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Abstract: We present an intensity based approach to modelling default risk and the term structure of credit risky bonds. Intensities of default are random but may depend on factors affecting the default-free term structure of interest rates. The framework is convenient for pricing derivative securities with risk of counterparty default and derivative securities written on credit spreads. An extension of a Markovian model proposed by Jarrow, Lando and Turnbull is given and we discuss a notion of conditionally non-systematic default-risk which is related to the use of emprically estimated hazard functions for pricing.

Key Words: Credit risk, Cox processes, term structure, derivative securities.

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1 Introduction

The aim of this paper is to illustrate how Cox processes - also known as doubly stochastic Poisson processes - provide a useful framework for modelling prices of financial instruments in which credit risk is a significant factor. Both the case where credit risk enters because of the risk of counterparty default and the case where some measure of credit risk such as a credit spread is used as an underlying variable in a derivative contract can be analyzed within our framework. The starting point is the modelling of prices of bonds issued by a party who may default. We will model the time of default (or any form of distress leading to a non-fulfillment of contractual obligations by a party) as the first jump time of a Cox process which for now can be thought of as a Poisson process with a random intensity.

The modelling framework is similar to that of Litterman and Iben (1991), Jarrow, Lando and Turnbull (1993), Jarrow and Turnbull (1992), Lando (1994) and Madan and Unal (1993) in that the event of default is not described explicitly as a first hitting time of some process modelling the value of the firm. Also, the recovery rate, i.e. the fraction of the face value of debt that bond holders receive in the event of default, is exogenously specified.

Let us briefly recall two classes of models for corporate debt which are based on modelling the value of the bond-issuing firm: The classical approach, originating in Black and Scholes (1973) and Merton (1974), uses option pricing theory to price corporate debt. At the maturity date of some zero coupon debt issue, default occurs if the value of the firm is less than the face value of the debt. The recovery rate is endogenously described as the ratio of the value of the firm to the face value of debt. A newer class of models declares default when the value of the firm crosses some boundary which may be either deterministic or random, but the recovery rate is then defined exogenously. Authors using this approach include

Nielsen, Saá-Requejo and Santa-Clara (1993) and Longstaff and Schwartz (1992). The model of Hull and White (1992) is also based on the notion of a first hitting time of a stochastic process, but this process need not be the value of the firm.

Modelling credit risk from the fundamental variable 'value of the firm' is conceptually very important but it presents some problems when trying to implement the models: First, the value of the firm is in general hard to observe and in the models we are aware of so is the boundary which the firm value must cross for default to occur. Second, first hitting times of processes are generally hard to compute.

In this paper default is modelled as an unpredictable, Poisson-type event. More specifically default is modelled as the first jump of a Cox process which is essentially a Poisson process with a random intensity. Observables which are known to influence the occurence of defaults govern the jump intensity of the Cox process and hence the likelihood of default may increase or decrease over time as state variables change. These state variables can be firm specific, i.e. describe the leverage of the firm or the industry to which the firm belongs, or they can be more general variables such as yields on treasury bonds.

The framework closely parallels that of intensity models in survival analysis with time dependent covariates and has the potential of using the rich set of statistical methods which is available for determining whether and to what extent certain covariates are relevant indicators of default risk. One important complication compared to survival analysis is that in a financial model we need to specify the probabilistic behavior of the covariates (i.e. state variables) in order to be able to calculate default probabilities. In survival analysis covariates are usually treated as exogenous variables and inference is done using conditional or partial likelihood which only incorporates the observed sample path of the covariates. Furthermore, even when the evolution of the state variables and the intensity of default are specified, we still need a way of pricing the default risk since there is no reason to expect that the 'risk neutral' probabilities are the ones by which the market values default risky bonds. We will return to this point later.

There are several works which consider intensity based models of default, such as for example Artzner and Delbaen (1993), Duffie, Schroder and Skiadas (1994), (1994), Jarrow and Turnbull (1992), Jarrow, Lando and Turnbull (1993), Chapter 2 of Lando (1994) and Madan and Unal (1993). But to our knowledge relatively little work has been done to produce continuous-time intensity based models which allow hazard rates of default to be correlated with the term structure of default-free bonds and which at the same time allow reasonably explicit calculation of risky bond prices and valuation of derivatives with credit risk elements. The independent work of Duffie and Singleton (1994) notes the Feynman-Kac representation of prices of credit risky bonds and derivatives in a Markovian setting, but apart from that the above mentioned references either consider an abstract model of the intensity process of default without adressing more specific models and computational issues or they use an assumption of independence between the default process and the evolution of the default-free term structure. This paper presents a framework for modelling intensities of default which does not require an independence assumption but still produces models with good properties from a computational and a statistical point of view.

The key assumption is that when conditioning on certain state variables default occurs at the first jump of a non-homogeneous Poisson process. Within this framework we have a fair amount of flexibility in how we specify the behavior of the state variables which simultaneously determine the default free term structure and the likelihood of default. When diffusions are used as state variables, the Feynman-Kac framework handles pricing of credit risky derivative securities in virtually the same way as it handles derivatives in which credit risk is not present. That particular setup is one of 'diffusion with killing' - a special case of the general framework. Another special case, which we think will be important when implementing models, is where a Markov chain modulates the intensity of default. We show that the model of Jarrow, Lando and Turnbull (1993) falls within our framework and how it can be generalized to include important covariates in addition to credit rating. In both of the special cases mentioned above we are able to let the default process depend on the same state variables as those governing the evolution of the riskless bond prices.

The structure of the paper is as follows: In section 2 we outline the basic construction of a Cox process, concentrating on the first jump time of such a process. We then show in section 3 how to calculate prices of bonds with default risk in our setting and in make a note of the direction of the bias in implied default probabilities when these are calculated as if default probabilities and the evolution of riskless bonds were independent.

Section 4 generalizes a Markovian model proposed by Jarrow, Lando and Turnbull (1993) to include transition rates between credit ratings which depend on the state variables. Section 5 deals with contingent claims pricing when the state variables are diffusions. It is seen that very general contingent claims with default risk can be handled through the Feynman-Kac formula. Finally, in section 6 we show that if default risk is non-systematic when we condition on the evolution of the state variables, we can use the empirically estimated hazard rates to calculate risky bond prices. An important distinction is made here: Even though we use empirically estimated hazard rates, this does not mean that the implied default probabilities are the same as the empirically observed ones. It means that the adjustment for risk takes place through the dependence of the intensity on the state-variables, not by changing the intensity function itself.

2 Construction of a Cox Process

Before setting up our model of credit risky bonds, we give both an intuitive and a formal description of how the default process is modelled.

Recall that an inhomogeneous Poisson process N with (non-negative) intensity function $l(\cdot)$ satisfies

$$P(N_t - N_s = k) = \frac{\left(\int_s^t l(u)\,du\right)^k}{k!} \exp\left(-\int_s^t l(u)\,du\right) \qquad k = 0, 1, \ldots.$$

In particular, assuming $N_0 = 0$, we have

$$P(N_t = 0) = \exp\left(-\int_0^t l(u)\,du\right).$$

A way of simulating the first jump τ of N is to let E_1 be a unit exponential random variable and define

$$\tau = \inf\{t : \int_0^t l(u)\,du \geq E_1\} \tag{2.1}$$

A Cox process is a generalization of the Poisson process in which the intensity is allowed to be random but in such a way that if we condition on a particular realization $l(\cdot, \omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $l(s, \omega)$.

In this paper we will write the random intensity on the form

$$l(s, \omega) = \lambda(X_s)$$

where X is an R^d–valued stochastic process and $\lambda : R^d \rightarrow [0, \infty)$ is a non-negative, continuous function.

The state variables will include interest rates on riskless debt and may include stock prices, credit ratings and other variables deemed relevant for predicting the likelihood of default. Heuristically, given that a firm has survived up to time t, and given X_t, the probability of defaulting within the next small time interval Δt is equal to $\lambda(X_t)\Delta t + o(\Delta t)$.

Formally, we have a probability space $(\Omega, \mathcal{F}, P)$ large enough to support an R^d-valued stochastic process $X = \{X_t : 0 \leq t \leq T_f\}$ which is right-continuous with left limits and a unit exponential random variable E_1 which is independent of X. Given also is a function $\lambda : R^d \to R$ which we assume is strictly positive and continuous. From these two ingredients we define the default time τ as follows:

$$\tau = \inf\{t : \int_0^t \lambda(X_s)\, ds \geq E_1\}. \tag{2.2}$$

This default time can be thought of as the first jump time of a Cox process with intensity process $\lambda(X_s)$. Note that this is an exact analogue to equation (2.1) with a random intensity replacing the deterministic intensity function.

When $\lambda(X_s)$ is large, the integrated hazard grows faster and reaches the level of the independent exponential variable faster, and therefore the probability that τ is small becomes higher. Sample paths for which the integral of the intensity is infinite over the interval $[0, T_f]$ are paths for which default always occurs.

From the definition we get the following key relationships:

$$P\left(\tau > t | \left(X_s\right)_{0\leq s\leq t}\right) = \exp\left(-\int_0^t \lambda\left(X_s\right) ds\right) \qquad t \in [0, T_f] \tag{2.3}$$

$$P\left(\tau > t\right) = E\exp\left(-\int_0^t \lambda\left(X_s\right) ds\right) \qquad t \in [0, T_f] \tag{2.4}$$

What we have modelled above, is only the first jump of a Cox process. When modelling a Cox process past the first jump one has to insure that the integrated intensity stays finite on finite intervals if explosions are to be avoided. One then proceeds as follows: Given a probability space $(\Omega, \mathcal{F}, P)$ large enough to support a standard unit rate Poisson process N with $N_0 = 0$ and a non-negative stochastic process $\lambda(t)$ which is independent of N and assumed to be right-continuous and integrable on finite intervals, i.e.

$$\Lambda(t) := \int_0^t \lambda(s)\, ds < \infty \qquad \text{a.s for all } t$$

Then $\Lambda(0) = 0$ and Λ has non-decreasing realizations. Now defining

$$\tilde{N}_t := N(\Lambda(t))$$

we have a Cox process with intensity measure Λ. For the technical conditions we need to check to see that this definition makes sense, see Grandell (1976) pp. 9-16. This approach will be relevant for extending the models presented in this paper to cases of repeated defaults by the same firm.

3 Bond Prices

In this section we construct a model of zero coupon bonds issued by a single firm (or agent) with credit risk. The intention is to obtain a term structure of credit risk for the firm which in turn can be used to price derivative contracts (such as swaps) in which the firm is engaged.

The initial setup for pricing bonds is the same as in Jarrow and Turnbull (1992) and Jarrow, Lando and Turnbull (1993): Denote by $p(t, T)$ the price at time t of a default free bond paying 1 unit of account at maturity T, and let $v(t, T)$ denote the time t price of a corporate bond maturing at T. The probability measure P is a martingale measure under which prices are calculated. We do not attempt to link the choice of martingale measure to a particular general equilibrium setting since we feel it would add little insight. An APT type argument is given at the end, however, to justify our choice of intensity function. Throughout, we let E_t denote conditional expectation under P given information $\mathcal{F}_t$ at time t.

Default is modelled by the random variable τ and in the event of default the risky firm is assumed to pay an exogenously given fraction $\delta \in [0,1)$ of its promised payment. We then have the following expressions for the money market account B and for bond prices:

$$
\begin{aligned}
B(t) &= \exp\left(\int_0^t R(X_s)\,ds\right) \\
p(t,T) &= E_t\left(\frac{B(t)}{B(T)}\right) \\
v(t,T) &= E_t\left(\frac{B(t)}{B(T)}\left(\delta 1_{\{\tau \le T\}} + 1_{\{\tau > T\}}\right)\right)
\end{aligned}
$$

To model τ we use the framework described in the previous section. We have a probability space as defined in the previous section which supports the state variable proccess X and a unit exponential random variable E_1 which is independent of X. The random variable τ is defined as in (2.2). X gives rise to a spot rate process $R(X_s)$ where R is a real-valued function on R^d. Define the following filtrations:

$$
\begin{aligned}
\mathcal{G}_t &= \sigma\{X_s : 0 \le s \le t\} \\
\mathcal{H}_t &= \sigma\{1_{\{\tau \le s\}} : 0 \le s \le t\} \\
\mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t
\end{aligned}
$$

Information available for computing prices at time t is represented by $\mathcal{F}_t$ which corresponds to knowing the evolution of the state variables up to time t and whether default has occured or not.

To compute prices on credit risky bonds we need the following

Lemma 3.1

$$E_t\left(\frac{B(t)1_{\{\tau>T\}}\}}{B(T)}\right) = 1_{\{\tau>t\}} E_t \exp\left(-\int_t^T R(X_s) + \lambda(X_s)\, ds\right). \tag{3.1}$$

Proof of Lemma 3.1. We first show that

$$E\left(1_{\{\tau>T\}}\middle|\mathcal{G}_T \vee \mathcal{H}_t\right) = 1_{\{\tau>t\}} \exp\left(-\int_t^T \lambda(X_s)\, ds\right) \tag{3.2}$$

Note that $\mathcal{H}_t$ is generated by the partition $\Omega = \{\tau \leq t\} \cup \{\tau > t\}$. Therefore (see for example Billingsley (1986) exercise 34.4)

$$\begin{aligned}
& E\left(1_{\{\tau>T\}}|\mathcal{G}_T \vee \mathcal{H}_t\right) \\
= \; & P\left(\tau > T|\mathcal{G}_T \vee \mathcal{H}_t\right) \\
= \; & 1_{\{\tau>t\}} \frac{P\left(\{\tau > T\} \cap \{\tau > t\}|\mathcal{G}_T\right)}{P\left(\tau > t|\mathcal{G}_T\right)} + 1_{\{\tau\leq t\}} \frac{P\left(\{\tau > T\} \cap \{\tau \leq t\}|\mathcal{G}_T\right)}{P\left(\tau \leq t|\mathcal{G}_T\right)} \\
= \; & 1_{\{\tau>t\}} \frac{P\left(\tau > T|\mathcal{G}_T\right)}{P\left(\tau > t|\mathcal{G}_T\right)} + 0 \\
= \; & 1_{\{\tau>t\}} \frac{\exp\left(-\int_0^T \lambda(X_s)\, ds\right)}{\exp\left(-\int_0^t \lambda(X_s)\, ds\right)} \\
= \; & 1_{\{\tau>t\}} \exp\left(-\int_t^T \lambda(X_s)\, ds\right)
\end{aligned}$$

which proves (3.2). To complete the proof of the lemma, we just iterate the expectation:

$$E_t\left(\frac{B(t)1_{\{\tau>T\}}}{B(T)}\right)$$

$$
\begin{aligned}
&= E_t\left(E\left(\frac{B(t)1_{\{\tau>T\}}\}}{B(T)}\middle|\mathcal{G}_T \vee \mathcal{H}_t\right)\right) \\
&= E_t\left(\exp\left(-\int_t^T R(X_s)\,ds\right)1_{\{\tau>t\}}\exp\left(-\int_t^T \lambda(X_s)\,ds\right)\right) \\
&= 1_{\{\tau>t\}}E_t \exp\left(-\int_t^T (R(X_s)+\lambda(X_s))\,ds\right).
\end{aligned}
$$

which is what we wanted to show. □

It is now easy to derive the expression for $v(t,T)$:

Proposition 3.1

$$
v(t,T) = \delta p(t,T) + 1_{\{\tau>t\}}(1-\delta)E_t \exp\left(-\int_t^T R(X_s)+\lambda(X_s)\,ds\right) \tag{3.3}
$$

Proof of Proposition 3.1.

$$
\begin{aligned}
v(t,T) &= E_t\left(\frac{B(t)}{B(T)}\left(\delta 1_{\{\tau\le T\}} + 1_{\{\tau>T\}}\right)\right) \\
&= E_t\left(\frac{B(t)}{B(T)}\left(\delta + (1-\delta)\,1_{\{\tau>T\}}\right)\right) \\
&= \delta p(t,T) + 1_{\{\tau>t\}}(1-\delta)E_t \exp\left(-\int_t^T R(X_s)+\lambda(X_s)\,ds\right)
\end{aligned}
$$

where we have used Lemma 3.1 in the last equality. □

Note that if default has already occured by time t, we have $v(t,T) = \delta\, p(t,T)$, so that after default, the term structure of the risky bonds collapses to that of the default free bonds. This assumption can be relaxed but we prefer to keep things simple by focusing on the distribution of the first time of default only.

Remark 3.1. The conditioning sigma field $\mathcal{F}_t$ may in fact be replaced by $\mathcal{G}_t$ in the expressions (3.1) and (3.3) above: For a measurable function $h : R^d \to [0,\infty)$

we have

$$(3.4) \qquad E_t \exp\left(-\int_t^T h(X_s)ds\right) = E\left(\exp\left(-\int_t^T h(X_s)ds\right)\bigg|\mathcal{G}_t\right).$$

To see this, recall that E_1 is a unit exponential random variable which is independent of the sigma field $\mathcal{G}_T$. In particular, E_1 is independent of the sigma field $\sigma\left(\exp\left(-\int_t^T h(X_s)ds\right)\right) \vee \mathcal{G}_t$ and therefore (see Williams (1991) 9.7(k))

$$(3.5) \qquad E\left(\exp\left(-\int_t^T h(X_s)ds\right)\bigg|\mathcal{G}_t \vee \sigma(E_1)\right)$$
$$= E\left(\exp\left(-\int_t^T h(X_s)ds\right)\bigg|\mathcal{G}_t\right).$$

But we also have the following inclusion among sigma fields:

$$(3.6) \qquad \mathcal{G}_t \subset \mathcal{G}_t \vee \mathcal{H}_t \subset \mathcal{G}_t \vee \sigma(E_1).$$

Combining (3.5) and (3.6) we conclude that

$$E\left(\exp\left(-\int_t^T h(X_s)ds\right)\bigg|\mathcal{G}_t \vee \mathcal{H}_t\right) = E\left(\exp\left(-\int_t^T h(X_s)ds\right)\bigg|\mathcal{G}_t\right).$$

which is what we wanted to show.

Remark 3.2. It is useful to specify precisely how in this framework we model independence between the default time and the evolution of default free bonds. The setup is often referred to as the *independence case* (see for example Hull (1993) for a similar notion). Let X^1 be state variable process from which we define the spot rate process $R(X_s^1)$ and let X^2 be a process independent of X_1 which governs the hazard rate of default $\lambda(X_s^2)$. Define the sigma fields

$$\mathcal{G}_t^1 = \sigma\{X_s^1 : 0 \le s \le t\}$$
$$\mathcal{G}_t^2 = \sigma\{X_s^2 : 0 \le s \le t\},$$

so that our assumption is that $\mathcal{G}_t^1$ and $\mathcal{G}_t^2$ are independent for all $t \in [0, T_f]$. Now let $X = (X^1, X^2)$ be the state variable process of our model. With this setup it can be shown using arguments similar to those of Remark 3.1 that

$$\begin{aligned}
v(t,T) &= \delta p(t,T) \\
&+ 1_{\{\tau>t\}}(1-\delta) E\left(\exp\left(-\int_t^T R(X_s^1)ds\right)\Big|\mathcal{G}_t^1\right) \\
&\times E\left(\exp\left(-\int_t^T \lambda(X_s^2)ds\right)\Big|\mathcal{G}_t^2\right) \\
&= \delta p(t,T) + 1_{\{\tau>t\}}(1-\delta)p(t,T)\frac{P(\tau > T|\mathcal{G}_t^2)}{P(\tau > t|\mathcal{G}_t^2)} \qquad (3.7)
\end{aligned}$$

In the special case where $\delta = 0$, we get

$$v(t,T) = 1_{\{\tau>t\}}p(t,T)\frac{P(\tau > T|\mathcal{G}_t^2)}{P(\tau > t|\mathcal{G}_t^2)}$$

which says that if default has not occured at time t, the price of the credit risky bond becomes a product of the price of a default free bond and a conditional survival probability computed under the martingale measure P.

It is clear from (3.7) that in the independence case we are able to calculate implied survival probabilities from bond prices, i.e. survival probabilities as calculated under the martingale measure. Assume for simplicity that $\delta = 0$. Then

$$P(\tau > T\,|\tau > 0) = \frac{v(0,T)}{p(0,T)}$$

and from this survival function we can derive the time 0 probability of survival past T conditionally on survival past t. This is the approach taken in Litterman and Iben (1991). When there is correlation between the spot rate and the default mechanism, this procedure does not give the survival probabilities under the martingale measure. Still assuming $\delta = 0$ straightforward calculations show that

$$\frac{v(0,T)}{p(0,T)} = E\exp\left(-\int_0^T \lambda(X_s)\, ds\right) + \frac{Cov\left(\exp\left(-\int_0^T R(X_s)\, ds\right), \exp\left(-\int_0^T \lambda(X_s)\, ds\right)\right)}{p(0,T)}$$

in which the first term is the survival probability and the second term produces bias when there is correlation.

If for example the processes $R(X)$ and $\lambda(X)$ are square integrable and

$$Cov\left(R(X_s), \lambda(X_t)\right) \geq 0 \text{ for all } s,t \in [0,T],$$

then

$$Cov\left(\exp\left(-\int_0^T R(X_s)\, ds\right), \exp\left(-\int_0^T \lambda(X_s)\, ds\right)\right) \geq 0.$$

A simple consequence of this is that if $Cov(R(X_s), R(X_t)) \geq 0$ and if the default intensity has the form $\lambda(R(X_t))$ with λ increasing (decreasing), then the bias term is positive (negative), causing survival probabilities to be overestimated (underestimated). So if a recession brings low interest rates and high default rates, as in the early 90's, we may overestimate the risk of default implied in corporate bond prices.

4 A Generalized Markovian Model

As a first example the case where the intensity of default is modulated by a continuous-time, finite state space Markov chain which is independent of the state variables governing the default-free bonds. This is the model described in Jarrow, Lando and Turnbull (1993).

In a slight change of notation (to save a large number of superscripts) define the state variable process

$$\tilde{X}_t = (X_t, \eta'_t) \qquad t \in [0, T_f],$$

and assume that X and η are independent. In the language of Remark 3.2 we have $X^1 = X$ and $X^2 = \eta'$. The spot rate process of default-free bonds is determined by X and denoted $R(X)$. The continuous-time Markov chain η' with state space $\{1, \ldots, K-1\}$ has generator matrix

$$A' = \begin{pmatrix} \lambda_1 & \lambda_{12} & \cdots & \lambda_{1,K-1} \\ \lambda_{21} & \lambda_2 & \cdots & \lambda_{2,K-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1} \end{pmatrix}$$

where

$$\lambda_{ij} \geq 0 \text{ for } i \neq j$$

and

$$\lambda_i = -\sum_{j=1, j\neq i}^{K-1} \lambda_{ij}, \; i = 1, \ldots, K-1.$$

Think of η' as modelling the credit rating of the bond issuer with state 1 corresponding to the highest rating (AAA in Standard and Poor's terminology) state $K-1$ is the lowest non-default rating.

Let the intensity of default when η' is in class i be given as $\lambda(i) := \lambda_{iK}$ (the K is added for reasons which will appear below). As in (2.4) the distribution of the time τ of default given that $\eta'_0 = i$ is given by

$$P^i(\tau > t) = E^i \exp\left(-\int_0^t \lambda\left(\eta'_s\right) ds\right)$$

An equivalent way of modelling the time to default is to add an extra state K to the Markov chain η', let this new Markov chain η have generator matrix

$$(4.1) \qquad A = \begin{pmatrix} \lambda_1 - \lambda_{1K} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1K} \\ \lambda_{21} & \lambda_2 - \lambda_{2K} & \lambda_{23} & \cdots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1} - \lambda_{K-1,K} & \lambda_{K-1,K} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and declare default the first time this Markov chain hits the absorbing state K. The reason that these two descriptions are equivalent is the following: From Rudemo (1973) we know that the bivariate process (η', N) with state space

$$\{1, \ldots, K-1\} \times \{0, 1, 2, \ldots\}$$

and transition rates given by

$$\begin{aligned} \lambda_{ij} &: \quad (i,n) \to (j,n) \text{ for } i \neq j, i,j \in \{1, \ldots, K-1\} \\ \lambda_{iK} &: \quad (i,n) \to (i,n+1) \text{ for } i \in \{1, \ldots, K-1\} \\ 0 &: \quad \text{otherwise} \end{aligned}$$

represents a (bivariate) Markov chain, and N_t is a Cox process whose intensity is modulated by the Markov chain η'. If we let all states (i, n) of the bivariate Markov chain in which $n > 0$ be absorbing, we get a new Markov chain with transition rates

$$\begin{aligned} \lambda_{ij} &: \quad (i,0) \to (j,0) \text{ for } i \neq j, i,j \in \{1, \ldots, K-1\} \\ \lambda_{iK} &: \quad (i,0) \to (i,1) \text{ for } i \in \{1, \ldots, K-1\} \\ 0 &: \quad \text{otherwise.} \end{aligned}$$

Given that the bivariate process starts in state $(i, 0)$ with $i \in \{1, \ldots, K-1\}$, it is evident that the time until the bivariate chain is absorbed, i.e. the first jump of the second component, can be modelled as a univariate chain with state space $\{1, \ldots, K\}$ which starts in state i and whose transition rate to a state $j \in \{1, \ldots, K-1\} \backslash \{i\}$ is λ_{ij} and to state K is λ_{iK}. But this is precisely the evolution of the Markov chain η.

The second description (in terms of the K-state Markov chain η) is the one used in Jarrow, Lando and Turnbull (1993).

We now turn to a generalization of this Markovian model in which the risk-adjusted transition probabilities of credit ratings depend on state variables. In the study of Fons and Kimball (1991) it is documented that default rates of lower rated firms show significant time variation and even for the top rated firms for which default is extremely rare the standard deviations of default rates may still be significant in relative terms. Whether this variation can be captured through variation in the state variables or whether a parametrization in terms of constant intensities as in Jarrow, Lando and Turnbull (1993) performs equally well is an interesting empirical issue. The goal here is to illustrate the flexibility of our framework while emphasizing computational tractability.

Let the matrix A be defined as in (4.1) and assume that this matrix can be diagonalized, so that

$$A = BDB^{-1}$$

where the diagonal matrix D consists of the eigenvalues $\alpha_1, \ldots, \alpha_{K-1}, 0$, and the columns of B are the eigenvectors of A.

Proposition 4.1 *Let $\mu : R^d \to [0, \infty)$ be a non-negative function defined on the state space of X and assume that for almost every sample path of X, we have*

$$\int_0^{T_f} \mu(X_s)\, ds < \infty.$$

For each path of X define the time dependent generator matrix as follows:

$$A_X(s) = \mu(X_s)A.$$

Also, define the diagonal matrix

$$E_X(s,t) = \begin{pmatrix} \exp\left(\alpha_1 \int_s^t \mu(X_u)\,du\right) & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \exp\left(\alpha_{K-1} \int_s^t \mu(X_u)\,du\right) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Then with

$$P_X(s,t) = B\,E_X(s,t)\,B^{-1},$$

$P_X(s,t)$ satisfies Kolmogorov's backward equation

$$\frac{\partial P_X(s,t)}{\partial s} = -A_X(s)P_X(s,t)$$

and $P_X(s,t)$ is the transition probability of an (inhomogeneous) Markov chain on $\{1,\ldots,K\}$.

Proof of Proposition 4.1.

Since $A_X(s)$ differs only from A by multiplication of a constant, it is clear that $A_X(s)$ has the same eigenvectors as A and the eigenvalues are given by $\alpha_1\mu(X_s),\ldots,\alpha_{K-1}\mu(X_s),0$. Let $D_X(s)$ denote the diagonal matrix generated by the eigenvalues of $A_X(s)$. Then we have

$$A_X(s)B = B\,D_X(s)$$

since the diagonal matrix just multiplies the columns of B, which are eigenvectors of $A_X(s)$, by the corresponding eigenvalue. Now we see that

$$\frac{\partial P_X(s,t)}{\partial s} = B\,(-D_X(s))E_X(s,t)B^{-1}$$

$$
\begin{aligned}
&= -A_X(s)B\,E_X(s,t)\,B^{-1} \\
&= -A_X(s)P_X(s,t)
\end{aligned}
$$

which shows that $P_X(s,t)$ is indeed a solution to the backward equation. From section 4.4 in Gill and Johansen (1990) we know that under our assumptions on $A_X(s)$ this equation has a unique solution, and the solution is a transition matrix for an inhomogeneous Markov chain with 'intensity measure' given by $A_X(s)$. Hence $P_X(s,t)$ defines a Markov chain, as was to be shown. □

Conditionally on X the probability of defaulting before t given no default at time s and given a credit rating of i at time s is then equal to the $(i,K)'$th entry of the matrix $P_X(s,t)$. This entry has the form

$$
P_X(s,t)_{i,K} = \sum_{j=1}^{K} b_{ij} \exp\left(\int_s^t \alpha_j \mu(X_u)\,du\right) b^{-1}_{jK},
$$

and by our special assumption that A has the K'th row equal to 0, we conclude that $\alpha_K = 0$ and $b_{iK} = b^{-1}_{KK} = 1$. Defining $\beta_{ij} = -b_{ij} b^{-1}_{jK}$ we may therefore write the probability of no default as

$$
1 - P_X(s,t)_{i,K} = \sum_{j=1}^{K-1} \beta_{ij} \exp\left(\int_s^t \alpha_j \mu(X_u)\,du\right)
$$

Using the conditioning argument once more, we are in a position to calculate prices of a bond issued by a firm in credit class i. We use superscript i in the expectation operator to indicate that the initial rating is i, but omit this superscript when the expectation only depends on the distribution of X. To simplify notation, assume that $\delta = 0$ and that we are at time 0. Then

$$
v^i(0,t) = E^i\left(\frac{1_{\{\tau>t\}}}{B(t)}\right)
$$

$$= E^i\left(1_{\{\tau>t\}}\exp\left(-\int_0^t R(X_s)\,ds\right)\right)$$

$$= E\left(\exp\left(-\int_0^t R(X_s)\,ds\right)E^i\left(1_{\{\tau>t\}}\middle|(X_s)_{0\le s\le t}\right)\right)$$

$$= \sum_{j=1}^{K-1}\beta_{ij}E\exp\left(\int_0^t(\alpha_j\mu(X_u)-R(X_u)\,du)\right)$$

So we have expressed the price of bond in credit class i as a linear combination of functionals of the form

$$E\exp\left(\int_0^t(\alpha_j\mu(X_u)-R(X_u)\,du)\right).$$

One should note that as the model is specified the parameters are not identified since a constant can be multiplied onto the function μ and divided out in the empirical generator matrix A without changing the prices. However, if we fix the entries of A to be those observed in a particular 'reference' year, then we have an identified model. A strong assumption of the model is that the intensities are multiplied by the same constant $\mu(X_s)$. This means that the total intensity of transitions may change from one year to the next, but the relative size of the intensities remains fixed. On the other hand this makes the inference very clear. We can use the relative intensities over all years to get our estimates of the empirical matrix A and then use for example instantaneous spreads to obtain estimates of the function $\mu(X_s)$. To get greater flexibility in matching initial term structures it is trivial to add time dependence making μ a function of both X_s and s.

In computing the prices explicitly, we could assume that the hazard rate and the spot rate are both affine in the state variables and make sure the state variables give rise to affine bond prices, see for example Duffie and Kan (1992). This will in some cases (for example when X is Gaussian) produce analytically tractable solutions, but at the expense of allowing negative interest rates and default intensities. Clearly, another option is use finite state space approximations of the state variable process. Finally, it is possible to derive PDEs using the Feynman-Kac formula - and this is the problem to which we now turn.

5 Contingent Claims Prices and the Feynman-Kac Formula

In models where the state variable governing interest rate movements is given by a diffusion one often has to rely on numerical solution of a partial differential equation to obtain prices for bonds and derivative securities. Especially if realistic features such as non-negative interest rates and several factors are included in the models some form of numerical technique is often inevitable. We show in this section that valuing credit risky derivatives in the Cox process framework easily fits within the framework of Feynman-Kac representations. The connection between 'diffusions with killing' and Feynman-Kac representations is of course well known, but the application to derivative securities subject to credit risk is to our knowledge new.

Throughout this section we have an R^d-valued diffusion process X with diffusion coefficient $\sigma(x,t)$ and drift $b(x,t)$ and we associate with the diffusion the infinitesimal generator

$$\mathcal{A}_t F(x) := \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(t,x)\frac{\partial^2 F(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial F(x)}{\partial x_i} \qquad F \in C^2(R^d)$$

Let $T = T_f$ and let $f : R^d \to R$, $g : [0,T] \times R^d \to R$ and $k : [0,T] \times R^d \to [0,\infty)$ be continuous functions. The fundamental equivalence which holds under regularity conditions on the diffusion coefficients and on the functions f, g, k is between solutions (satisfying an exponential growth condition) to the Cauchy problem

$$
\begin{aligned}
-\frac{\partial v}{\partial t} + kv &= \mathcal{A}_t v + g \quad \text{in } [0,T] \times R^d \\
v(T,x) &= f(x) \quad x \in R^d
\end{aligned}
$$

and expectations of the form

$$
\begin{aligned}
v(t,x) &= E^{t,x}\left(f\left(X_T\right)\exp\left(-\int_t^T k(u,X_u)\,du\right)\right) \\
&+ E^{t,x}\left(\int_t^T g(s,X_s)\exp\left(-\int_t^s k(u,X_u)\,du\right)\,ds\right) \qquad (5.1)
\end{aligned}
$$

There are a number of continuity and/or smoothness conditions that need to be satisfied for this equivalence to hold. We refer to Karatzas and Shreve (1988) and Duffie (1992) for descriptions of and references to some of the literature on which combinations of conditions one can impose to obtain equivalence. Note that in this section we will explicitly write time dependence of payments, spot rates and hazard rates. In previous sections time dependence is possible simply by letting one of the state variables be time - here we prefer to separate it from the diffusion components.

Since in our framework credit spreads are given as functions of the state variable process involving of course the hazard rate λ and the spot rate function R it is clear that the Feynman-Kac formula can be used to derive PDEs for contingent claims written on credit spreads. In simple cases we may even be able to obtain analytical solutions.

More importantly, the risk of counterparty default is also handled quite generally within our framework. Consider three types of contingent claims which arise naturally in applications:

1. A payment at a fixed date T of the form $\tilde{f}(X_T)1_{\{\tau>T\}}$.

2. A payment at a rate $\tilde{g}(s, X_s)1_{\{\tau>s\}}$.

3. A payment at the time of default of the form $\tilde{h}(\tau, X_\tau)$.

The first two types of payments are to be thought of as payments similar to ordinary claims except payment only occurs as long as there is solvency. The third type of payment is meant to capture a settlement payment at the time of default. For a swap it could be a partial repayment by a distressed party based on the remaining value of the swap, which in turn can be calculated by the value of the state variables. When the expectation of these payments exists, it is straightforward to verify that they are of the form (5.1) also:

Theorem 5.1 *Let $R(s, X_s)$ denote the spot rate and let $\lambda(s, X_s)$ denote the hazard rate of default, and assume that both are non-negative. Assume that $E|f(X_T)|$, $E|g(s, X_s)|$ and $E|h(s, X_s)\lambda(s, X_s)|$ are all finite. Then*

1. *The value of the claim $\tilde{f}(X_T)1_{\{\tau>T\}}$ is given by 5.1 with $f = \tilde{f}$, $g = 0$ and $k = R + \lambda$.*

2. *The value of the claim paying at the rate $\tilde{g}(s, X_s)1_{\{\tau>s\}}$ is given by 5.1 with $f = 0$, $g = \tilde{g}$ and $k = R + \lambda$.*

3. *The value of the claim $\tilde{h}(\tau, X_\tau)$ is given by 5.1 with $f = 0$, $g = \tilde{h}\lambda$ and $k = R + \lambda$.*

Proof of of Theorem 5.1. The proof of (1) is exactly like the proof of Lemma 3.1 with the payoff changed from 1 to $\tilde{f}(X_T)$ at maturity. The proof of (2) needs

Fubini's theorem to pull the conditional expectation under the integral sign but is otherwise identical:

$$E\left(\int_t^T \tilde{g}\,(s, X_s)\, 1_{\{\tau>s\}} \exp\left(-\int_t^s R(u, X_u)\, du\right) ds\right) =$$

$$E\int_t^T E\left(\tilde{g}\,(s, X_s)\, 1_{\{\tau>s\}} \exp\left(-\int_t^s R(u, X_u)\, du\right)\Big|\, (X_u)_{0\le u\le T}\right) ds =$$

$$E\int_t^T \tilde{g}\,(s, X_s) \exp\left(-\int_t^s \lambda(u, X_u)\, du\right) \exp\left(-\int_t^s R(u, X_u)\, du\right) ds =$$

$$E\int_t^T \tilde{g}\,(s, X_s) \exp\left(-\int_t^s (R+\lambda)(u, X_u)\, du\right) ds$$

which is the desired result.

For the proof of (3) note that conditionally on X the density of the default time is given by

$$\frac{\partial}{\partial s}P(\tau \le s) = \lambda(s, X_s) \exp\left(-\int_0^s \lambda(u, X_u)\, du\right).$$

Hence we get the following

$$E\left(\tilde{h}\,(\tau, X_\tau) \exp\left(-\int_t^\tau R(u, X_u)\, du\right)\right)$$

$$= E\left(E\left(\tilde{h}\,(\tau, X_\tau) \exp\left(-\int_t^\tau R(u, X_u)\, du\right)\Big|\, (X_t)_{0\le t\le T}\right)\right)$$

$$= E\left(\int_0^T \tilde{h}\,(s, X_s) \exp\left(-\int_t^s R(u, X_u)\, du\right) \lambda(s, X_s) \exp\left(-\int_0^s \lambda(u, X_u)\, du\right) ds\right)$$

$$= E\int_0^T (\tilde{h}\lambda)\,(s, X_s) \exp\left(-\int_t^s (R+\lambda)(u, X_u)\, du\right) ds$$

and this is what we wanted to prove. □

As a further example of how the framework presented here resembles that of ordinary term structure modelling, consider the claim $f(X_T)1_{\{\tau>T\}}$ and note that the 'change of numeraire technique' (see for example El Karoui, Myneni and Viswanathan (1992)) is simple to apply in order to factor out the price into an expectation and the price of corporate bonds. Define $k = R + \lambda$ and let $\hat{k}(s)$ be defined as the forward rate structure of risky bonds:

$$v(0,T) = \exp\left(-\int_0^T \hat{k}(s)\, ds\right)$$

Recalling that $v(0,T) = E\exp(-\int_0^T k(s, X_s)\, ds)$ it is easy to check that with

$$Z_T = \exp\left(-\int_0^T \left(k(s, X_s) - \hat{k}(s)\right) ds\right),$$

and

$$\frac{d\tilde{P}}{dP} = Z_T$$

we define an equivalent probability measure $\tilde{P}$ on $\mathcal{F}_T$.

Now recall the identity (see for example (3.9) page 155 of Jacod and Shiryaev (1987)):

$$\tilde{E}\left(f\left(X_T\right)|\mathcal{F}_t\right) = \frac{1}{Z_t} E\left(f(X_T) Z_T | \mathcal{F}_t\right).$$

For simplicity let $t = 0$. Then

$$\begin{aligned}
\tilde{E}\left(f\left(X_T\right)\right) &= \frac{1}{Z_0} E\left(f(X_T) Z_T\right) \\
&= E\left(f(X_T) \exp\left(-\int_0^T \left(k(s, X_s) - \hat{k}(s)\right) ds\right)\right) \\
&= \exp\left(\int_0^T \hat{k}(s)\, ds\right) E\left(f(X_T) \exp\left(-\int_0^T k\left(X_s\right) ds\right)\right)
\end{aligned}$$

from which we see that the price of the contingent claim $f(X_T)1_{\{\tau>T\}}$ is given by

$$\begin{aligned} & E\left(f(X_T)1_{\{\tau>T\}}\exp\left(-\int_0^T R(s,X_s)\,ds\right)\right) \\ = \; & E\left(f(X_T)\exp\left(-\int_0^T k(s,X_s)\,ds\right)\right) \\ = \; & \exp\left(-\int_0^T \hat{k}(s)\,ds\right)\tilde{E}\left(f(X_T)\right) \\ = \; & v(0,T)\tilde{E}f(X_T) \end{aligned}$$

which gives the desired decomposition.

6 Empirical Hazard Rates

From an econometric viewpoint and in an attempt to explain risk premia better it would be appealing if the statistical theory for estimating hazard rates could be brought into play in the theory of corporate bond pricing. Given the sample paths of the state variable process X, well developed techniques for estimating the empirical hazard rate $\lambda(X_s)$ exist, see for example Andersen et al. (1993). For a hazard rate λ the empirical default probability is given by $E^Q\exp\left(-\int_0^t \lambda(X_s)\,ds\right)$ where Q denotes the measure governing the true behavior of the state variable process. However, empirical evidence seems to suggest that the empirically observed default probabilities are small compared to the implied default probabilities, and this in turn suggests that some sort of risk premium is present. For example, based on the empirical default probabilities reported in Standard and Poor's Creditreview (1993), Jarrow, Lando and Turnbull (1993) (Figure 5) calculate an instantaneous forward rate spread for a AAA-rated bond with five years to maturity close to 10 basis points. The yield spread corresponding to the same maturity will be considerably less since the instantaneous forward rate curve for AAA-rated bonds based on empirical default probabilities is upward sloping, and yields are obtained by averaging the instantaneous forward rates. Litterman and Iben (1991) (Figure 3) report spreads from traded bonds which are typically around 50 basis points on bonds issued by AAA-rated firms.

What we argue in the following is that a risk adjusted default probability of the form

$$P(\tau < t) = E \exp\left(- \int_0^t \lambda\left(X_s\right) \right)$$

computed under the risk adjusted measure for the state-variable process X using the empirically observed hazard function has intuitive appeal. It captures the economic intuition that state variables which carry risk premia and influence the likelihood of default cause the default probability of firms to carry a risk premium but the actual event of default (triggered by litigation, project failure or something truly firm specific) represents diversifiable risk.

The mathematical formulation of this intuition is as follows: Given n firms with the same hazard rate of default. Defaults of n firms are given by the stopping times

$$\tau_i = \inf\{t : \int_0^t \lambda(X_s)\, ds \geq E_i\}.$$

where $E_1, \ldots, E_n$ are independent exponential random variables. This says that the general environment of the economy as specified by the state variable X influences the likelihood of default for the different firms and indeed makes likelihood of default correlated across the firms. However, conditionally on a particular evolution of the environment defaults are independent: Firms default for firm specific reasons. Assuming $\delta = 0$ the payoff on a portfolio consisting of $\frac{1}{n}$ of each bond is given by

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau_i > t\}}.$$

The key result to note then is the following

Proposition 6.1

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau_i > t\}} \to \exp\left(-\int_0^t \lambda(X_s)\,ds\right) \tag{6.1}$$

almost surely and in L^p, $1 \le p < \infty$.

Proof of Proposition 6.1. Let $\mathcal{G}_t = \sigma\{X_s : 0 \le s \le t\}$. Define the event

$$A = \left\{\omega : \frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau^i > t\}} \to \exp\left(-\int_0^t \lambda(X_s)\,ds\right)\right\}.$$

Then since $\tau_1, \ldots, \tau_n$ are independent given $\mathcal{G}_t$ we have by the strong law of large numbers that

$$P(A|\mathcal{G}_t) = 1,$$

and therefore, taking expectations on both sides,

$$P(A) = 1.$$

By dominated convergence (everything is dominated by the constant 1) we have convergence in L^p also. □

This result shows that a diversified portfolio of corporate bonds with similar empirical hazard rate λ is close to the random variable $\exp\left(-\int_0^t \lambda(X_s)\,ds\right)$ whose price is given by the already priced state variables as

$$E\exp\left(-\int_0^t R(X_s) + \lambda(X_s)\,ds\right).$$

In other words, when the systematic component of risk in corporate bonds is given solely through the effect of state variables on hazard rates, the price of corporate bonds should be calculated using the empirically observed hazard rate but computing the expectation with respect to the risk adjusted measure.

7 Conclusion

We have laid out a framework which is convenient for analyzing financial instruments subject to credit risk through counterparty default and derivatives whose underlying is a credit risk variable such as a credit spread. We take into account the possible correlation between default free bonds and default probabilities by letting interest rates and hazard rates be governed by common state variables. The main feature of the framework is that it reduces the technical issues of modelling credit risk to the same issues we face when modelling the ordinary term structure of interest rates. Some explicit constructions are provided that illustrate this analogy.

There is still much work to be done. We have only sketched the possibilities of doing statistical inference in these models and a natural first question is whether the default premium can indeed be modelled through the empirical hazard function adjusting for risk only through state variables.

Applications to foreign currency instruments, in which the risk of devaluation is an issue, to differentials between LIBOR and US Treasury rates and to municipal bonds are also topics of future research.

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