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6 Preprint
July
1994

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Abstract *The asymptotic power of the likelihood ratio test for cointegration is investigated for the error correction model with a drift term, which allows for a linear trend in the variables. The likelihood ratio test is compared with a detrended version of the test. In the detrended version the variables have been corrected for mean and linear trend before testing for cointegration. The two tests asymptotic distributions under the null hypothesis of cointegration and under local alternatives are found. By comparing local power properties of the two tests, it is argued that the detrending procedure leads to a loss in asymptotic power.*[1]

[1] I would like to thank Søren Johansen for his very helpful suggestions and comments. Also thanks to Bent Nielsen who kindly modified the simulation program [13] for the power simulations.

1 Introduction

Since Granger [3] in 1981 introduced the concept of cointegration much effort has been devoted to deriving tests for cointegration. In this paper two such tests are investigated for a multivariate system with variables integrated of order one and with a linear trend.

The one test studied is the likelihood ratio (LR) test for the number of cointegrating relations in the p-dimensional error correction model with a deterministic drift term. A derivation of this test is found in Johansen [6], where it is shown that the likelihood analysis leads to calculation of the canonical correlations between the demeaned first differences and the demeaned first lags of the p-dimensional process (possibly corrected for short term dynamics). Demeaning refers to correction for the average.

Another approach is simply to detrend the variables analysed before testing for cointegration. That is to correct for both mean and trend by ordinary least squares, and then apply the canonical correlations between the first differences and first lags to test for cointegration. The test derived this way is referred to as the DLR test and may be viewed as one way to apply the principle that "any known deterministic components can be subtracted before the analysis is begun" (Engle and Granger p.256 [2]). It is shown that the asymptotic distribution of the DLR test is similar with respect to the drift parameter, which is analogous to the idea of Kiviet and Phillips [14]. This contrasts the LR test which is not similar with respect to the drift parameter.

Both tests are consistent in the sense that the asymptotic power tends to one under fixed alternatives, and the asymptotic power is therefore derived under local alternatives (cf. Pitman [17], ch.7). From the local power properties of the DLR test it will be argued that detrending as described leads to a loss in asymptotic power when compared to the LR test. This merely reflects the redundant regression performed by the detrending.

The paper is organized as follows. In Section 2 the LR test and the DLR test for the number, r, of cointegrating relations in the error correction model are presented. For notational purposes and reference a brief summary of the theory of Johansen [7] is given. The two tests asymptotic behaviour under the null-hypothesis of cointegration is investigated. Next in section 3 the local power functions of the LR test are derived, and the local power of the DLR test is investigated in section 4. Finally section 5 contains some concluding remarks and the two tests are compared by means of local power properties.

The proofs of the results of this paper are given in the appendix and rely on the theory of weak convergence of near-integrated processes, applied in the papers Phillips [15] and Johansen [9].

2 Testing for Cointegration

In this section the LR test and the DLR test are presented. The hypothesis of cointegration is formulated within the p-dimensional error correction (EC) model with a deterministic drift term μ, which allows for a linear trend. The asymptotic distribution of the DLR test is derived under the hypothesis of cointegration and Table 1 shows the simulated distribution. For the LR test the results are from Johansen [6], [7], [10] and [11], and in order to present the notation involved, a summary of the above mentioned likelihood analysis is given.

2.1 The Cointegration Hypothesis

The model considered is the p-dimensional EC model with Gaussian errors given by

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t \,, \tag{2.1}$$

where $t = 1 \dots T$ and $\varepsilon_t \sim iidN_p(0, \Omega)$. The $(p \times p)$ matrix Π is denoted the impact matrix and the $(p \times p)$ matrices Γ_i are the short term dynamics coefficient matrices. The drift term μ is a $(p \times 1)$ vector and allows for a linear trend. Finally the covariance matrix Ω is assumed to be positive definite. The null-hypothesis $H(r)$ of at most r cointegrating relations is given by rank$(\Pi) \leq r$ or equivalently

$$H(r) : \Pi = \alpha\beta' \text{ where } \alpha, \beta \text{ are } (p \times r) \text{ matrices} \,. \tag{2.2}$$

From Johansen's representation theorem below explicit conditions on the parameters in the model (2.1) can be stated for $(X_t)_{t=1\dots T}$ to be integrated of order at most one and for $\beta' X_t$ to be stationary corresponding to the cointegration hypothesis $H(r)$. The assumptions are given in terms of the characteristic polynomial, $A(z)$ and the matrix

$$\Gamma = \left(-\tfrac{dA(z)}{dz}|_{z=1}\right) = I + \Pi - \textstyle\sum_1^{k-1} \Gamma_i \,.$$

Assumption 2.1 *Assume that* $\text{rank}(\Pi) = r < p$ *and the roots of* $A(z)$ *are either outside the unit circle or at* 1. *Furthermore assume that* $\text{rank}\,(\alpha'_{\perp}\Gamma\beta_{\perp}) = p - r$.

Here and in the following for any $(p \times r)$ matrix m of full rank r, $m_{\perp}$ will be defined to mean a $(p \times (p - r))$ matrix of full rank such that $m'm_{\perp} = 0$ so that $\text{span}(m, m_{\perp}) = R^p$.

These assumptions provide the necessary and sufficient restrictions on the parameters in the model (2.1) to guarantee the above mentioned properties of X_t stated in the following theorem.

Theorem 2.1 (Johansen's Representation Theorem)
Under the Assumptions 2.1 the processes ΔX_t *and* $\beta' X_t$ *can be given initial distributions such that they become stationary. Furthermore with* $C = \beta_{\perp}(\alpha'_{\perp}\Gamma\beta_{\perp})^{-1}\alpha'_{\perp}$, $(X_t)_{t=1...T}$ *has the representation*

$$X_t = C \textstyle\sum_1^t \varepsilon_i + C\mu t + Y_t + A\,, \tag{2.3}$$

where Y_t *is a stationary process defined in terms of the* ε_t*'s and* $\beta' A = 0$.

Thus when Π has reduced rank and the assumptions are satisfied, the process X_t consists of a random walk, a linear trend and a stationary part. The linear trend coefficient is given by the term

$$\tau = C\mu = \beta_{\perp}(\alpha'_{\perp}\Gamma\beta_{\perp})^{-1}\alpha'_{\perp}\mu\,, \tag{2.4}$$

and it follows that if $\alpha'_{\perp}\mu = 0$ the trend is absent. The purpose of this paper is to investigate two tests for cointegration in the presence of a linear trend. Hence in the following it will be assumed that

Assumption 2.2 (Linear trend presence) $\alpha'_{\perp}\mu \neq 0$.

Note that under $H(r)$ the p-dimensional process X_t can have linear trend in all or some of the components, whereas the linear combinations $\beta' X_t$ are truly *stationary* as opposed to *trend stationary.*

2.2 The Likelihood Analysis

The statistical models generated by the sequence $(H(r))_{r=0\ldots p}$ are nested in the following simple way

$$H(0) \subset \ldots \subset H(r) \subset \ldots \subset H(p),$$

and consider here the likelihood ratio test of $H(r)$ ($\leq r$ cointegrating relations) against $H(p)$ ($\leq p$ cointegrating relations).

In model (2.1) under $H(r)$ the parameters $((\Gamma_i)_{i=1\ldots k-1}, \mu, \alpha, \beta, \Omega)$ all vary freely. By regression of ΔX_t and X_{t-1} on the lagged differences, $(\Delta X_{t-i})_{i=1\ldots k-1}$, and the constant the likelihood function is concentrated with respect to the parameters $((\Gamma_i)_{i=1\ldots k-1}, \mu)$. Note that the regressions on the constant amounts to correcting ΔX_t and X_{t-1} for their average, i.e. demean, even though the model allows for a linear trend in X_t.

From the initial regressions one obtains the residuals R_{0t} and R_{1t}, in terms of which the concentrated likelihood function is given by

$$L_{max}^{-2/T}(\alpha, \beta, \Omega) = |\Omega| \, exp\left\{T^{-1}\sum_{t=1}^{T}(R_{0t} - \alpha\beta' R_{1t})'\Omega^{-1}(R_{0t} - \alpha\beta' R_{1t})\right\}. \tag{2.5}$$

For fixed β, the maximum likelihood estimators of α and Ω are then found by ordinary regression, leading to the definition of the residual product moment matrices

$$S_{ij} = T^{-1}\textstyle\sum_1^T R_{it}R'_{jt} \quad (i,j = 0,1)\,. \tag{2.6}$$

By reduced rank regression it follows that β is estimated as the r largest canonical correlations between essentially the demeaned first differences and first lags of X_t. More precisely the following theorem can be stated.

Theorem 2.2 (The LR test, Johansen) *In the error correction model given by* (2.1) *the* LR *test of at most r cointegrating relations against the hypothesis of at most p is given by*

$$\text{LR}(H(r)|H(p)) = -T\sum_{i=r+1}^{p}\ln(1-\hat{\lambda}_i). \tag{2.7}$$

Here the ordered eigenvalues $1 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_p > 0$, *solve*

$$\left|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}\right| = 0. \tag{2.8}$$

Furthermore under $H(r)$ the maximum likelihood estimator of β is given by $(\hat{v}_1, \ldots \hat{v}_r)$, that is, the suitably normalized eigenvectors corresponding to the r largest eigenvalues.

The sequential testing strategy is presented in Johansen [7] and $H(r)$ is accepted only if $H(0)\dots H(r-1)$ are all rejected. It is therefore natural to consider the distribution of the likelihood ratio test for $H(r)$ under the assumption that the rank of Π is indeed r rather than less than or equal r. Invoking the results of Theorem 2.1 then leads to

Theorem 2.3 (Asymptotic distribution of the LR test, Johansen)
Under the Assumptions 2.1 and 2.2 as $T \to \infty$,

$$LR(H(r)|H(p)) \xrightarrow{w} tr\left\{\int_0^1 dWG'(\int_0^1 GG'du)^{-1}\int_0^1 GdW'\right\}, \tag{2.9}$$

where W is a $(p-r)$ dimensional standard brownian motion, and G is given by,

$$G_i(u) = \begin{cases} u - \frac{1}{2} & \text{i=1,} \\ W_i(u) - \overline{W}_i & \text{i=2, \dots ,p-r} \end{cases} \tag{2.10}$$

Here $\overline{W}_i = \int_0^1 W_i(u)du$ and $u \in [0,1]$.

A table with simulations of the nonstandard distribution in (2.9) is found in Johansen and Juselius [12]. Note that G consists of a deterministic part reflecting the trend of X_t, and of a brownian motion part reflecting the random walk. Also note the correction for mean in G which reflects the demeaning in the likelihood analysis.

It should be emphasized that the limit distribution is dependent on the assumption of a linear trend. Indeed if $\alpha'_\perp\mu = 0$, then G should be replaced by $W - \overline{W}$. Thus the LR test is not similar with respect to the drift parameter, μ.

2.3 The DLR statistic

Assuming that the (observed) process posess a linear trend, the idea of the DLR test is to detrend before testing for cointegration. In the framework of section 2.2, the statistical calculations remain the same except that the residuals R_{0t}, R_{1t} are replaced by $\tilde{R}_{ot}$ and $\tilde{R}_{1t}$ respectively. The latter are obtained by regression of ΔX_t and X_{t-1} on the lagged differences, a constant and a linear trend. That is apart from correction for short term dynamics, one detrends. Denoting the residual product moment matrices by $\tilde{S}_{ij}$, the following definition can be given.

Definition 2.1 The DLR test

The DLR test for at most r cointegrating relations is given by

$$DLR = -T \sum_{i=r+1}^{p} \ln(1-\tilde{\lambda}_i), \tag{2.11}$$

where the ordered eigenvalues, $1 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_p > 0$*, solve the eigenvalue problem,*

$$\left|\tilde{\lambda}\tilde{S}_{11} - \tilde{S}_{10}\tilde{S}_{00}^{-1}\tilde{S}_{01}\right| = 0.$$

The asymptotic distribution of the DLR test under the hypothesis of r cointegrating relations is given in Theorem 2.4 and it is seen that by construction the asymptotic distribution of the DLR test is independent of the drift parameter. As was the case for the LR test, the distribution is non standard and quantiles of the simulated distribution are listed in Table 1 below.

Theorem 2.4 (Asymptotic Distribution of the DLR test)

Under the Assumptions 2.1 and 2.2 as $T \to \infty$

$$DLR \xrightarrow{w} tr\left\{\int_0^1 dW\mathcal{F}(W)'(\int_0^1 \mathcal{F}(W)\mathcal{F}(W)'dt)^{-1}\int_0^1 \mathcal{F}(W)dW'\right\}, \tag{2.12}$$

where W *is a* $(p-r)$ *dimensional brownian motion and* $\mathcal{F}(W)$ *is* W *corrected for mean and linear trend.*

For the simulations here and in the rest of the paper a modified version of the simulation program [13] was used. The principle of the simulations is the same as in Johansen and Juselius ([12]), and amounts to simulating the brownian motion, W by a random walk applying Donsker's Theorem (cf. Billingsley [1]) $1/\sqrt{T}\sum_{i=1}^{[Tu]}\varepsilon_i \xrightarrow{w} W(u)$. The number of simulations is set to 6000 and the number of steps, T in the random walk is 400 with $u = 0, \frac{1}{T} \ldots \frac{T}{T}$.

In order to prove Theorem 2.4 note that because of the detrending, the asymptotic analysis can be performed in the EC model (2.1) with μ set to 0. That is by construction the asymptotic distribution of the DLR test is independent of the drift parameter as already noted. To see this, let $X_t^{(\mu)}$ be generated by (2.1). Then the process $X_t^{(\mu)}$ has the representation given by Theorem 2.1,

$$X_t^{(\mu)} = C\sum_{1}^{t}\varepsilon_i + C\mu t + Y_t^{(\mu)} + A^{(\mu)} \tag{2.13}$$

Table 1:

Quantiles of the asymptotic distribution of the DLR test statistic for r cointegrating vectors among p variables,

$$tr\left\{\int_0^1 dW\mathcal{F}(W)'(\int_0^1 \mathcal{F}(W)\mathcal{F}(W)'dt)^{-1}\int_0^1 \mathcal{F}(W)dW'\right\},$$

where W is a $(p-r)$ dimensional brownian motion and $\mathcal{F}(W)$ is W corrected for mean and linear trend. The number of simulations is 6000 with the number of observations T set to 400.

Dimension	Quantiles						Sample	
$p-r$	5%	10%	50%	90%	95%	97.5%	Mean	Variance
1	0.9	1.5	4.7	9.8	11.4	13.3	5.2	11.0
2	6.9	8.1	13.6	20.9	23.4	25.9	14.2	26.0
3	17.0	18.9	26.7	36.2	39.1	41.7	27.1	46.4
4	31.2	33.4	43.1	54.8	58.6	61.5	43.8	70.6

whereas with $\mu = 0$ this reduces to

$$X_t^{(0)} = C\sum_1^t \varepsilon_i + Y_t^{(0)} + A^{(0)}. \tag{2.14}$$

Note that C is the same in (2.13) and (2.14), whereas the index on A and Y in (2.13) signifies their dependence on the parameter μ. The correction for mean and trend may be represented by F,

$$F(X_t) = X_t - \overline{X} - \frac{\sum_1^T(X_t-\overline{X})(t-\overline{t})}{\sum_1^T(t-\overline{t})^2}(t-\overline{t}), \tag{2.15}$$

where for any X, $\overline{X} = \frac{1}{T}\sum X_t$. It follows that apart from stationary terms $F(X_t^{(0)}) = F(X_t^{(\mu)})$ and $F(\Delta X_t^{(0)}) = F(\Delta X_t^{(\mu)})$. Hence for the asymptotic analysis the process X_t may be considered as generated by the EC model for $X_t^{(0)}$. In Johansen [11] it is shown that in the case of $\mu = 0$, the asymptotic distribution of the LR test is given by Theorem 2.3, (2.9) but with G replaced by the $(p-r)$-dimensional brownian motion, W. That is, the asymptotic distribution is given by

$$tr\left\{\int_0^1 dWW'(\int_0^1 WW'dt)^{-1}\int_0^1 WdW'\right\}. \tag{2.16}$$

Now the mapping $\mathcal{F}: D[0,1]^p \mapsto D[0,1]^p$, given by

$$\mathcal{F}(x)(u) = x(u) - \overline{x} - \frac{\int_0^1[x(u)-\overline{x}][u-1/2]du}{\int_0^1[u-1/2]^2du}[u-1/2], \tag{2.17}$$

which corrects for mean and trend, is uniformly continuous. Here $\overline{x} = \int_0^1 x(u)du$ and $D[0,1]^p$ denotes the space of p dimensional cadlag functions. Using that $\mathcal{F}(X_{[Tu]}) = F(X_{[Tu]})$ and the continuous mapping theorem (see Lemma A.1, Appendix A), the result follows by mimicking the proof of (2.16).

3 The Power Function of the Likelihood Ratio Test

In this section the power function is found for the LR test given by (2.7). For a fixed alternative to the null hypothesis of at most r cointegrating relations the power tends to one and the power is therefore investigated in a neighbourhood of the null. This involves the theory of local alternatives or near-integrated processes applied in Phillips [15] and Johansen [9].

3.1 The Local Alternatives

The alternative considered to the model (2.1) under the null, $H(r)$, is the inclusion of one or more additional cointegrating relations. The focus on how well the test captures extra cointegrating relations reflects the sequential testing where, as already emphasized, $H(r)$ is accepted only if $H(0)\ldots H(r-1)$ are all rejected. The number s of extra cointegrating relations is restricted by $s \leq p-r$, where r and p refer to $H(r)$ and the dimension of the EC system respectively.

In order to see that the LR test is consistent, consider the fixed alternative of possibly s additional cointegrating relations given by

$$H(r+s) : \Pi = (\alpha, \alpha_1)(\beta, \beta_1)' = \alpha\beta' + \alpha_1\beta_1',$$

where (α_1, β_1) are $(p \times s)$ matrices. The LR test for $H(r)$ against $H(p)$ is given in (2.7) and amounts to calculate T times the sum of the $(p-r)$ smallest eigenvalues which solve the eigenvalue problem $\left|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}\right| = 0$. Under $H(r)$ and the Assumptions 2.1 the $(p-r)$ eigenvalues tend to zero at the rate of T, and the result of Theorem 2.3 holds. Whereas under $H(r+s)$, assuming that $\text{rank}(\Pi) = r+s$ (and the further Assumptions 2.1 in terms of $H(r+s)$), s of the $(p-r)$ eigenvalues do *not* tend to zero, only $(p-(r+s))$ of them do. Hence the LR test tends to infinity under $H(r+s)$, and is therefore consistent.

When investigating the distribution of the LR test under local alternatives the parameters of interest are Π and μ in (2.1). Under the null-hypothesis, $H(r)$, $\Pi = \alpha\beta'$, where α and β are $p \times r$ matrices, and μ is a p-dimensional vector.

Consider the local alternatives allowing s extra cointegrating relations, β_1, with small loadings $\alpha_1/T^{3/2}$ and with μ varying unrestrictedly, i.e.

$$H_T(r+s) : \Pi = \alpha\beta' + \alpha_1\beta_1'/T^{3/2}. \tag{3.1}$$

The rate at which the alternative $H_T(r+s)$ approaches the null $H(r)$ is $T^{\frac{3}{2}}$ since for a rate less (greater) than $T^{\frac{3}{2}}$ the power tends to one (the size of the test). In comparison the local alternative for the LR test in the EC model with no drift term as treated in Johansen [9] takes the normalisation T rather than $T^{3/2}$. Thus preliminary this indicates that in the model with drift term, the local power of picking up the extra cointegrating relations is higher, when compared to the model without drift due to the normalisations $T^{3/2}$ and T respectively.

As already emphasized the asymptotic distribution of the LR test under the null-hypothesis depends on whether or not $C\mu$ equals zero, or equivalently whether or not $\alpha_\perp'\mu$ equals zero. The interest is in the case of a linear trend and hence it is assumed that $\alpha_\perp'\mu \neq 0$. Consider now the local alternative where μ tends to zero in the directions corresponding to the extra cointegrating relations, β_1. The sequence of local alternatives then take the form,

$$H_T^\mu(r+s) : \Pi = \alpha\beta' + \alpha_1\beta_1'/T \quad \text{and} \quad \beta_1'C\mu = \mu^{(b)}/T^{1/2}, \tag{3.2}$$

where the parameter $\mu^{(b)}$ is a s-dimensional vector. Note that under H_T^μ, s has to be strictly less than $(p-r)$ in order not to invalidate the assumption that $\alpha_\perp'\mu \neq 0$ under the null-hypothesis. Furthermore normalising μ by $T^{1/2}$ in the s directions corresponding to β_1 leads to loadings α_1/T rather than $\alpha_1/T^{3/2}$, as was the case under H_T. This will be clear form the proofs in the appendix, together with the explicit parametrisation of μ under H_T^μ given below.

When deriving the local power of the LR test for the hypothesis of r cointegrating relations under the alternatives (3.1) and (3.2), the short term dynamics will for simplicity be omitted in the model (2.1). The process will be denoted X_t under the null-hypothesis

$H(r)$, $X_t^{(T)}$ under $H_T(r+s)$, and finally $X_t^{(T,\mu)}$ under $H_T^{\mu}(r+s)$. Now $X_t^{(T)}$ is given by,

$$\Delta X_t^{(T)} = \Pi_T X_{t-1}^{(T)} + \mu + \varepsilon_t\,, \tag{3.3}$$

$$\Pi_T = \alpha\beta' + \alpha_1\beta_1'/T^{3/2}, \tag{3.4}$$

where (α, β) are $(p \times r)$ matrices, (α_1, β_1) are $(p \times s)$ matrices and ε_t are $iidN_p(0, \Omega)$.

Next in order to parametrize the model under H_T^{μ} introduce some notation. With a, b any $n \times m$ matrices of full row rank m and such that $(b'a)$ has full rank, let $a_b = a(b'a)^{-1}$ and $b_{a\perp} = b_{\perp_{a_\perp}}$. Then $(b, a_\perp)$ spans R^n or equivalently, $I_n = a_b b' + b_{a_\perp} a_\perp'$. With this notation $X_t^{(T,\mu)}$ is given by,

$$\Delta X_t^{(T,\mu)} = \Pi_T X_{t-1}^{(T,\mu)} + \mu_T + \varepsilon_t\,, \tag{3.5}$$

$$\Pi_T = \alpha\beta' + \alpha_1\beta_1'/T, \tag{3.6}$$

$$\mu_T = \alpha_\beta \mu^{(\beta)} + \beta_{\alpha_\perp} a_b \mu^{(b)}/T^{1/2} + \beta_{\alpha_\perp} b_{a_\perp}\gamma\,. \tag{3.7}$$

Here $a = \alpha_\perp'\alpha_1$ $(p-r \times s)$, $b = (\beta_\perp'\alpha_\perp)^{-1}\beta_\perp'\beta_1$ $(p-r \times s)$ and hence $T^{-1/2}\mu^{(b)} = \beta_1' C\mu$, with the impact matrix C defined in Johansen's representation Theorem 2.1. Furthermore $\mu^{(\beta)}$ is a r vector, $\mu^{(b)}$ is a s vector corresponding to the s extra cointegrating relations and $\gamma = a_\perp'\alpha_\perp'\mu$ is a $(p-r-s)$ vector. That a, b and $b'a$ above have full rank is a consequence of Assumption 3.1 below. For a proof of this see Johansen [9].

As in Section 2.1 in addition to the assumptions on the roots of the characteristic polynomial, conditions on the parameters $(\alpha, \beta, \alpha_1, \beta_1)$ are needed to ensure that X_t is at most $I(1)$ under both the null and the alternatives. Note that the conditions, which are stated below, when compared with Assumptions 2.1 are simplified due to the omission of the short term dynamics.

Assumption 3.1 *Assume that the roots of the characteristic polynomial for* X_t *under* $H(r)$, $H_T(r+s)$ *and* $H_T^{\mu}(r+s)$ *are either outside the unit circle or at 1. Furthermore assume that*

$$\begin{aligned} &\operatorname{rank}(\alpha) = \operatorname{rank}(\beta) = r, \quad \operatorname{rank}(\alpha_\perp'\beta_\perp) = p - r, \\ &\operatorname{rank}(\alpha_1) = \operatorname{rank}(\beta_1) = s, \quad \operatorname{rank}((\alpha, \alpha_1)_\perp'(\beta, \beta_1)_\perp) = p - (r+s). \end{aligned} \tag{3.8}$$

3.2 Asymptotics under the local Alternatives

When deriving the asymptotic distribution of the likelihood ratio test under $H_T(r+s)$ and $H_T^{\mu}(r+s)$ the idea is to study the asymptotic behaviour of the processes in a properly chosen coordinate system. It follows that under Assumption 3.1 $(\beta, \alpha_\perp)$ spans R^P and the asymptotics will initially be studied in these two directions. Basicly by the representation Theorem 2.1 this separates the behaviour into a *"near-stationary"* and a *"near-integrated"* direction, each leading to different asymptotics.

The next Lemma 3.1 states the asymptotic behaviour of $X_t^{(T)}$ and $X_t^{(T,\mu)}$ in the *"near-stationary"* direction.

Lemma 3.1 *Under* Assumption 3.1,

$$\beta' X_t^{(T)} = \beta' X_t + R_{t\beta}\,, \tag{3.9}$$

$$\beta' X_t^{(T,\mu)} = \beta' X_t + R_{t\beta}^{\mu} \tag{3.10}$$

where $\mathrm{Max}_{t \leq T} E|R_{t\beta}|^2$ *and* $\mathrm{Max}_{t \leq T} E|R_{t\beta}^{\mu}|^2$ *are bounded by a constant times* T^{-1}.

Thus apart from terms which are $O_p(T^{-1/2})$ both processes are asymptotically stationary. A proof of the lemma is found in the Appendix. With the notation introduced earlier it follows by (3.3) and (3.5) that in the $\alpha_\perp$ direction,

$$\alpha_\perp' \Delta X_t^{(T)} = a\beta_1' X_{t-1}^{(T)}/T^{3/2} + \alpha_\perp' \mu + \alpha_\perp' \varepsilon_t\,, \tag{3.11}$$

$$\alpha_\perp' \Delta X_t^{(T,\mu)} = a\beta_1' X_{t-1}^{(T,\mu)}/T + a_b \mu^{(b)}/T^{1/2} + b_{a\perp}\gamma + \alpha_\perp' \varepsilon_t. \tag{3.12}$$

For $\alpha_\perp' X_t^{(T)}$ the term given by $\tau = \alpha_\perp' \mu$ dominates, and the asymptotics is therefore investigated in the the two directions τ and $\tau_\perp$. As for $\alpha_\perp' X_t^{(T,\mu)}$ the asymptotics is investigated in the $a_\perp$ direction and in the direction b. Clearly these are not orthogonal, but as noted a, b and $b'a$ have full rank and therefore $(b, a_\perp)$ span R^{p-r}. For $a_\perp' \alpha_\perp' X_t^{(T,\mu)}$ the γ term dominates asymptotically and hence the asymptotics is derived in the γ and the $\gamma_\perp$ directions.

Lemma 3.2 *As* $T \to \infty$, *for* $u \in [0,1]$ *and under* Assumption 3.1, *then for* $X_t^{(T)}$,

$$\alpha_\perp' X_{[Tu]}^{(T)}/T \stackrel{w}{\rightarrow} \tau u\,, \tag{3.13}$$

$$\tau_\perp{}' \alpha_\perp' X_{[Tu]}^{(T)}/T^{1/2} \stackrel{w}{\rightarrow} \tau_\perp{}' a b' \tau \frac{u^2}{2} + \tau_\perp{}' \alpha_\perp' B(u), \tag{3.14}$$

while for $X_t^{(T,\mu)}$, *as* $T \to \infty$ *and for* $u \in [0,1]$,

$$b'\alpha'_\perp X_{[Tu]}^{(T,\mu)}/T^{1/2} \overset{w}{\to} \mathcal{U}(u) \tag{3.15}$$

$$a'_\perp\alpha'_\perp X_{[Tu]}^{(T,\mu)}/T \overset{w}{\to} \gamma u\,, \tag{3.16}$$

$$\gamma'_\perp a'_\perp\alpha'_\perp X_{[Tu]}^{(T,\mu)}/T^{1/2} \overset{w}{\to} \gamma'_\perp a'_\perp \alpha'_\perp B(u) \tag{3.17}$$

where $\mathcal{U}$ *is the s-dimensional Ornstein-Uhlenbeck process satisfying,*

$$d\mathcal{U} = \left(\mu^{(b)} + b'a\mathcal{U}\right) du + b'\alpha'_\perp dB. \tag{3.18}$$

Here B *is a p-dimensional Brownian Motion with covariance matrix* Ω.

From the previous it follows that the asymptotic behaviour of the process $X_t^{(T)}$ is derived in the directions given by,

$$V_T = (\beta, \alpha_\perp\tau/T, \alpha_\perp\tau_\perp/T^{1/2}) \tag{3.19}$$

where the normalisations correspond to the rate of convergence. While for $X_t^{(T,\mu)}$,

$$V_T = (\beta, \alpha_\perp b/T^{1/2}, \alpha_\perp a_\perp\gamma/T, \alpha_\perp a_\perp\gamma_\perp/T^{1/2})\,. \tag{3.20}$$

To investigate the asymptotic behaviour of the likelihood ratio test in Theorem 2.2 the asymptotic properties of the product moment matrices $(S_{ij})_{i,j=0,1}$ are needed. Under H_T the S_{ij} matrices are given by

$$S_{11} = \tfrac{1}{T}\textstyle\sum_{t=1}^T (X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)})(X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)})', \tag{3.21}$$

$$S_{00} = \tfrac{1}{T}\textstyle\sum_{t=1}^T (\Delta X_t^{(T)} - \Delta\overline{X}^{(T)})(\Delta X_t^{(T)} - \Delta\overline{X}^{(T)})', \tag{3.22}$$

$$S_{10} = \tfrac{1}{T}\textstyle\sum_{t=1}^T (X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)})(\Delta X_t^{(T)} - \Delta\overline{X}^{(T)})', \tag{3.23}$$

using the notation that for any process Y_t, $\overline{Y} = \frac{1}{T}\sum_{t=1}^T Y_t$ and $\overline{Y}_{-1} = \frac{1}{T}\sum_{t=1}^T Y_{t-1}$. Similarly for the product moment matrices in terms of $X_t^{(T,\mu)}$. By the stationarity of $\beta' X_t$ and ΔX_t under Assumptions 2.1 define

$$\mathrm{Var}(\Delta X_t) = \Sigma_{00},\ \mathrm{Var}(\beta' X_t) = \Sigma_{\beta\beta},\ \mathrm{Cov}(\Delta X_t, \beta' X_t) = \Sigma_{0\beta}. \tag{3.24}$$

With the just defined variance and product moment matrices the following holds.

Lemma 3.3 *With V_T defined in* (3.19) *and* (3.20) *for $X_t^{(T)}$ and $X_t^{(T,\mu)}$ respectively, then under* Assumption 3.1 *and for $u \in [0,1]$, as $T \to \infty$,*

$$V_T' S_{11} V_T \xrightarrow{w} \begin{pmatrix} \Sigma_{\beta\beta} & 0 \\ 0 & \int_0^1 \mathcal{F}\mathcal{F}' du \end{pmatrix}. \tag{3.25}$$

Here $\mathcal{F}$ is $(p-r)$-dimensional, and for $X_t^{(T)}$ it is given by

$$\mathcal{F}(u) = \begin{cases} \tau'\tau(u - \frac{1}{2}) \\ \tau_\perp{}' a b' \tau \frac{(u^2 - \frac{1}{3})}{2} + \tau_\perp{}' \alpha_\perp'(B(u) - \overline{B}) \end{cases} \tag{3.26}$$

while for $X_t^{(T,\mu)}$, $\mathcal{F}$ is given by

$$\mathcal{F}(u) = \begin{cases} \mathcal{U}(u) - \overline{\mathcal{U}} \\ \gamma'\gamma(u - \frac{1}{2}) \\ \gamma_\perp' a_\perp' \alpha_\perp'(B(u) - \overline{B}) \end{cases} \tag{3.27}$$

For a proof of Lemma 3.3 see the Appendix, where also a proof of the following lemma is found. The lemma gives the asymptotics for the remaining product moment matrices.

Lemma 3.4 *With V_T and $\mathcal{F}$ given in* Lemma 3.3 *then for $u \in [0,1]$ and as $T \to \infty$,*

$$\sqrt{T}\, V_T' \{S_{10} - S_{11}\Pi_T'\} \xrightarrow{w} \begin{bmatrix} N(0, \Sigma_{\beta\beta} \otimes \Omega) \\ \int_0^1 \mathcal{F} dB' \end{bmatrix}. \tag{3.28}$$

Furthermore with Σ_{00} and $\Sigma_{\beta 0}$ defined in (3.24),

$$S_{00} \xrightarrow{P} \Sigma_{00}, \tag{3.29}$$

$$\beta' S_{10} \xrightarrow{P} \Sigma_{\beta 0}. \tag{3.30}$$

These lemmae provide the necessary background for the main Theorem 3.1 of this section, in which the asymptotic distribution of the LR test for r cointegrating relations is stated under the local alternatives.

Theorem 3.1 (Local Power of the LR test) *Under* Assumption 2.2 *and* Assumption 3.1 *the asymptotic distribution of the LR test for the hypothesis $H(r)$ against $H(p)$, is under the local alternatives H_T and H_T^μ, given by*

$$tr\left\{\int_0^1 dZF' [\int_0^1 FF'du]^{-1} \int_0^1 FdZ'\right\}. \tag{3.31}$$

Under H_T the $(p-r)$-dimensional process Z is given by

$$Z_i(u) = \begin{cases} c_1\frac{u^2}{2} + W_i(u) & \text{i=1,} \\ c_2\frac{u^2}{2} + W_i(u) & \text{i=2,} \\ W_i(u) & \text{i=3, \ldots ,p-r,} \end{cases} \tag{3.32}$$

while F equals $Z - \overline{Z}$, but with $Z_1(u) - \overline{Z}_1$ replaced by the linear trend $(u - \frac{1}{2})$. The $(p-r)$ dimensional process W is a standard brownian motion and for any process Y, $\overline{Y} = \int_0^1 Y(u)du$. The scalar constants in Z are defined as

$$c_1 = (\mu'\Sigma\mu)^{-1/2}(\mu'\Sigma\alpha_1)(\beta_1'C\mu), \quad c_2 = \left\{(\mu'C\beta_1)(\alpha_1'\Sigma\alpha_1)(\beta_1'C\mu) - c_1^2\right\}^{1/2}, \tag{3.33}$$

where $C = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp'$ and $\Sigma = \alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\alpha_\perp'$.
Under H_T^μ the $(p-r)$-dimensional process Z satisfies the stochastic differential equation

$$dZ = (\tilde{a}\tilde{b}'Z + \tilde{a}(\tilde{b}'\tilde{a})^{-1}\mu^{(b)})du + dW, \tag{3.34}$$

and F equals

$$F = \begin{cases} (u - \frac{1}{2}) \\ \tilde{b}'(Z - \overline{Z}) \\ \gamma_\perp'\tilde{a}_\perp'(Z - \overline{Z}) \end{cases} \tag{3.35}$$

The parameters in Z and F are given by,

$$\mu^{(b)} = \beta_1'C\mu\sqrt{T} \tag{3.36}$$

$$\tilde{a} = (\alpha_\perp'\Omega\alpha_\perp)^{-1/2}a = (\alpha_\perp'\Omega\alpha_\perp)^{-1/2}\alpha_\perp'\alpha_1 \tag{3.37}$$

$$\tilde{b} = (\alpha_\perp'\Omega\alpha_\perp)^{+1/2}b = (\alpha_\perp'\Omega\alpha_\perp)^{+1/2}(\beta_\perp'\alpha_\perp)^{-1}\beta_\perp'\beta_1 \tag{3.38}$$

$$\gamma = a_\perp'\alpha_\perp'\mu \tag{3.39}$$

Under H_T and H_T^μ it follows that the asymptotic power depends *"on how the extra loadings (α_1) and cointegrating relations (β_1) are related to the α and β assumed under $H(r)$"* (cf. Johansen [9]). And apart from the dependence on the number of dimensions $(p-r)$ for the extra cointegrating relations to hide in, the power depends on the term $\beta_1'C\mu$. The term represents the angle between μ and β_1 and is different from zero as β_1 cannot lie in the space spanned by β.

Table 2:

Asymptotic power of the LR test for r cointegrating vectors among p variables under the local alternative of s extra cointegrating vectors. The distribution of the LR test under $H_T(r+s)$ is given by,

$$tr\left\{\int_0^1 dZF' \left[\int_0^1 FF'du\right]^{-1} \int_0^1 FdZ'\right\},$$

where the $(p-r)$-dimesional processes F, Z are defined in Theorem 3.1. For each simulated distribution under $H_T(r+s)$ identified by the parameters $(c_1, c_2, (p-r))$, the power is found as the tail probability of the 95% quantiles of the distribution under $H(r)$. A table with simulated 95% quantiles under $H(r)$ is given in Johansen and Juselius ([12]). The number of simulations is 6000 with the number of observations T set to 400.

The Power of the LR test under H_T at a 5% level:

$p-r=1$							
	$c_1=0$	$c_1=-3$	$c_1=-6$	$c_1=-9$	$c_1=-12$	$c_1=-15$	$c_1=-18$
$c_2=0$	4.5	13.9	41.8	73.2	93.0	99.1	99.9

$p-r=2$							
	$c_1=0$	$c_1=-3$	$c_1=-6$	$c_1=-9$	$c_1=-12$	$c_1=-15$	$c_1=-18$
$c_2=0$	4.8	7.7	19.6	42.6	69.9	90.5	98.2
$c_2=4$	11.0	14.9	29.0	51.4	77.2	93.5	98.8
$c_2=8$	39.7	44.1	57.1	75.3	88.5	96.7	99.5
$c_2=12$	76.3	78.7	85.6	91.1	96.5	99.1	99.8
$c_2=16$	93.8	95.0	95.9	98.0	99.3	99.8	99.9

$p-r=3$							
	$c_1=0$	$c_1=-3$	$c_1=-6$	$c_1=-9$	$c_1=-12$	$c_1=-15$	$c_1=-18$
$c_2=0$	4.6	6.4	13.4	28.3	52.8	77.1	93.2
$c_2=4$	8.5	10.6	18.6	35.7	58.9	81.1	94.9
$c_2=8$	25.9	29.5	39.6	56.8	75.7	90.8	97.4
$c_2=12$	61.7	63.9	72.4	81.9	91.2	96.9	99.2
$c_2=16$	88.2	88.8	91.7	95.6	97.8	99.2	99.0
$c_2=20$	97.9	98.0	98.8	99.1	99.8	99.9	100.0

Under H_T it is clear that the scalar parameters c_1 and c_2 depend on the length of μ, the size of α_1 and that they are independent of the choice of $\alpha_\perp$ and $\beta_\perp$. The limit distribution is non standard except for the case $(p-r)=1$, where the distibution is a noncentral χ^2_1 with noncentrality parameter $\frac{c_1}{\sqrt{12}}$. Thus in order to investigate the distribution for the case where $(p-r)>1$ the distribution is simulated. For each simulated distribution under $H_T(r+s)$ identified by the parameters $(c_1, c_2, (p-r))$, the power is found as the tail probability of the 95% quantiles of the distribution under $H(r)$. For each set of parameters $(c_1, c_2, (p-r))$ the number of simulations is set to 6000, and Table 2 shows a selection of the simulated power function. It is clear from the table that the power decreases as the dimension $(p-r)$ increases which confirms the result of Johansen [9]. Note that by definition $c_2 \geq 0$ and that the distribution is symmetric in c_1. The symmetry in c_1 follows by the invariance to change of sign of the brownian motion.

Under H_T^μ there are too many parameters for a tabulation of the power function in the general case. Instead only the case with $(p-r)=2$ and $s=1$ is tabulated in Table 4 in section 5. This is used for a comparison of the LR and the DLR test.

4 The power function of the DLR test

The discussion in Section 3.1 regarding the fixed alternative of s additional cointegrating relations immediately carries over. Thus also the DLR test is consistent and the power of the test is investigated in a neighbourhood of the null-hypothesis.

But as was argued section 2.3, the correction for mean and trend in X_t implies that the parameter μ plays no role in the asymptotics and can be ignored. This influences the normalisation of the local alternatives, and as mentioned in Section 3.1, it follows by Johansen [9] that the normalisation is T for the EC model without μ. That is, the sequence of local alternatives to be considered is given by

$$\tilde{H}_T(r+s) : \Pi_T = \alpha\beta' + \alpha_1\beta_1'/T. \tag{4.1}$$

Thus for a normalisation greater than T, e.g. $T^{3/2}$ as was the case before for the LR test under H_T, the power of the test tends to the asymptotic size of the test and the DLR test has therefore less (local) power than the LR test against alternatives with μ varying unretrictedly. However under H_T^μ or equivalently in the neighbourhood of the parameter point where μ is zero in the directions corresponding to the additional cointegrating relations β_1, the rate of convergence is the same for both tests. A further discussion of the simulated asymptotic power of the DLR test compared with the LR test is given in Section 5.

The local power function of the DLR test is stated in Theorem 4.1 below and the simulated power is given in Table 3. The result in Theorem 4.1 follows as in section 2.3 by mimicking the proof of Johansen [9], where the power function is investigated for the likelihood ratio test in the model with $\mu = 0$. It is shown that $LR(\tilde{H}(r)|\tilde{H}(p))$ under $\tilde{H}_T(r+s)$ is asymptotically distributed as

$$tr\left\{\int_0^1 dZZ'\,[\int_0^1 ZZ'du]^{-1}\int_0^1 ZdZ'\right\}, \tag{4.2}$$

with Z given in Theorem 4.1 below.

Theorem 4.1 (Local Power of the DLR test)

Under the Assumption 2.1 and the Assumption 3.1 in terms of $\tilde{H}(r)$ and $\tilde{H}_T(r+s)$, the asymptotic distribution of the DLR test for the hypothesis of r cointegrating vectors is under the local alternative, $\tilde{H}_T(r+s)$ (cf. (4.1)), given by

$$tr\left\{\int_0^1 dZ\mathcal{F}(Z)'\,[\int_0^1 \mathcal{F}(Z)\mathcal{F}(Z)'du]^{-1}\int_0^1 \mathcal{F}(Z)dZ'\right\}. \tag{4.3}$$

The $(p-r)-$dimensional Ornstein-Uhlenbeck process Z satisfies the stochastic differential equation,

$$dZ = \tilde{a}\tilde{b}'Zdu + dW, \tag{4.4}$$

and $\mathcal{F}(Z)$ is Z corrected for mean and linear trend. The $(p-r)\times s$ matrices $\tilde{a}$, $\tilde{b}$ are defined in Theorem 3.1 and W is a $(p-r)$ dimensional standard brownian motion.

Note the resemblance between the Z given by (4.4) and the Z given by (3.34) under H_T^μ. The difference is the drift term involving $\mu^{(b)}$, which by construction the DLR test does not depend on.

As noted in Johansen [9] a tabulation of the power function involves $2(p-r)s$ parameters, but by rotation of the brownian motion W as described in Johansen [9] the following Corollary can be stated for the simple case, where the number of extra cointegrating relations, s equals 1.

Table 3:

Asymptotic power of the DLR test for r cointegrating vectors among p variables under the local alternative of 1 extra cointegrating vector. The distribution of the DLR test under $\tilde{H}_T(r+1)$ is given by,

$$tr\left\{\int_0^1 dZ\mathcal{F}(Z)'\left[\int_0^1 \mathcal{F}(Z)\mathcal{F}(Z)'du\right]^{-1}\int_0^1 \mathcal{F}(Z)dZ'\right\},$$

where the $(p-r)$-dimesional process Z is defined in Theorem 4.1, and $\mathcal{F}$ corrects for mean and linear trend. For each simulated distribution under $\tilde{H}_T(r+s)$ identified by the parameters $(f_1, f_2, (p-r))$, the power is found as the tail probability of the 95% quantiles of the asymptotic distribution of the DLR test under $\tilde{H}(r)$. Table 1 shows the simulated quantiles. The number of simulations is 6000 with the number of observations T set to 400.

The Power of the DLR test at a 5% level:

$p-r=1$							
	$f_1=0$	$f_1=-3$	$f_1=-9$	$f_1=-15$	$f_1=-21$	$f_1=-27$	$f_1=-30$
$f_2=0$	5.0	6.9	16.9	39.8	69.1	89.4	94.8

$p-r=2$							
	$f_1=0$	$f_1=-3$	$f_1=-9$	$f_1=-15$	$f_1=-21$	$f_1=-27$	$f_1=-30$
$f_2=0$	5.0	6.0	10.3	20.2	36.1	56.0	65.2
$f_2=6$	9.0	11.0	15.0	25.3	40.4	59.0	67.5
$f_2=12$	36.4	35.0	33.0	40.1	53.2	68.7	75.8
$f_2=18$	73.5	71.4	64.3	65.6	72.5	81.5	86.4

$p-r=3$							
	$f_1=0$	$f_1=-3$	$f_1=-9$	$f_1=-15$	$f_1=-21$	$f_1=-27$	$f_1=-30$
$f_2=0$	5.0	5.9	8.2	13.3	22.3	35.8	43.0
$f_2=6$	8.2	8.9	11.1	16.3	25.8	38.0	45.0
$f_2=12$	29.2	27.4	23.8	27.1	35.4	46.4	52.5
$f_2=18$	61.0	59.0	48.6	46.8	51.8	60.3	65.3

Corollary 4.1 *Under the Assumptions in Theorem 4.1 the assymptotic distribution of the DLR test under the alternative of one extra cointegrating vector, $\tilde{H}_T(r+1)$, is given by* (4.3) *with*

$$Z_i(u) = \begin{cases} -f_1 \int_0^u Z_1(v)dv + Z_1(u) = W_1(u) & \text{i=1}, \\ -f_2 \int_0^u Z_1(v)dv + Z_2(u) = W_2(u) & \text{i=2}, \\ Z_i(u) = W_i(u) & \text{i=3}, \ldots, \text{p-r}. \end{cases} \tag{4.5}$$

The scalars f_1, f_2 are given by

$$f_1 = \tilde{a}'\tilde{b} < 0, \tag{4.6}$$

$$f_2 = (\tilde{a}'\tilde{a}\tilde{b}'\tilde{b} - f_1^2)^{1/2}, \tag{4.7}$$

with the vectors $\tilde{a}$, $\tilde{b}$ defined in Theorem 3.1.

The power function derived from simulations of the distribution in Corollary 4.1 is given in Table 3. As before, for each distibution, identified by the parameters $(f_1, f_2, (p-r))$, the power is found as the tail probability of the 95% quantiles of the simulated distribution under $\tilde{H}(r)$, given in Table 1. It is clear from the table that the power decreases as $(p-r)$ increases, which was also the case for the LR test.

5 Concluding Remarks

The LR and the DLR tests asymptotic properties under the null-hypothesis of cointegration and under local alternatives of s extra cointegrating relations have been investigated. It follows that the LR test is asymptotically most powerfull under local alternatives where the drift parameter, μ varies unrestrictedly. This is demonstrated by the fact that the extra cointegrating relations have loadings of order $T^{-3/2}$ for the LR test, whereas the loadings are of order T^{-1} for the DLR test. Thus under local alternatives of s extra cointegrating relations with loadings appoaching zero at the rate of T, the LR test has asymptotic power 1, while the DLR test has power less than 1 determined by the parameters given in Theorem 4.1.

An important role for the power properties of the LR test is played by the drift term in the directions corresponding to the extra cointegrating relations given by $\beta_1'C\mu$. This reflects the fact that the LR test is not invariant with respect to μ and contrasts the DLR test which by construction is asymptotically invariant. Therefore the LR tests asymptotic behaviour was studied under local alternatives with loadings α_1/T and $\beta_1' C\mu = \mu^{(b)}/T^{1/2}$. From the Example 5.1 below it follows that also under these alternatives the LR test is asymptotically most powerfull.

However it should be emphasized that this investigation is based on asymptotic analysis and therefore only gives an indication of the performance for finite samples.

Example 5.1 In this example the case of $p-r=2$ and $s=1$ is studied for the two tests under local alternatives where the trend tends to zero in the β_1 direction. Thus let $\alpha=\beta=(I_r,0)'$, $\Omega=I$, $\alpha_1=(0,\dots,0,\pi)'$ and $\beta_1=(0,\dots,0,1)'$. This choice is the simplest system compatible with the parameters given in Theorem 3.1 and Corollary 4.1.

From the choice of parameters above let (x,y) denote the last $p-r=2$ components of X_t under the alternative H_T^μ and consider the process given by

$$\Delta x_t = 1+\varepsilon_{xt} \tag{5.1}$$

$$\Delta y_t = \frac{\pi}{T} y_{t-1} + \mu/T^{1/2} + \varepsilon_{yt}\,. \tag{5.2}$$

The power of the DLR test is by Corollary 4.1 determined by the parameters

$$\begin{aligned} f_1 &= \tilde{a}'\tilde{b} = \pi \\ f_2 &= (\tilde{a}'\tilde{a}\tilde{b}'\tilde{b} - \pi)^{-1/2} = 0. \end{aligned}$$

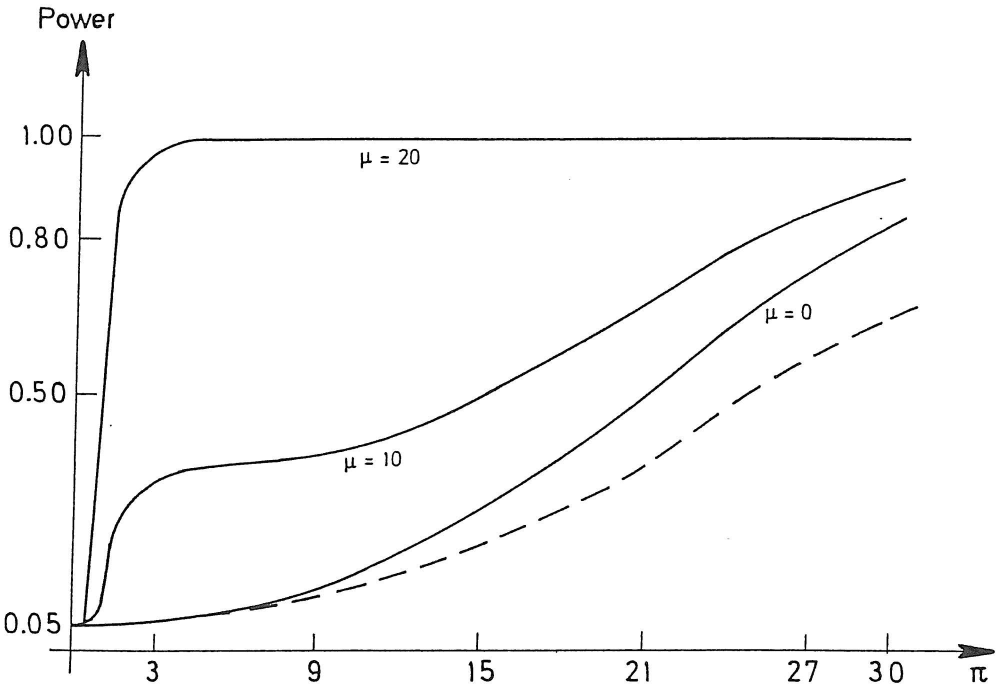
The power of the LR test is by Theorem 3.1 given by the parameters

$$\begin{aligned} \mu^{(b)} &= \mu \\ \tilde{a} &= (0,\pi)'. \end{aligned}$$

In Figure 1 the simulated power functions are shown as functions of (π,μ), and Table 4 gives the simulated power of the LR test. The LR test is seen to be asymptotically most powerfull.

Figure 1:

The power function of the LR test and the DLR test under the local alternative of one extra cointegrating relation and the trend tend to zero in the direction corresponding to the extra cointegrating relation, cf. Example 5.1.



The power of the LR test (–) is shown as a function of π for different values of μ. The power of the DLR test (– –) is independent of μ, and is shown as a function of π.

Table 4:

Asymptotic power of the LR test for r cointegrating vectors among p variables under the local alternative of $s = 1$ extra cointegrating vector in the case where $p - r = 2$. The distribution of the LR test under H_T^μ is given by,

$$tr\left\{\int_0^1 dZF' \left[\int_0^1 FF'du\right]^{-1} \int_0^1 FdZ'\right\},$$

where the $(p-r)$-dimesional processes F, Z are defined in Theorem 3.1. For each simulated distribution under H_T^μ identified by the parameters $(\pi, \mu, (p-r) = 2)$ (cf. Example 5.1), the power is found as the tail probability of the 95% quantiles of the distribution under $H(r)$. A table with simulated 95% quantiles under $H(r)$ is given in Johansen and Juselius ([12]). The number of simulations is 6000 with the number of observations T set to 400.

The Power of the LR test under H_T^μ at a 5% level:

$p-r=2, s=1$							
	$\pi = 0$	$\pi = -3$	$\pi = -9$	$\pi = -15$	$\pi = -21$	$\pi = -27$	$\pi = -30$
$\mu = 0$	4.7	5.4	11.6	27.6	50.2	74.4	83.5
$\mu = 5$	4.7	8.5	16.3	32.3	54.8	77.4	85.8
$\mu = 10$	5.2	33.9	36.6	49.8	68.1	85.2	91.1
$\mu = 15$	5.0	81.4	77.8	77.6	85.9	93.7	96.1
$\mu = 20$	4.8	99.2	98.7	96.7	97.0	98.4	99.1

Appendix

A Weak Convergence and some Matrix Results

The proofs in the paper rely on results from the theory of weak convergence on the p dimensional product space of cadlag functions endowed with the Skorokhod topology, $(D[0,1]^p, \mathcal{D}^{\otimes p})$. An introduction to the theory can be found in Billingsley [1]. From the definition of the Skorokhod topology it follows that "..it relativized to C coincides with the uniform topology" (Billingsley [1], p.112). Here $(C = C[0,1]^p, \mathcal{C}^{\otimes p})$ denotes the product space of continuous functions endowed with the uniform topology. As a consequence the following corollary to the Continuous Mapping Theorem can be stated.

Lemma A.1 (Continuous Mapping Theorem)
If (Z_T) is a sequence of random elements of the p-dimensional space of cadlag functions, $(D[0,1]^p, \mathcal{D}^{\otimes p})$, and Z a random element with support on the space of continuous functions, $(C[0,1]^p, \mathcal{C}^{\otimes p})$, then

$$Z_T \xrightarrow{w} Z \quad \textit{implies} \quad \mathcal{F}(Z_T) \xrightarrow{w} \mathcal{F}(Z) \tag{A.1}$$

if the mapping $\mathcal{F}$ is continuous in the uniform topology. Here $\mathcal{F} : D[0,1]^p \mapsto D[0,1]^p$ or $\mathcal{F} : D[0,1]^p \mapsto R^{p\times p}$, the space of $(p \times p)$ matrices.

The next lemma provides the necessary result for convergence of autoregressive processes under local alternatives to the Ornstein-Uhlenbeck process. A result presented in e.g. Jacobsen [5].

Lemma A.2 (Weak Convergence to the Ornstein-Uhlenbeck process)
Consider the s-dimensional near-integrated process $(Z_t^{(T)})_{t=1\ldots T}$, given by $Z_0^{(T)} = 0$ and

$$\Delta Z_t^{(T)} = DZ_{t-1}^{(T)}/T + \delta/T^{1/2} + \varepsilon_t, \tag{A.2}$$

where δ is a s-vector and D a $(s \times s)$ matrix, while ε_t are $iidN_s(0,\Sigma)$. Then as $T \to \infty$,

$$T^{-1/2}Z^T \xrightarrow{w} Z, \tag{A.3}$$

where Z^T is the cadlag version of $Z^{(T)}$ and Z is the Ornstein-Uhlenbeck process, solving the stochastic differential equation,

$$dZ = (\delta + DZ)du + dB.$$

Note that $Z(u) = \int_0^u \exp((u-s)D)(\delta ds + dB(s))$. The process B is a s-dimensional brownian motion with covariance matrix Σ. The result extends to the case where the error process is replaced by any error process (η_t), with the property that $T^{-1/2}\sum_{t=1}^{[T\cdot]}\eta_t \xrightarrow{w} B(\cdot)$, as $T\to\infty$ on $D[0,1]^s$.

The continuous mapping theorem and Lemma A.2 provide the background of the proofs in combination with the invariance principle and the theory of weak convergence of product moment matrices of linear processes to stochastic integrals as presented in e.g. Johansen [8], Hansen [4] and Phillips and Durlauf [16].

Lemma A.3 (Brownian Motion and the Stochastic Integral)
Suppose that the p-dimensional processes U_t and V_t are given by

$$U_t = C(L)\varepsilon_t = \sum_0^\infty C_i\varepsilon_{t-i} \quad , \quad V_t = D(L)\varepsilon_t = \sum_0^\infty D_i\varepsilon_{t-i}$$

where ε_t are $iidN_p(0,\Omega)$ and $C(z)$,$D(z)$ are convergent for $|z|\leq 1+\delta$ for some $\delta>0$. Then the invariance principle states that

$$T^{-1/2}\sum_{t=1}^{[T\cdot]} U_t \xrightarrow{w} C(1)B(u), \tag{A.4}$$

where B is a p-dimensional brownian motion with covariance matrix Ω. Furthermore

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t-1} U_iV_t' \xrightarrow{w} C(1)\int_0^1 BdB'D(1)' + \Lambda, \tag{A.5}$$

where $\Sigma = \sum_{h=1}^\infty Cov(U_t, V_{t+h})$. The result (A.5) extends to the case with $\sum_1^t U_i$ replaced by the near-integrated process, $Z_t^{(T)}$ given in (A.2) and the limit differs.

The idea in the proofs of the lemmae in Section 3 is to evaluate the difference between the process X_t under $H(r)$ and the alternatives. In order to do so, some results on matrices are needed. With A a $(p_1\times p_2)$ matrix, $\|A\| = \sqrt{\mathrm{tr}\{A'A\}}$ denotes the norm. Similarly for a a p vector, the norm is given by $|a| = \sqrt{a'a}$. The following lemma is from the Appendix in Johansen [8].

Lemma A.4 (Powers of Small Matrices)

If the eigenvalues $(\lambda_i)_{i=1\ldots p}$ *of the* $(p \times p)$ *matrix* Λ, *are less than one in absolute value, then for* $\lambda = max_i |\lambda_i|$,

$$\|\Lambda^n\| \leq C_\lambda \lambda^n \,, \tag{A.6}$$

$$\sum_{n=0}^{\infty} \Lambda^n = (I - \Lambda)^{-1} \tag{A.7}$$

where C_λ *is a positive constant.*

Finally the binomial formula for matrices.

Lemma A.5 (The Binomial Formula)

With A, B $(p \times p)$ *(noncommutative) matrices*

$$(A + B)^n = \sum_{m=0}^{n} \sum_{i_1+\ldots+i_{m+1}=n-m} A^{i_1} B A^{i_2} B \ldots B A^{i_{m+1}} \tag{A.8}$$

where m *equals the number of times* B *occurs in the inner sum. The inner sum is over* $i_1, \ldots, i_{m+1} \in [0, \ldots, n]$ *and* $\sum_{i_1+\ldots+i_{m+1}=n-m} = \binom{n}{m}$.

B Local Asymptotics

With the notation introduced in Section 3.1 it follows that with $A = (I + \alpha\beta')$, $B_T = T^{-3/2}\alpha_1\beta_1'$ and $B_T^\mu = T^{-1}\alpha_1\beta_1'$,

$$X_t = \textstyle\sum_0^{t-1} A^i(\varepsilon_{t-i} + \mu)\,, \tag{B.1}$$

$$X_t^{(T)} = \textstyle\sum_0^{t-1}(A + B_T)^i(\varepsilon_{t-i} + \mu)\,, \tag{B.2}$$

$$X_t^{(T,\mu)} = \textstyle\sum_0^{t-1}(A + B_T^\mu)^i(\varepsilon_{t-i} + \mu_T) \tag{B.3}$$

where $X_0^{(T)} \equiv X_0^{(T,\mu)} \equiv X_0 \equiv 0$. The next Lemma gives bounds for the matrices in the formulae above. These preliminary results are used when deriving the weak convergence results for the process in the following.

Lemma B.1 *With $A = (I + \alpha\beta')$, $B_T = T^{-3/2}\alpha_1\beta_1'$ and $B_T^\mu = T^{-1}\alpha_1\beta_1'$ and under* Assumption 3.1, *then*

$$\|A^i\| \le C_A\,, \tag{B.4}$$

$$\|\beta' A^i\| \le C_\lambda \lambda^i\,, \tag{B.5}$$

$$\|\beta'(A + B_T)^i - \beta' A^i\| \le C_\beta \|B_T\|\,, \tag{B.6}$$

$$\|\textstyle\sum_{m=n}^{i} \sum_{i_1+\ldots i_{m+1}=i-m} A^{i_1} B_T \ldots B_T A^{i_{m+1}}\| \le C_{AB,n} T^n \|B_T\|^n \tag{B.7}$$

where $\lambda \in (0,1)$ and $C_A, C_\lambda, C_\beta, (C_{AB,n})_{n=0\ldots i}$ are positive constants. The results also hold with B_T replaced by B_T^μ.

Proof: First note that $\beta' A^i = (I + \beta'\alpha)^i \beta'$ and that $\alpha_\perp' A^i = \alpha_\perp'$. Under Assumption 3.1 the roots of the characteristic polynomial for X_t under $H(r)$ are outside the unit circle or at 1, and as a consequence the eigenvalues of $(I + \beta'\alpha)$ are inside the unit circle. Thus (A.6) imply that $\|\beta' A^i\| \le C_\lambda \lambda^i$, which is the result of (B.5).
As to (B.4) use that $I = (\alpha_\beta, \beta_{\alpha\perp})(\beta, \alpha_\perp)'$ then

$$\|A^i\| = \|(\alpha_\beta, \beta_{\alpha\perp})([(I + \beta'\alpha)^i\beta']', \alpha_\perp)'\| \le \|\alpha_\beta(I + \beta'\alpha)^i\beta'\| + \|\beta_{\alpha\perp}\alpha_\perp'\| \le C_A$$

To prove (B.6) use the binomial formula and the just proved results to get,

$$\begin{aligned} \|\beta'(A + B_T)^i - \beta' A^i\| &= \|\textstyle\sum_{m=1}^{i} \sum_{i_1+\ldots+i_{m+1}=i-m} \beta' A^{i_1} B_T A^{i_2} B_T \ldots B_T A^{i_{m+1}}\| \\ &\le \textstyle\sum_{m=1}^{i} \|B_T\|^m C_A^m C_\lambda \sum_{i_1+\ldots+i_{m+1}=i-m} \lambda^{i_1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{i} \|B_T\|^m C_A^m C_\lambda \sum_{i_1=0}^{i-m} \lambda^{i_1} \binom{i-i_1-1}{m-1} \\
&\leq \sum_{m=1}^{i} \|B_T\|^m C_A^m C_\lambda \frac{(i-1)^{m-1}}{(m-1)!} \sum_{i_1=0}^{i-m} \lambda^{i_1} \\
&\leq \|B_T\| C_A C_1 \sum_{m=0}^{i-1} \frac{(\|B_T\| C_A (i-1))^m}{m!} \\
&\leq \|B_T\| C_A C_1 \exp(C_A T \|B_T\|) \leq C_\beta \|B_T\|,
\end{aligned}
$$

since $T\|B_T\|$ is bounded. The constant C_1 is obtained by using that λ^{i_1} is summable. As to (B.7),

$$
\begin{aligned}
\|\sum_{m=n}^{i} \sum_{i_1+\ldots i_{m+1}=i-m} A^{i_1} B_T \ldots B_T A^{i_{m+1}}\| &\leq \sum_{m=n}^{i} \|B_T\|^m C_A^{m+1} \frac{i^m}{m!} \\
&\leq T^n \|B_T\|^n K \exp(T C_A \|B_T\|)
\end{aligned}
$$

where K is a constant. The proofs hold for B_T replaced with B_T^μ. □

Next follow the proofs of the results in Section 3.

Proof of Lemma 3.1: Consider first $\beta' X_t^{(T)}$ which by the representation (B.1) is given by

$$
\begin{aligned}
\beta' X_t^{(T)} &= \beta' \sum_{i=0}^{t-1} (A + B_T)^i (\varepsilon_{t-i} + \mu) \\
&= \sum_{i=0}^{t-1} \beta' A^i (\varepsilon_{t-i} + \mu) + \sum_{i=0}^{t-1} (\beta'(A + B_T)^i - \beta' A^i)(\varepsilon_{t-i} + \mu) \\
&= \beta' X_t + R_{t\beta}
\end{aligned}
$$

Denote by Λ_i the term $\beta'(A + B_T)^i - \beta' A^i$ and henceforth let $(K_i)_{i=1\ldots10}$ denote positive constants. Then $E(R_{t\beta}) = \sum_0^{t-1} \Lambda_i \mu$ and by application of (B.6) it follows that

$$
|E(R_{t\beta})| \leq K_1 T \|B_T\| |\mu| \tag{B.8}
$$

Next $\operatorname{Var}(R_{t\beta}) = \sum_0^{t-1} \Lambda_i \Omega \Lambda_i'$ and again by (B.6) it follows that

$$
\|\operatorname{Var}(R_{t\beta})\| \leq K_2 T \|B_T\|^2.
$$

Hence by

$$
E|R_{t\beta}|^2 = \operatorname{tr}\{\operatorname{Var}(R_{t\beta})\} + |E(R_{t\beta})|^2 \leq p \|\operatorname{Var}(R_{t\beta})\| + |E(R_{t\beta})|^2, \tag{B.9}
$$

it follows that $|E(R_{t\beta})|^2$ dominates and (3.9) follows. However with B_T replaced by B_T^μ the argument gives that $|E(R_{t\beta}^\mu)|^2$ is of order 1, since $T\|B_T^\mu\||\mu_T|$ is bounded by a constant. This shows that a more careful evaluation of the mean of $R^{\mu}_{t\beta}$ is needed, where the parametrization of μ_T is used. Application of the binomial formula gives,

$$R^{\mu}_{t\beta} = \textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum_{i_1+\ldots+i_{m+1}=i-m} \beta' A^{i_1} B^{\mu}_T A^{i_2} \ldots A^{i_m} B^{\mu}_T A^{i_{m+1}} (\varepsilon_{t-i} + \mu_T).$$

By the decomposition $I = \alpha_\beta \beta' + \beta_{\alpha\perp} \alpha'_\perp$, this may be represented as the sum of R^{μ}_1 and R^{μ}_2, where

$$\begin{aligned} R^{\mu}_1 &= \textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum \beta' A^{i_1} B^{\mu}_T A^{i_2} \ldots B^{\mu}_T \beta_{\alpha\perp} \alpha'_\perp (\varepsilon_{t-i} + \mu_T), \\ R^{\mu}_2 &= \textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum \beta' A^{i_1} B^{\mu}_T A^{i_2} \ldots B^{\mu}_T \alpha_\beta \beta' A^{i_{m+1}} (\varepsilon_{t-i} + \mu_T). \end{aligned}$$

Regarding R^{μ}_1 note that by definition $B^{\mu}_T \beta_{\alpha\perp} \alpha'_\perp \mu_T = T^{-1} \alpha_1 \beta'_1 C \mu_T$, where C is the impact matrix given by $\beta_\perp (\alpha'_\perp \beta_\perp)^{-1} \alpha'_\perp$, cf. Theorem 2.1. Then by mimicking the proof of (B.6),

$$\begin{aligned} |E(R^{\mu}_1)| &= |\textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum \beta' A^{i_1} B^{\mu}_T A^{i_2} \ldots A^{i_m} \alpha_1 \beta'_1 C \mu_T T^{-1}| \\ &\leq K_3 \|\beta'_1 C \mu_T\| \exp(T C_A \|B^{\mu}_T\|) \leq K_4 \|\beta'_1 C \mu_T\| \end{aligned}$$

Thus applying the parametrization of μ_T given in Section 3.1 (cf. (3.7)), $\beta'_1 C \mu_T = \mu^{(b)}/T^{-1/2}$ and therefore $|E(R^{\mu}_1)|$ is of order $T^{-1/2}$. Next ,

$$\begin{aligned} |E(R^{\mu}_2)| &= |\textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum \beta' A^{i_1} B^{\mu}_T A^{i_2} \ldots B^{\mu}_T \alpha_\beta \beta' A^{i_{m+1}} \mu_T| \\ &\leq K_5 \textstyle\sum_{i=1}^{t-1} \sum_{m=1}^{i} \sum_{i_1+\ldots+i_{m+1}=i-m} \lambda^{i_1+i_{m+1}} \|B^{\mu}_T\|^m C_A^{m-1} \\ &= K_5 \textstyle\sum_{i=1}^{t-1} \left[\lambda^{i-1} i \|B^{\mu}_T\| + \sum_{m=2}^{i} \|B^{\mu}_T\|^m C_A^{m-1} \sum \binom{i-i_1-i_{m+1}-2}{m-2} \lambda^{i_1+i_{m+1}} \right] \\ &\leq K_6 \|B^{\mu}_T\| + K_7 \textstyle\sum_{i=2}^{t-1} \sum_{m=2}^{i} \|B^{\mu}_T\|^m C_A^{m-1} \frac{(i-2)^{m-2}}{(m-2)!} \sum_{i_1+i_{m+1} \leq i-m} \lambda^{i_1+i_{m+1}} \\ &\leq K_6 \|B^{\mu}_T\| + K_8 T \|B^{\mu}_T\|^2 \exp(T C_A \|B^{\mu}_T\|) \leq K_9 \|B^{\mu}_T\|, \end{aligned}$$

using that $i\lambda^i$ is summable. Finally

$$|E(R^{\mu}_{t\beta})| \leq |E(R^{\mu}_1)| + |E(R^{\mu}_2)| \leq K_{10} T^{-1/2},$$

and the result (3.10) follows by (B.9). □

Proof of Lemma 3.2: Consider first $\alpha'_\perp X^{(T)}_t$ which by the representation (B.1) and the binomial formula is given by,

$$\begin{aligned} X^{(T)}_t &= \textstyle\sum_{i=0}^{t-1} \sum_{m=0}^{i} \sum A^{i_1} B_T \ldots B_T A^{i_{m+1}} (\varepsilon_{t-i} + \mu) \\ &= \textstyle\sum_{i=0}^{t-1} A^i (\varepsilon_{t-i} + \mu) + \sum_{i=1}^{t-1} \sum_{j=0}^{i-1} A^j B_T A^{i-j-1} (\varepsilon_{t-i} + \mu) \\ &\quad + \textstyle\sum_{i=2}^{t-1} \sum_{m=2}^{i} \sum A^{i_1} B_T \ldots B_T A^{i_{m+1}} (\varepsilon_{t-i} + \mu) \\ &\equiv X_t + Y^{(T)}_t + R^{(T)}_{tX}. \end{aligned} \qquad \text{(B.10)}$$

Now $\alpha_\perp' A^i = \alpha_\perp'$ and therefore

$$\alpha_\perp' X_{[Tu]} \;=\; \alpha_\perp' \textstyle\sum_1^{[Tu]} \varepsilon_t + \alpha_\perp' \mu [Tu] = \alpha_\perp' S_{[Tu]} + \tau [Tu]. \tag{B.11}$$

The invariance principle and the continous mapping theorem applied on the mapping $x \mapsto \max_{t \leq T}|x|$, imply that $T^{-1/2}\max_{t\leq T}|S_t| = O_p(1)$. Therefore $\sup_{u\in[0,1]} T^{-1} S_{[Tu]} \xrightarrow{P} 0$, and it follows that

$$T^{-1}\alpha_\perp' X_{[Tu]} \xrightarrow{w} \alpha_\perp' \mu \, u = \tau \, u \, .$$

Henceforth let B denote a brownian motion with covariance Ω. Multiplication by $\tau_\perp$ in (B.11) leads by the invariance principle to the result,

$$T^{-1/2}\tau_\perp' \alpha_\perp' X_{[Tu]} \xrightarrow{w} \tau_\perp' \alpha_\perp' B(u).$$

For the $Y_t^{(T)}$ apply the decomposition $I = \alpha_\beta \beta' + \beta_{\alpha\perp}\alpha_\perp'$ and get

$$\begin{aligned}
\alpha_\perp' Y_{[Tu]}^{(T)} &= \alpha_\perp' B_T \textstyle\sum_{i=1}^{[Tu]-1} \sum_{j=0}^{i-1} A^{i-1-j} (\varepsilon_{[Tu]-i} + \mu) \\
&= \alpha_\perp' B_T \left[\alpha_\beta \textstyle\sum_{i=1}^{[Tu]-1} \sum_{j=0}^{i-1} \beta' A^{i-1-j} + \beta_{\alpha\perp} \sum_{i=1}^{[Tu]-1} i \alpha_\perp' \right] (\varepsilon_{[Tu]-i} + \mu) \\
&= Y_{1t} + \alpha_\perp' B_T \beta_{\alpha\perp} \textstyle\sum_{i=1}^{[Tu]-1} i \alpha_\perp' (\varepsilon_{[Tu]-i} + \mu) \, .
\end{aligned} \tag{B.12}$$

Using (B.7) and mimicking the proof of Lemma 3.1, $\max_{t\leq T} E|\alpha_\perp' Y_{1t}|^2 = O(T^{-1})$ and therefore $T^{-1/2}\max_t Y_{1t} \xrightarrow{P} 0$. By definition B_T equals $T^{-3/2}\alpha_1\beta_1'$ and hence,

$$T^{-1/2}\alpha_\perp' Y_{[Tu]}^{(T)} \;\xrightarrow{w}\; \alpha_\perp' \alpha_1 \beta_1' \beta_{\alpha\perp} \alpha_\perp' \mu \frac{u^2}{2} = a b' \tau \frac{u^2}{2}. \tag{B.13}$$

Finally the results (3.13) and (3.14) follow by showing that

$$T^{-1/2}\max_t R_{tX}^{(T)} \xrightarrow{P} 0 \, .$$

To see this note that (B.7) with K a constant, $|R_{tX}^{(T)}| \leq K(T^{-1}\sum_1^T |\varepsilon_i| + |\mu|)$, and the result immediately follows since the ε_i are identically and independently $N_p(0, \Omega)$ distributed.

Regarding $\alpha_\perp' X_t^{(T,\mu)}$, it follows by (3.12) that

$$a_\perp' \alpha_\perp' \Delta X_t^{(T,\mu)} = \gamma + a_\perp' \alpha_\perp' \varepsilon_t, \tag{B.14}$$

from which (3.16) and (3.17) follow. For the b direction,

$$\begin{aligned}
b' \alpha_\perp' \Delta X_t^{(T,\mu)} &= T^{-1} b' a \beta_1' X_{t-1}^{(T,\mu)} + T^{-1/2}\mu^{(b)} + b' \alpha_\perp' \varepsilon_t \\
&= T^{-1} b' a b' \alpha_\perp' X_{t-1}^{(T,\mu)} + T^{-1/2}\mu^{(b)} + b' \alpha_\perp' \varepsilon_t + T^{-1} b' a \beta_1' \alpha_\beta \beta' X_{t-1}^{(T,\mu)},
\end{aligned}$$

using the definition of b. Then the result (3.15) follows by Lemma A.2, with $Z_t^{(T)} \equiv b'\alpha'_\perp X_t^{(T,\mu)}$, $\delta \equiv \mu^{(b)}$, $D \equiv b'a$ and finally

$$\eta_t \equiv b'\alpha'_\perp \varepsilon_t + b'a\beta'_1\alpha_\beta\beta' X_{t-1}^{(T,\mu)}/T. \tag{B.15}$$

That $T^{-1/2}\sum_{t=1}^{[T\cdot]}\eta_t \xrightarrow{w} b'\alpha'_\perp B(\cdot)$, where B is a p-dimensional brownian motion with covariance Ω, follows by the invariance principle if

$$T^{-3/2}\text{sup}_{u\in[0,1]}\sum_{t=1}^{[Tu]}\beta' X_{t-1}^{(T,\mu)} \xrightarrow{P} 0\,.$$

As before $T^{-3/2}\text{sup}_{u\in[0,1]}\sum_{t=1}^{[Tu]}\beta' X_{t-1} \xrightarrow{P} 0$, by the invariance principle and the continous mapping theorem. Finally $T^{-3/2}\text{sup}_{u\in[0,1]}\sum_{t=1}^{[Tu]}R_{t\beta}^{\mu} \xrightarrow{P} 0$, since by Lemma 3.1, $\max_t E|R_{t\beta}^{\mu}|^2 = O(T^{-1})$. □

Proof of Lemma 3.3: From the definition of S_{11} and Lemma 3.1 it follows that under H_T,

$$\beta' S_{11}\beta = \tfrac{1}{T}\textstyle\sum_1^{T-1}\left[\beta'(X_t - \overline{X}) + R_{t\beta} - \overline{R}_\beta\right]\left[\beta'(X_t - \overline{X}) + R_{t\beta} - \overline{R}_\beta\right]',$$

using the notation that for any process Y_t, $\overline{Y} = T^{-1}\sum_1^T Y_t$. By Lemma 3.1, $\max_{t\le T}E|R_{t\beta}|^2$ is $O(T^{-1})$ and hence $T^{-1}\sum(R_{t\beta} - \overline{R}_\beta)(R_{t\beta} - \overline{R}_\beta)' \xrightarrow{P} 0$. Likewise the cross product terms involving $R_{t\beta}$ tend to zero by Hölder's inequality using that $\max_{t\le T}E|\beta' X_t|^2$ is $O(1)$. Finally the law of large numbers for ergodic processes implies that $\beta' S_{11}\beta \xrightarrow{P} \Sigma_{\beta\beta}$. Similarly under H_T^{μ}.

As for the other directions let $V_T = (\beta, \tilde{V}_T)$, where $\tilde{V}_T = \left(\alpha_\perp\tau/T, \alpha_\perp\tau_\perp/T^{1/2}\right)$ under H_T and $\tilde{V}_T = \left(\alpha_\perp b/T^{1/2}, \alpha_\perp a_\perp\gamma/T, \alpha_\perp a_\perp\gamma_\perp/T^{1/2}\right)$ under H_T^{μ}. Then by the Continuous Mapping Theorem applied on the mapping $(z \mapsto \int_0^1(z(u) - \int_0^1 z(s)ds)(z(u) - \int_0^1 z(s)ds)'du)$, the results in Lemma 3.2 immediately gives

$$\begin{aligned}\tilde{V}'_T S_{11}\tilde{V}_T &= \tfrac{1}{T}\textstyle\sum_{u=1/T}^{T/T}\left[\tilde{V}'_T(X_{[Tu]}^{(T)} - \tfrac{1}{T}\sum_{s=1/T}^{T/T}X_{[Ts]}^{(T)})\right]\left[(X_{[Tu]}^{(T)} - \tfrac{1}{T}\sum_{s=1/T}^{T/T}X_{[Ts]}^{(T)})'\tilde{V}_T\right] \\ &\xrightarrow{w} \textstyle\int_0^1 \mathcal{F}(u)\mathcal{F}(u)'du.\end{aligned} \tag{B.16}$$

It remains to be shown that

$$\begin{aligned}\tilde{V}'_T S_{11}\beta &= \tfrac{1}{T}\textstyle\sum_1^T \tilde{V}'_T(X_t^{(T)} - \overline{X}^{(T)})(\beta'(X_t^{(T)} - \overline{X}^{(T)}))' \\ &= \tfrac{1}{T}\textstyle\sum_1^T \tilde{V}'_T(X_t^{(T)} - \overline{X}^{(T)})(\beta' X_t + R_{t\beta})' \xrightarrow{P} 0.\end{aligned}$$

To see this apply first the result that $\sqrt{T}\tilde{V}_T' \frac{1}{T}\sum_1^T(X_t^{(T)} - \overline{X}^{(T)})(\beta' X_t)'$ is $O_P(1)$ by Lemma A.3. Next note that $\max_{t\leq T} E|\alpha_\perp' X_t|^2$ is $O(T^2)$ but $\max_{t\leq T} E|\tau_\perp' \alpha_\perp' X_t|^2$ is only $O(T)$. Also $\max_{t\leq T} E|R_{tX}^{(T)}|^2$ is $O(1)$ and finally $\max_{t\leq T} E|\alpha_\perp' Y_t^{(T)}|^2$ is $O(T)$. Application of Hölder's inequality then gives that the remaining terms tend to zero. Similarly for $X_t^{(T,\mu)}$. □

Proof of Lemma 3.4: Concentrating the likelihood function with respect to the drift parameter μ, leads to

$$\Delta X_t^{(T)} - \Delta \overline{X}^{(T)} = \Pi_T(X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)}) + (\varepsilon_t - \overline{\varepsilon}),$$

and therefore under H_T,

$$S_{10} - S_{11}\Pi_T' = \tfrac{1}{T}\textstyle\sum_1^T(X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)})(\varepsilon_t - \overline{\varepsilon})' = \tfrac{1}{T}\sum_1^T(X_{t-1}^{(T)} - \overline{X}_{-1}^{(T)})\varepsilon_t' \equiv S_{1\varepsilon}. \quad \text{(B.17)}$$

As previously noted $V_T = (\beta, \tilde{V}_T)$. Consider first the β direction, where by the Central Limit Theorem for Martingale Differences

$$\sqrt{T}\beta' S_{1\varepsilon} = \textstyle\sum_1^T\left[\beta'(X_{t-1} - \overline{X}_{-1}) + R_{t-1\beta} - \overline{R}_\beta\right]\varepsilon_t'/\sqrt{T} \xrightarrow{w} N\left(0, \Sigma_{\beta\beta}\otimes\Omega\right), \quad \text{(B.18)}$$

since $\sum_1^T(R_{t-1\beta} - \overline{R}_\beta)\varepsilon_t'/\sqrt{T} \xrightarrow{P} 0$. Next in the $\tilde{V}_T$ direction by Lemma A.3,

$$\sqrt{T}\tilde{V}_T' S_{1\varepsilon} = \textstyle\sum_{1/T}^{T/T}(\tilde{V}_T'(X_{[Tu]-1}^{(T)} - \overline{X}_{-1}^{(T)}))\varepsilon_{[Tu]}'/\sqrt{T} \xrightarrow{w} \int_0^1 \mathcal{F}dB'. \quad \text{(B.19)}$$

As to (3.29) rewrite S_{00} as

$$S_{00} = S_{\varepsilon\varepsilon} + S_{\varepsilon 1}\Pi_T' + \Pi_T S_{1\varepsilon} + \Pi_T S_{11}\Pi_T', \quad \text{(B.20)}$$

where $\Pi_T = \alpha\beta' + \alpha_1\beta_1'/T^{3/2}$ under H_T. Applying the Law of Large Numbers for ergodic processes it follows that $S_{\varepsilon\varepsilon} \xrightarrow{P} \Omega$. Furthermore $\beta' S_{11}\beta \xrightarrow{P} \Sigma_{\beta\beta}$ and $\beta' S_{1\varepsilon} \xrightarrow{P} 0$. For the term $\beta_1' S_{1\varepsilon}$ use again the decomposition, $I_p = \alpha_\beta\beta' + \beta_{\alpha\perp}\alpha_\perp'$, Lemma 3.1 and Lemma 3.2 to see that $\beta_1' S_{1\varepsilon}$ is dominated by $b'\alpha_\perp' S_{1\varepsilon}$. Hence by Lemma 3.2 $T^{-3/2}\beta_1' S_{1\varepsilon}$ tends to zero. Likewise $T^{-3}\beta_1' S_{11}\beta_1$ tends to zero. This implies $S_{00} \xrightarrow{P} \alpha\Sigma_{\beta\beta}\alpha' + \Omega = \Sigma_{00}$. Applying similar arguments,

$$\beta' S_{10} = \beta' S_{11}\Pi_T' + \beta' S_{1\varepsilon} \xrightarrow{P} \Sigma_{\beta\beta}\alpha' = \Sigma_{\beta 0}, \quad \text{(B.21)}$$

and (3.30) follows. Similarly under H_T^μ. □

Proof of Theorem 3.1: The underlying structure of the proof is the same as presented in Johansen [7], p.1569 Proof of Theorem 2.1. The proof is split into three parts, where the first part contains results valid under both H_T and H_T^{μ}.

Proof of Theorem 3.1: (Part 1) From (2.7) it follows that the likelihood ratio test of $H(r)$ against $H(p)$ is given by $-T\sum_{i=r+1}^{p}\ln(1-\hat{\lambda}_i)$,where $(\hat{\lambda}_i)_{i=r+1\ldots p}$ are the $(p-r)$ ordered *smallest* eigenvalues solving the eigenvalue problem (2.8)

$$\left|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}\right| = 0\,. \tag{B.22}$$

From Johansen [7] it follows that the eigenvalues are continuous functions of the coefficient matrices. Post and premultiplication in (B.22) by $V_T = (\beta, \tilde{V}_T)$ then gives by application of Lemma 3.3 and 3.4 that $(\hat{\lambda}_i)_{i=1\ldots p}$ converge to the ordered eigenvalues of

$$\lambda^{p-r}|\textstyle\int_0^1 \mathcal{F}\mathcal{F}'du||\lambda\Sigma_{\beta\beta} - \Sigma_{\beta 0}\Sigma_{00}^{-1}\Sigma_{0\beta}| = 0\,. \tag{B.23}$$

Thus the $(p-r)$ smallest eigenvalues tend to zero, corresponding to the near-(co)integrated and the non-stationary components. Next define $\rho = \lambda T$.
Post and premultiplication in (B.22) with $(\beta, T^{1/2}\tilde{V}_T)$ gives as $T\to\infty$ by Lemma 3.3, that the $(p-r)$ smallest eigenvalues normalized by T satisfy in the limit

$$\left|\rho\begin{bmatrix} 0 & 0 \\ 0 & \int\mathcal{F}\mathcal{F}'du\end{bmatrix} - wlim\left\{(\beta, T^{1/2}\tilde{V}_T)'S_{10}S_{00}^{-1}S_{01}(\beta, T^{1/2}\tilde{V}_T)\right\}\right| = 0\,. \tag{B.24}$$

By (3.29), (3.30) and the formula for the determinant of a block matrix (B.24) can be rewritten as

$$|\Sigma_{\beta 0}\Sigma_{00}^{-1}\Sigma_{0\beta}||\rho\int\mathcal{F}\mathcal{F}'du - wlim\left\{T^{1/2}\tilde{V}_T'S_{10}S_{00.\beta}^{-1}S_{01}T^{1/2}\tilde{V}_T\right\}| = 0 \tag{B.25}$$

with

$$\begin{aligned} S_{00.\beta} &= S_{00}^{-1} - S_{00}^{-1}S_{01}\beta(\beta S_{10}S_{00}^{-1}S_{01}\beta)^{-1}\beta' S_{10}S_{00}^{-1} \\ &\xrightarrow{P} \Sigma_{00}^{-1} - \Sigma_{00}^{-1}\Sigma_{0\beta}(\Sigma_{\beta 0}\Sigma_{00}^{-1}\Sigma_{0\beta})^{-1}\Sigma_{\beta 0}\Sigma_{00}^{-1} = \alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\alpha_\perp'\,, \end{aligned}$$

where the last equality follows from Johansen ([7] Lemma A.1, p.1567). Hence what is of interest is

$$\begin{aligned} wlim\left\{T^{1/2}\tilde{V}_T'S_{10}\alpha_\perp\right\} &= wlim\left\{T^{1/2}\tilde{V}_T'S_{1\varepsilon}\alpha_\perp\right\} + wlim\left\{T^{1/2}\tilde{V}_T'S_{11}\Pi_T'\alpha_\perp\right\} \\ &= \int_0^1 \mathcal{F}dB'\alpha_\perp + wlim\left\{T^{1/2}\tilde{V}_T'S_{11}\Pi_T'\alpha_\perp\right\}, \end{aligned} \tag{B.26}$$

by Lemma A.3, see (B.17). This and the remaining part of the proof will be treated seperately for H_T and H_T^{μ} in the next two parts.

Proof of Theorem 3.1: (Part 2) Under H_T it follows that

$$T^{1/2}\tilde{V}_T'S_{11}\Pi_T'\alpha_\perp = \tfrac{1}{T}\tilde{V}_T'S_{11}\beta_1 a' \xrightarrow{w} \textstyle\int_0^1 \mathcal{F}(u)udu(ab'\tau)', \tag{B.27}$$

by the continuous mapping theorem and using the decomposition, $I_p = \alpha_\beta\beta' + \beta_{\alpha\perp}\alpha_\perp'$, to see that $T^{-1/2}\beta_1'X_{[Tu]}^{(T)}$ equals $T^{-1/2}b'\alpha_\perp'X_{[Tu]}^{(T)} + o_P(1)$. Hence with the $(p-r)$ dimensional process $\mathcal{Z}$ defined by

$$\mathcal{Z}(u) \;=\; ab'\tau\frac{u^2}{2} + \alpha_\perp' B(u), \tag{B.28}$$

it follows that (B.26) equals $\int_0^1 \mathcal{F}d\mathcal{Z}'$. Note that $\mathcal{F}$ equals (cf. (3.26)),

$$\mathcal{F}_i(u) \;=\; \begin{cases} \tau'\tau(u-\frac{1}{2}) & \text{i=1,} \\ \tau_\perp{}'(\mathcal{Z}_i - \overline{\mathcal{Z}}_i) & \text{i=2, \ldots, p-r,} \end{cases} \tag{B.29}$$

where for any process Z, $\overline{Z} = \int_0^1 Z(u)du$. By definition $\Omega_{\alpha_\perp} = \alpha_\perp'\Omega\alpha_\perp$ is the covariance of $\mathcal{Z}$ and alltogether by (B.25), ρ satisfies in the limit,

$$\left|\rho \textstyle\int_0^1 \mathcal{F}\mathcal{F}'\,du - \int_0^1 \mathcal{F}d\mathcal{Z}'\Omega_{\alpha_\perp}^{-1}\int_0^1 d\mathcal{Z}\mathcal{F}'\right| = 0. \tag{B.30}$$

Equivalently,

$$|\rho \textstyle\int (N\mathcal{F})(N\mathcal{F})'\,du - \int (N\mathcal{F})d(M\mathcal{Z})'(M\Omega_{\alpha_\perp}M')^{-1}\int d(M\mathcal{Z})(N\mathcal{F})'| = 0, \tag{B.31}$$

for any square $(p-r)$ matrices of full rank, which shows that linear transformations of $\mathcal{F}, \mathcal{Z}$ are allowed. The process $\mathcal{F}$ is only dependent on $\mathcal{Z}$ through the linear combinations $\tau_\perp{}'\mathcal{Z}$, which are independent of the linear combination $\tau'\Omega_{\alpha_\perp}^{-1}\mathcal{Z}$. Thus decompose $\mathcal{Z}$ into two independent processes given by,

$$\begin{aligned} \mathcal{Z}_1 &= \tau'\Omega_{\alpha_\perp}^{-1}\mathcal{Z}, \\ \mathcal{Z}_2 &= \tau_\perp{}'\mathcal{Z}. \end{aligned}$$

The process $\mathcal{Z}_2$ has quadratic trends in the direction given by $\phi = \tau_\perp{}'ab'\tau$ and none in the orthogonal directions $\phi_\perp$. As before $\phi_\perp'\tau_\perp'\mathcal{Z}$ is independent of the linear combination $\phi'(\tau_\perp'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\tau_\perp'\mathcal{Z}$, and the decomposition of $\mathcal{Z}$ becomes

$$\begin{aligned} \mathcal{Z}_1 &= \tau'\Omega_{\alpha_\perp}^{-1}\mathcal{Z}, \\ \mathcal{Z}_2 &= \phi'(\tau_\perp'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\tau_\perp'\mathcal{Z}, \\ \mathcal{Z}_3 &= \phi'\tau_\perp{}'\mathcal{Z}, \end{aligned}$$

where the $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_3$ are independent. Summarising, define M as,

$$M = \left(\Omega_{\alpha_\perp}^{-1}\tau, \tau_\perp(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\phi, \tau_\perp\phi_\perp\right)', \tag{B.32}$$

and next N by,

$$\operatorname{diag}\{(\tau'\tau)^{-1}, \begin{bmatrix} (\phi'(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\phi)^{-1/2}\phi'(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1} \\ (\phi_\perp'\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp\phi_\perp)^{-1/2}\phi_\perp' \end{bmatrix}\}. \tag{B.33}$$

Furthermore set $F = N\mathcal{F}$, $Z = (M\Omega_{\alpha_\perp}M')^{-1/2}M\mathcal{Z}$ and finally denote the $(p-r)$ dimensional Brownian Motion $(M\Omega_{\alpha_\perp}M')^{-1/2}M\alpha_\perp' B(u)$ by $W(r)$. Then ρ in the limit satisfies

$$|\rho \textstyle\int FF'du - \int FdZ' \int dZF'| = 0, \tag{B.34}$$

where

$$Z(u) = \begin{cases} \left(\tau'\Omega_{\alpha_\perp}^{-1}\tau\right)^{-1/2}\tau'\Omega_{\alpha_\perp}^{-1}ab'\tau\frac{u^2}{2} + W_i(u) & \text{i=1,} \\ (\phi'(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\phi)^{1/2}\frac{u^2}{2} + W_i(u) & \text{i=2,} \\ W_i(u) & \text{i=3, \ldots, p-r,} \end{cases} \tag{B.35}$$

while

$$F(u) = \begin{cases} (u - \frac{1}{2}) & \text{i=1,} \\ (\phi'(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\phi)^{1/2}\frac{(u^2-\frac{1}{3})}{2} + (W_i(u) - \overline{W_i}) & \text{i=2,} \\ (W_i(u) - \overline{W_i}) & \text{i=3, \ldots, p-r.} \end{cases} \tag{B.36}$$

With $\Sigma = \alpha_\perp(\alpha_\perp'\Omega_{\alpha_\perp}\alpha_\perp)^{-1}\alpha_\perp'$ and using the definitions of τ, a and b,

$$\begin{aligned} c_1 &= (\tau'\Omega_{\alpha_\perp}^{-1}\tau)^{-1/2}\tau'\Omega_{\alpha_\perp}^{-1}ab'\tau \\ &= (\mu'\Sigma\mu)^{-1/2}(\mu'\Sigma\alpha_1)(\beta_1'C\mu), \end{aligned}$$

where the impact matrix C equals $\beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp'$. Furthermore using the definition of ϕ and the decomposition, $I_{p-r} = \Omega_{\alpha_\perp}\tau_\perp(\tau_\perp'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\tau_\perp' + \tau(\tau'\Omega_{\alpha_\perp}^{-1}\tau)^{-1}\tau'\Omega_{\alpha_\perp}$, it follows that,

$$\begin{aligned} c_2^2 &= \left(\phi'(\tau_\perp{}'\Omega_{\alpha_\perp}\tau_\perp)^{-1}\phi\right)^{1/2} \\ &= (\mu'C\beta_1)(\alpha_1'\Sigma\alpha_1)(\beta_1'C\mu) - c_1^2. \end{aligned}$$

Hence by $-T\sum_{i=r+1}^{p}\ln(1-\hat{\lambda}_i) = \sum_{i=r+1}^{p}T\hat{\lambda}_i + o_p(1)$, using $\rho = T\lambda$ the result (3.31) in Theorem 3.1 follows.

Proof of Theorem 3.1: (Part 3) Under H_T^{μ} it follows as in (B.27) that

$$wlim\left\{T^{1/2}\tilde{V}_T'\gamma_T' S_{10}\alpha_\perp\right\} = \int \mathcal{F}(\alpha_\perp' dB + a\mathcal{U}du)', \tag{B.37}$$

where the s-dimensional process $\mathcal{U}$ satisfies, (cf. (3.18))

$$d\mathcal{U} = \left(\mu^{(b)} + b'a\mathcal{U}\right) du + b'\alpha_\perp' dB, \tag{B.38}$$

and the $(p-r)$-dimensional process $\mathcal{F}$ equals (cf. (3.27)),

$$\mathcal{F}(u) = \begin{cases} \mathcal{U}(u) - \overline{\mathcal{U}} \\ \gamma'\gamma(u - \frac{1}{2}) \\ \gamma_\perp' a_\perp' \alpha_\perp'(B(u) - \overline{B}) \end{cases} \tag{B.39}$$

The p-dimensional brownian motion, B has covariance matrix Ω. Define the $(p-r)$-dimensional process, $\mathcal{Z}$, by $b'\mathcal{Z} = \mathcal{U}$. Then

$$d\mathcal{Z} = ab'\mathcal{Z}du + a_b\mu^{(b)}du + \alpha_\perp' dB, \tag{B.40}$$

and by $\int \mathcal{F}du = 0$, the limit in (B.37) equals $\int \mathcal{F}d\mathcal{Z}'$. And $\mathcal{F}$ equals,

$$\mathcal{F}(u) = \begin{cases} b'\mathcal{Z}(u) - \overline{\mathcal{Z}} \\ \gamma'\gamma(u - \frac{1}{2}) \\ \gamma_\perp' a_\perp' \alpha_\perp'(\mathcal{Z}(u) - \overline{\mathcal{Z}}) \end{cases} \tag{B.41}$$

The result in Theorem 3.1 follows by setting $Z = (\alpha_\perp'\Omega\alpha_\perp)^{-1/2}\mathcal{Z}$. □

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