My research
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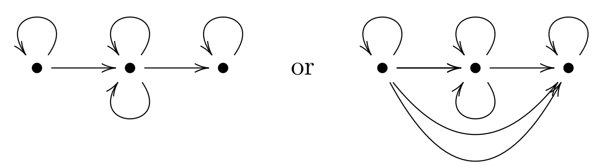
The quest for classifying C^*-algebras by their K-theory goes back to the late 1970's, and in the four decades in which it has been a major ambition for operator algebraists to prove such results, there has been a strong Copenhagen presence in this area. Indeed, when George Elliott proved the first seminal results, he was employed here as an associate professor, and he has directly or indirectly (in my own case: Very directly as he was a *de facto* PhD advisor for me) influenced most of the operator algebra group into contributing over the years. In particular, the area in which I presently work is highly influenced by the early work of Mikael Rørdam, although I have to concede that he was employed at Odense at the time he did it.

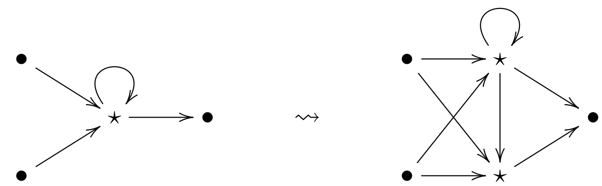
It would take me too far to define all the objects involved, but the idea is very easy to convey. C^*-algebras are functional analytic objects, infinite normed vector spaces, which originate in quantum physics but have proven useful in a multitude of settings. To each C^*-algebra one associates a number of K-groups – ordered abelian groups – and it turns out that the axioms defining C^*-algebras are so inherently rigid that the K-groups in many cases are complete invariants in the sense that the C^*-algebras in a certain class are the same if and only if their K-groups are the same. Since the K-groups are often easy to compute and compare, results of this nature are extremely useful in applications.

Recently, I have worked on understanding the classification theory for the class of Cuntz-Krieger C^*-algebras which are associated to finite graphs such as

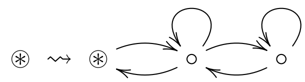


and are directly related to topological dynamics. Mikael proved that those Cuntz-Krieger algebras which are simple are classified by their K-theory, and this was generalized to all Cuntz-Krieger algebras with finitely many ideals by Copenhagen student Gunnar Restorff in his *speciale* in 2003. Working with Gunnar (who is now in Torshavn) as well as Efren Ruiz (Hawaii) and Adam Sørensen (also from here, now in Oslo) we developed over the course of several years the point of view that rather than working directly with K-theory, one should attempt to explain isomorphisms of the Cuntz-Krieger algebra in terms of "moves" of the graph in a way inspired by Reidemeister moves on knots: Two graphs give the same C^*-algebra if and only if one can be transformed into the other by a finite sequence of such moves. The moves in question look like

or

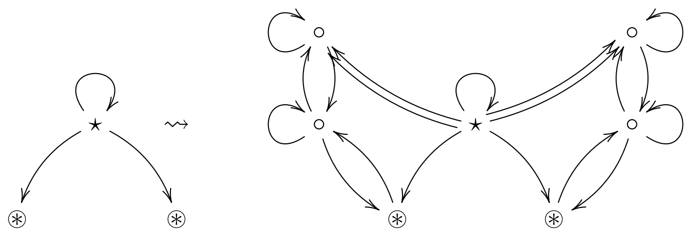


or even



where the latter move is exactly the one that Mikael proved invariance of. The circle on the anchor vertex of the move indicates that this vertex must satisfy certain conditions for the move to be “legal”.

And a couple of years ago, inspired by computer experiments which told us precisely where to look (the pair of graphs on the previous page!), we found an entirely new such move, which allows the mutation of a graph as



It was very challenging to prove that this move does not change the C^*-algebra, but in fact pretty easy to see then that this new move makes the list of moves complete in the sense that the C^*-algebras are the same precisely when there are moves leading from one graph to another. And as an easy corollary one sees that indeed all Cuntz-Krieger algebras are classified by K-theory, which leads to the conclusion that there is an algorithm to decide whether or not two such C^*-algebras are isomorphic.

When we proved this result I must admit I thought with some relief I was done with Cuntz-Krieger algebras, but something very interesting has recently happened which has made me revise this sentiment. Indeed, Toke Carlsen (the last Copenhagener I am going to mention, also now in Torshavn) and James Rout (Technion) showed that if one takes some additional structure of the Cuntz-Krieger algebra into account, then they actually remember the isomorphism class of the underlying dynamics. When one combines this astonishing result with a classical one by Krieger, one learns that the (in)famous Williams problem from symbolic dynamics can be completely recast as a question in C^*-algebras. So now my coauthors and I are getting back to work on this, trying among many other things to find moves describing those isomorphisms that preserve this added structure, in the hope that a C^*-algebraic approach may eventually be able to provide new tools towards answering the remaining open question of Williams', namely whether or not isomorphism among such dynamical systems is a decidable property.