MY RESEARCH
HOLOMORPHIC DYNAMICS

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I have been working in and with (discrete) holomorphic dynamics, complex analysis and potential theory all of my professional life. Holomorphic dynamical systems are dynamical systems with phase space a Riemann surface $\mathbb{X}$, often the complex line $\mathbb{C}$ or the extended complex plane $\widehat{\mathbb{C}}$ also known as the Riemann's sphere or the complex projective line and dynamical law given by iteration of one or more holomorphic maps f from $\mathbb{X}$ to itself. Prime examples are iteration of one quadratic polynomial with phase space $\widehat{\mathbb{C}}$. Even though the dynamical law of such a dynamical system is very simple, the dynamics is generally very complicated. One reason is that the degree of the iterates diverge to infinity, in fact does so exponentially fast. The typical questions are "what is the long term behavior of orbits?" and "how does the dynamics change, when we change the dynamical law?" The first question about individual dynamical systems necessarily has a non-simple answer, because the phase space always contains an invariant compact set J_f called the Julia set on which dynamics is chaotic. The second question may or may not have a complicated answer depending on the family or parameter space one is looking at and where in the parameterspace one is looking. The family of quadratic polynomial dynamics is conveniently parametrized up to affine conjugacy by the complex line of maps $Q_c(z) = z^2 + c$, $c \in \mathbb{C}$. Even for the quadratic family we do not have a complete picture. The Mandelbrot set $M = \{c \mid J_c = J_{Q_c} \text{ is connected } \}$ is a geographers chart of the world of dynamical systems given by quadratic polynomials. It is a compact Dirichlet regular continuum. It has a very complicated topological and geometric structure. it contains for instance quasi-conformal copies of it self on all scales. In fact such copies are dense in its boundary (see also Figure 1 below).

The central cardioid bounds a hyperbolic component H_0 in which all maps have an attracting fixed point. The fixed point eigenvalue map on the hyperbolic component coincides with the Riemann map sending sending 0 to 0 with positive derivative. It extends as a homeomorphism between the closures, so that boundary maps have a fixed-point eigenvalue of the form $e^{i2\pi\theta}$, $\theta \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. For a large class of θ-values the map has a Siegel disk, a topological disk on which the map is linearizeable, i.e. conformally conjugate to the rigid rotation of angle θ. These dynamical systems are interesting in their own right, but the whole discussion of wether a map with given rotation number θ is linearizeable or not depends sensitively on how well θ is approximated by rational numbers and is related to KAM-theory. In my early career I first solved the problems, does the corresponding Julia sets have zero area and are they locally connected if θ is at least of bounded type? The answer turned out to be yes on both occasions. Here bounded type means the coefficients in the continued fraktion

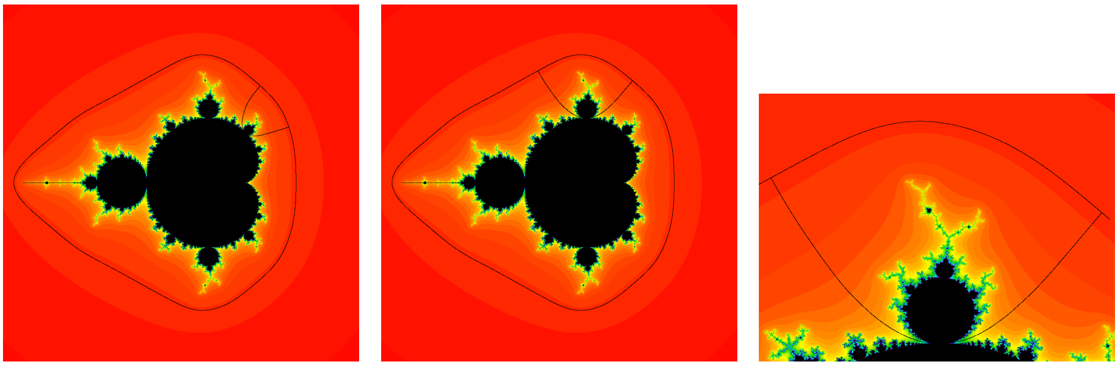

FIGURE 1. Images of the Mandelbrot set M (in black) with colocoding of the complement according to levels of the Green's function for the complement. Technically each common boundary curve of two neighbouring color bands is an equipotential curve for the Green's function. One of these equipotential curves is also colored black together with segments of two field lines descending to the root of the $1/4$-limb (left), to the $1/3$-limb (center) and a blow-up of the $1/3$-limb right.

expansion of θ are bounded [13]. Local connectivity implies we have faithfull topological models of the Julia sets and hence of the dynamics. This was used by Curtis T. McMullen to understand also the geometry of such Julia sets and to prove that their Hausdorff dimension is strictly less than 2, [12]. Some years later I generalised in collaboration with Saeed Zakeri at CUNY these results to a full (Lebesgue) measure set of rotation numbers, [14].

The points of the cardioid for which the rotation number is rational, say $\theta = p/q$, $(p, q) = 1$ are special separation points of M called root points. The complement in M of such a point falls in two connected components the central component containing 0 and a peripheral component, which together with the disconnecting root point is denoted the p/q-limb or the limb of rotation number p/q. The root of the limb is also the root (meaning corresponding to the cusp point of the cardioid) of a homeomorphic copy of M, which has a smoothly bounded central hyperbolic component instead of a cusp. It was the standing conjecture for more than 20 years that these topological copies should be mutually quasi-conformally homeomorphic. I disproved the full generality of this conjecture together with Luna Lomonaco (a former Ph.D. student) now at IMPA in Rio by showing it is false if the rotation numbers of the limbs have different denominators, [10]. And we have just completed writing a proof that the conjecture restricted to satellite copies in limbs with the same denominator are q.c.-homeomorphic, thus settling the problem completely [11].

The dynamics of cubic polynomials is by far less understood than that of quadratic polynomials. Bodil Branner and John Hamal Hubbard were the first to define and study the cubic connectedness locus $\mathcal{C}$, [2], [3]. There has since then been several deep results on $\mathcal{C}$, but no general clear picture of the topology and geometry of $\mathcal{C}$. For the Mandelbrot set M the notion of limbs and the dynamical property underlying their definition was first described by Adrien Douady and Hubbard. It played a crucial role in

Jean-Christofe Yoccoz construction of Markov partitions for the (quadratic) polynomials in the limb and subsequently to his celebrated theorems on local connectivity of the Mandelbrot set at almost all (for the equilibrium measure on M) parameters. These results were a giant step forward leaving only the infinitely renormalizeable parameters as white dots or perhaps spots on the geographers chart. (A parameter c is Infinitely renormalizeable if it belongs to the intersection of infinitely many nested copies of M in M.) In a just finished joint work with Zakeri we have made a first large step towards a limb-structure description of the cubic connectedness locus $\mathcal{C}$, [15]. This gives a unified approach to several of the previous results about $\mathcal{C}$. Moreover we expect that a fully developed limb structure of the cubic connectedness locus can be the basis of a divide an conquer strategi to understand the cubic connectedness locus, in the spirit of Yoccoz approach to the Mandelbrot set.

Over the past 10 years I have used my expertise in holomorphic dynamics to shed new light on families of extremal polynomials, such as the sequence of normalized Chebyshev polynomials for a given non-polar compact set K and the sequence of orthonormal polynomials for a probability measure μ on $\mathbb{C}$ with compact non-polar support. If K is the support of μ those polynomials are found as the minimal monic polynomial in each degree normalized to norm 1 in $L^p(\mu)$ for $\mu = \infty, 2$ respectively. More generally there is for each $p > 0$ and n a unique normalized minimal polynomial in $L^p(\mu)$ of degree n. A priori there was no tangible connection between such extremal polynomials and dynamical systems other than in both cases we are interested in particular sequences of polynomials. However together with my research group Henrik Laurberg Pedersen, Jacob Stordal Christiansen, Christian Henriksen we have shown that the Julia sets and the filled Julia sets, the polynomially convex hulls of the Julia set of the Chebyshev polynomial in some precise sense converge towards the outer boundary of K and the filled K, [5]. And similarly for the orthonormal polynomials for μ under som weak conditions on μ, [4]. These results has afterwards been generalized by Bayraktar and Efe [1] to $L^p(\mu)$, $p > 0$.

The following excerpt from [4] illustrates our convergence theorems for orthonormal polynomials: Fig. 2 on page 4 illustrates our convergence theorems for sequences of orthonomal polynomials for μ the equilibrium measure for the boundary of the boomerang-shaped white set K in the top left image. The black fractal sets in the other images are the Julia sets J_n for $n = 10, 15$ and 20 (which in these cases appear to be equal to the filled Julia sets K_n). As is custom in holomorphic dynamics, the Green's functions are visualised by coloring alternating intervals of level sets blue and red. The green curve is the support of the measure (i.e., the outer boundary J of K). Note that the equilibrium measures belong to a special class of measures, the so-called *n-th root regular measures*.

Over the past two years Henriksen and I together with Eva Uhre have made progress on understanding the properties of more general families of polynomials of strictly increasing degrees sharing the basic properties of sequences of extremal polynomials. More precisely we impose two conditions the first satisfied by all sequences of extremal polynomials and the second satisfied for a large class of extremal polynomials.

Let Ω denote the unbounded connected component of the compact set K. We say a sequence of polynomials is root-sparse in Ω if the root loci are uniformly bounded and

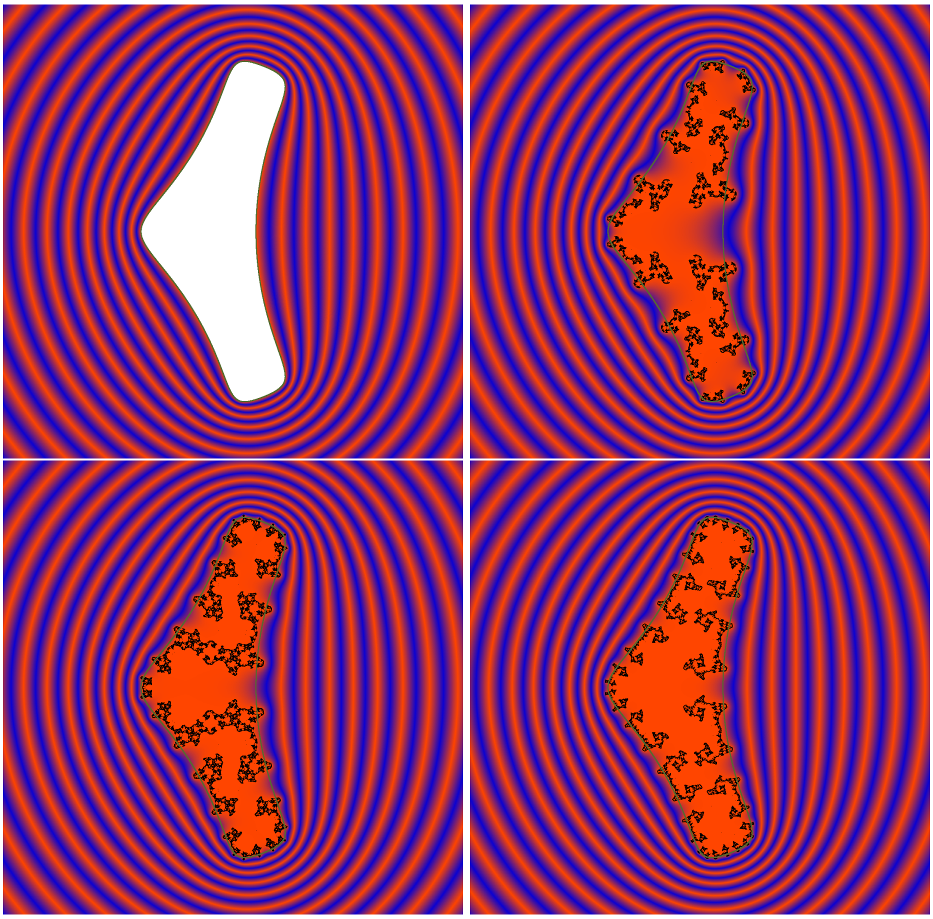

FIGURE 2. Top left a Jordan curve bounding the white region K followed by successive images of the filled Julia sets K_{10}, K_{15} and K_{20}. The alternating blue and red areas indicate successive bands of levels for the Green's function for the complement of K and of the filled Julia sets respectively. As the degree of the polynomials are rather high, the coloring scheme of equipotentials is finer than the standard coloring scheme, which simply colors according to the number of iterations it takes for the center of the pixel to escape to a fixed large potential (Images by Christian Henriksen).

for any compact subset $L \subset \Omega$ there exists $M(L)$ such that each polynomial has at most $M(L)$ roots in L.

Moreover we say that the sequence of polynomials $(q_k)_k$ of degrees $n_k \nearrow \infty$ is (n-th root) regular if and only if $\frac{1}{n_k} \log |q_k|$ converges to the Green's function for Ω.

Inspired by the situation in holomorphic dynamics we have proven that several "characteristic" features of extremal polynomial sequences are shared by all regular and root-sparse sequence's of polynomials. For instance regularity and root sparsity are inherited by the sequences of derivatives of any fixed order [6], [7], [8], [9].

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