so are all crystals on the infinitesimal site. This is known for smooth projective varieties (joint with Mehta, *Inventiones* 2010). De Jong generalized this conjecture in characteristic $p > 0$ to isocrystals in the crystalline site. Small progress has been made with Shiho, but nothing really definitive. Likely the reason for the stalemate relies on the fact that the proof goes through characteristic $p > 0$ while the problem is p-adic. It would be wishful to use the topological p-adic viewpoint mentioned in 2) to understand the dynamic of the Frobenius to approach de Jong's conjecture (discussions with Groechenig).