MY RESEARCH AREA

Classical descriptive set theory is a field concerned with studying the properties of concretely definable subsets of the real line $\mathbb{R}$, and other similar topological spaces. It originated in the work of Cantor, Lebesgue, Lusin and Suslin, among others. The archetypical example of a concretely definable set is a Borel set: Such a set can be built from open intervals by repeatedly taking countable unions and complements.

A major rethinking of descriptive set theory occurred from late 1950s to the early 1970s, when the field *effective descriptive set theory* was developed by people like Addison, Moschovakis, Kechris and Solovay, among many others. From this point of view, descriptive set theory is established as a theory parallel to computability theory of functions on the natural numbers in the sense of Turing and Church. That is, a computability theory of the continuous. Pushing aside the intricacies, and lying only slightly, the upshot is this: Borel functions on $\mathbb{R}$, and similar topological spaces, are quite a good analogue of computable functions.

While descriptive set theorists of the 1950s and 1960s were busy developing effective descriptive set theory, Mackey, Thoma, Glimm and Effros were studying classification problems of (usually infinite dimensional) unitary representations of groups and C^*-algebras.

For Abelian countable groups, Pontryagin's duality provides a satisfying classification of irreducible representations in terms of characters, but what about the irreducible representations of general countable groups?

From one point of view this question is trivial, by way of the following trick: There are at most continuum many irreducible unitary representations of a countable group up to isomorphism, so we can in principle assign to each isomorphism class a unique real number as an invariant.

This, however, is a deeply unsatisfying classification because it does not give us a concrete way of *computing* from a given irreducible unitary representation the invariant. In fact, if we accept the tenet that in the continuous setting Borel means computable, then nothing like the above "trick classification" of irreducible representations is possible in general.

The troublemaker is the *Vitali equivalence relation* E_v on $\mathbb{R}$: Two real numbers $x, y \in \mathbb{R}$ are E_v-equivalent iff $x - y$ is rational. There is no Borel (or measurable) function $f : \mathbb{R} \to \mathbb{R}$ such that xE_vy if and only if $f(x) = f(y)$ (it is a fun exercise to show this). That is, there is no way of classifying real numbers up to rational translation by assigning to each class a single real number as a complete invariant, *if* we require that this assignment be Borel (or just measurable).

The combined work of Effros, Glimm and Thoma in the 1960s showed that if G is a countable group that is not Abelian-by-finite, then E_v can be embedded in a Borel way into the isomorphism problem for irreducible unitary representations of G, thus obstructing a concrete classification.

To Mackey and his contemporaries, the idea that Borel means computable was taken to be intuitively true; they were unaware of the development of effective descriptive set theory, but were vindicated by it. It also seems that they thought that once E_v was shown to be embedded into an isomorphism problem, then all hope was lost.

In the late 1980s, descriptive set theorists realized that a better view is that E_v represents just one of many *degrees of complexity* that a classification problem can have; and it is a rather low one.

The key definition is the following: Given a space X whose points are interesting "objects" we wish to classify (e.g., unitary representations), and denoting the isomorphism relation among the objects by E (which is then an equivalence relation in X), and given a space Y of potential invariants, and an equivalence relation F on Y among those invariants, it would be natural to say that the points of X can be classified up to E-equivalence by a computable (!) assignment of invariants in Y, modulo F-equivalence, if there is a Borel $f : X \to Y$ such that

$$xEy \iff f(x)Ff(y).$$

In this case we write $E \leq_B F$ (read "E is Borel reducible to F"), and think of E as being "at most as complicated" as F. If both $E \leq_B F$ and $F \leq_B E$, then we write $E \equiv_B F$ (read "E is Borel bi-reducible to F"). The $\equiv_B$ classes may then be thought of as "degrees of complexity" for classification problems.

The results of Effros, Glimm and Thoma then show that when G is a countable group which is not Abelian-by-finite, then E_v is Borel reducible to the isomorphism relation on irreducible unitary representations of G.

One may wonder if the converse is true, in which case things wouldn't be so bad at all, because it would seem like a really nice theorem to be able to classify irreducible unitary representations by real numbers modulo rational differences!

Alas, the news is very bad: In the late 1990s, Gregory Hjorth developed a "theory of turbulence" for equivalence relations, which allowed him to prove that there are many natural equivalence relations that are far, far more complex than E_v, including the isomorphism relation on irreducible representations of groups that are not Abelian-by-finite.

Hjorth's theory is a watershed in the study of continuous actions of "large" topological groups, such as the unitary group of infinite dimensional Hilbert space, or the permutation group of the natural numbers; and it has been applied far and wide by now. For instance, we now know that the main classification problem in ergodic theory, of ergodic measure-preserving transformations of the unit interval, is turbulent (Foreman-Weiss, 2003). This despite the success of Ornstein in classifying the very special class of Bernoulli shifts, or the classification of transformations with pure point spectrum by von Neumann.

Operator algebras is, of course, an area ripe with classification problems of central interest, and here the news is bad, too. The isomorphism problem on nuclear, simple, separable, unital C*-algebras is turbulent (Farah-Toms-Törnquist, 2011), and while there is also a natural upper bound on the complexity (Elliott-Farah-Paulsen-Rosendal-Toms-Törnquist, 2013), it was shown that this upper bound is in fact achieved by this isomorphism problem (Marcin, 2014).

Von Neumann algebras fare little better: The isomorphism problem on separably acting von Neumann factors (in every type class, except type I) is turbulent (Sasyk-Törnquist, 2009). We also identified an upper bound on the complexity of classifying von Neumann factors, but it remains open if this upper bound is achieved. Answering this is closely related to understanding certain complicated continuous actions of the group of unitary operators on an infinite dimensional separable Hilbert space. And this is emblematic of much of descriptive set theory today: New results are achieved by studying large topological groups and their continuous actions, and this in turn is related to understanding the confluence of combinatorics (finite and infinite) and analysis.

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