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BARTLETT CORRECTIONS FOR UNIT ROOT TEST STATISTICS

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Abstract

Bartlett correction for the log likelihood ratio, testing for a unit root in an autoregressive process of order one or two, is studied. The correction is numerically calculated for order one, as well as for order two in the special case of a zero nuisance parameter.

1. Introduction

Consider the AR(2) model

(1.1) $$X_t = \rho_1 X_{t-1} + \rho_2 X_{t-2} + \varepsilon_t, \quad t = 1, \ldots, T,$$

where the ε_t:s are independent and normally distributed with mean zero and variance σ^2, and $X_0 = X_{-1} = 0$. Our object is to test the hypothesis $H_0 : \rho_1 + \rho_2 = 1$ against $\neg H_0$. We may also rewrite (1.1) in error correction form, i.e.

(1.2) $$\Delta X_t = \pi X_{t-1} + \gamma \Delta X_{t-1} + \varepsilon_t,$$

where $\pi = \rho_1 + \rho_2 - 1$ and $\gamma = -\rho_2$. Now, the null hypothesis is $H_0 : \pi = 0$, and γ is a nuisance parameter for this test. We say that we test for a unit root of the process.

Now, let us for a moment consider the multivariate version of (1.2), i.e. let X_t and ε_t be p-dimensional vectors and let π and γ be $p \times p$ matrices. In this situation, an important issue is to test $H(r) : \text{rank}(\pi) = r < p$ against e.g. $H(p) : \text{rank}(\pi) = p$. This is a multivariate version of the unit root test.

Performing this test in practice, the common thing to do is to use a table of the asymptotic distribution of the likelihood ratio test statistic (the Dickey-Fuller distribution). This is a well-known functional of the vector-valued Brownian motion, which has been simulated by several authors (see e.g. Johansen (1988)). However, if a very large amount of data is not at hand, it has recently been found that (see e.g. Jacobsson (1992)) straightforward use of these tables could be very misleading. Thus, there seems to be a need of small sample correction for the asymptotic test, and it is the purpose of our work to find such corrections. We start by studying the relatively simple scalar model (1.2), but in the future, our aim is to generalize our results to the multivariate case.

2. Bartlett correction

In a pioneering paper (Bartlett (1937)), Bartlett introduced a small sample correction technique, later known as Bartlett correction. The idea is that, instead of looking directly at the test statistic, say S_T (with an unknown distribution), which tends to S_∞ (with known distribution) as $T \to \infty$, we look at the distribution of $\frac{S_T}{ES_T}$, which of course tends to the distribution of $\frac{S_\infty}{ES_\infty}$ as $T \to \infty$. Thus,

$$S_T \approx ES_T \frac{S_\infty}{ES_\infty},$$

an approximation which (at least in "standard" cases) turns out to be useful also for moderately large T values. However, a problem is that we might not know ES_T, but if we can find a series expansion like

$$ES_T = ES_\infty + \frac{R}{T} + O\left(\frac{1}{T^2}\right),$$

we get

$$S_T \approx \left(ES_\infty + \frac{R}{T}\right)\frac{S_\infty}{ES_\infty}.$$

This is called the Bartlett correction. In "standard" cases, this correction has been shown to correct also higher moments and fractiles (cf Jensen (1993)) for an overview of the subject).

Testing H_0 in (1.2), the log likelihood ratio test statistic is

$$-2\log Q_T = -T\log(1-M_T) = TM_T + O\left(\frac{1}{T}\right) \qquad \text{as} \quad T\to\infty,$$

where

$$M_T = \frac{\left(\sum X_{t-1}\Delta X_t - \frac{\sum \Delta X_t \Delta X_{t-1}}{\sum(\Delta X_{t-1})^2}\sum X_{t-1}\Delta X_{t-1}\right)^2}{\left(\sum(\Delta X_t)^2 - \frac{\left(\sum \Delta X_t\Delta X_{t-1}\right)^2}{\sum(\Delta X_{t-1})^2}\right)\left(\sum X_{t-1}^2 - \frac{\left(\sum X_{t-1}\Delta X_{t-1}\right)^2}{\sum(\Delta X_{t-1})^2}\right)}. \tag{2.1}$$

(If nothing else is said, the summation goes from $t=1$ to $t=T$.) It follows that

$$TM_T \xrightarrow{d} \frac{\left(\int_0^1 W_t dW_t\right)^2}{\int_0^1 W_t^2 dt} \stackrel{\text{def}}{=} Z \qquad \text{as} \quad T\to\infty,$$

where W_t is a standard Wiener process (Brownian motion). In the following, we will derive the expansion

$$ETM_T = EZ + \frac{R(\gamma)}{T} + O\left(\frac{1}{T^2}\right), \quad R(\gamma) = R_1 + R_2(\gamma). \tag{2.2}$$

Indeed, looking at the corresponding AR(1) test statistic

$$Z_T \stackrel{\text{def}}{=} T\frac{\left(\sum S_{t-1}\varepsilon_t\right)^2}{\sum \varepsilon_t^2 \sum S_{t-1}^2}, \quad S_t \stackrel{\text{def}}{=} \sum_{i=1}^{t}\varepsilon_i, \tag{2.3}$$

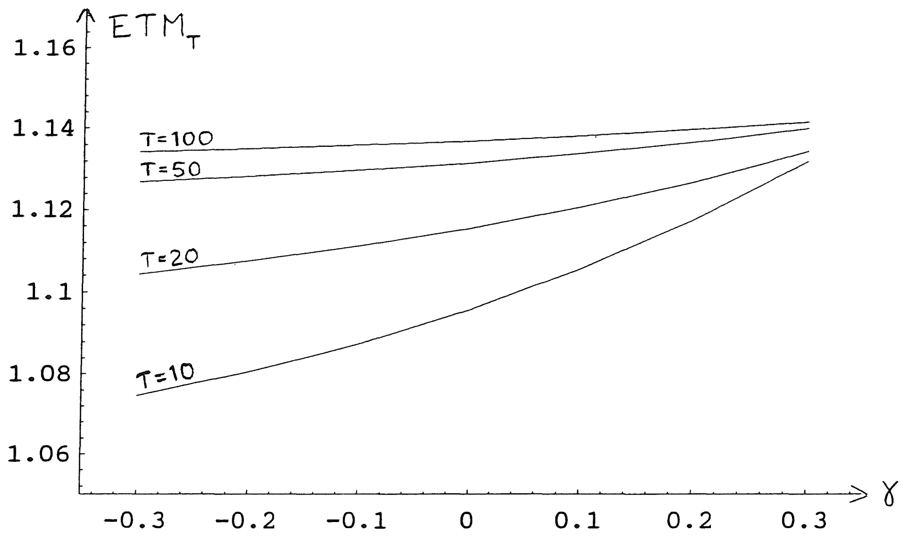
we have

$$EZ_T = EZ + \frac{R_1}{T} + O\left(\frac{1}{T^2}\right), \tag{2.4}$$

the Bartlett correction for the AR(1) test. Accordingly, we may view the term $\frac{R_2(\gamma)}{T}$ as a correction from the AR(1) to the AR(2) test. (Naturally, this is where the nuisance parameter γ enters.) We will be able to calculate R_1 and $R_2(0)$ numerically.

To get a feeling for the shape of $R_2(\gamma)$, we have performed some simulations of ETM_T for $T=10$, 20, 50 and 100 with 1,000,000 replications, which are displayed in figure 1. (The upper curve corresponds to $T=100$, the next to upper to $T=50$, and so on.) From this figure, we see that, for $|\gamma|\le 0.3$, the approximation $R_2(\gamma) \approx R_2(0)$ is fairly accurate for $T \geq 20$, whereas for lower T values we might have to consider the linear approximation $R_2(\gamma) \approx R_2(0) + \gamma R_2'(0)$.

Figure 1:



3. A representation of $R_2(0)$

If $\gamma = 0$, it follows from (1.2) that under $H_0 : \pi = 0$, $X_t = \sum_{i=1}^{t} \varepsilon_i = S_t$, i.e. $\Delta X_t = \varepsilon_t$, implying $\sum \Delta X_t \Delta X_{t-1} = \sum \varepsilon_t \varepsilon_{t-1}$ and $\sum X_{t-1} \Delta X_{t-1} = \sum S_{t-1} \varepsilon_{t-1}$. Multiplying out the main term in (2.1), we have

$$TM_T = T \frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum \varepsilon_t{}^2 \sum S_{t-1}^2} \left(1 - \frac{\sum \varepsilon_t \varepsilon_{t-1} \sum S_{t-1}\varepsilon_{t-1}}{\sum S_{t-1}\varepsilon_t \sum \varepsilon_{t-1}{}^2}\right)^2 \left(1 - \frac{(\sum \varepsilon_t \varepsilon_{t-1})^2}{\sum \varepsilon_t{}^2 \sum \varepsilon_{t-1}{}^2}\right)^{-1} \left(1 - \frac{(\sum S_{t-1}\varepsilon_{t-1})^2}{\sum S_{t-1}^2 \sum \varepsilon_{t-1}{}^2}\right)^{-1}.$$

Now, since (for convenience, we assume $\sigma^2 = 1$ in the following) $\sum \varepsilon_t{}^2 = T + O_p(1)$, $\sum \varepsilon_{t-1}{}^2 = T + O_p(1)$, $\sum \varepsilon_t \varepsilon_{t-1} = O_p(\sqrt{T})$, $\sum S_{t-1}\varepsilon_t = O_p(T)$, $\sum S_{t-1}\varepsilon_{t-1} = O_p(T)$ and $\sum S_{t-1}^2 = O_p(T^2)$ (the notation $X_T = O_p(T^\alpha)$ means that $\frac{X_T}{T^\alpha}$ converges in distribution to a "non-degenerate" random variable as $T \to \infty$), Taylor expansion yields

$$TM_T = T \frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum \varepsilon_t{}^2 \sum S_{t-1}^2} \left(1 - \frac{2}{T} \frac{\sum \varepsilon_t \varepsilon_{t-1} \sum S_{t-1}\varepsilon_{t-1}}{\sum S_{t-1}\varepsilon_t} + \frac{1}{T^2} \left(\frac{\sum \varepsilon_t \varepsilon_{t-1} \sum S_{t-1}\varepsilon_{t-1}}{\sum S_{t-1}\varepsilon_t}\right)^2 + \right.$$

$$\left. + \frac{1}{T^2} \left(\sum \varepsilon_t \varepsilon_{t-1}\right)^2 + \frac{1}{T} \frac{(\sum S_{t-1}\varepsilon_{t-1})^2}{\sum S_{t-1}^2} + O_p(T^{-\frac{3}{2}})\right)$$

and so, since $\sum S_{t-1}\varepsilon_{t-1} = \sum \varepsilon_{t-1}{}^2 + \sum S_{t-2}\varepsilon_{t-1} = T + \sum S_{t-1}\varepsilon_t + o_p(T)$, we have in view of (2.2) and (2.4)

$$(3.1)\qquad \frac{R_2(0)}{T} = -2E\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t\left(1+\frac{1}{T}\sum S_{t-1}\varepsilon_t\right)}{\sum S_{t-1}^2}\right)+$$

$$+\frac{\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2\left(1+\frac{2}{T}\sum S_{t-1}\varepsilon_t+\frac{2}{T^2}\left(\sum S_{t-1}\varepsilon_t\right)^2\right)}{\sum S_{t-1}^2}+TE\left(\frac{\left(\sum S_{t-1}\varepsilon_t\right)^2\left(1+\frac{1}{T}\sum S_{t-1}\varepsilon_t\right)^2}{\left(\sum S_{t-1}^2\right)^2}\right).$$

(We will come back to the calculation of R_1 in chapter 6.) Now, we claim that the three terms in the r.h.s. of (3.1) are $O(T^{-1})$, i.e. that $R_2(0) = O(1)$. In view of the orders of magnitude of the sums, this is evidently true for the second and third terms. However, by the same reasons the first term appears to be of order $T^{-\frac{1}{2}}$, but this is a false statement. This is so, since as is shown in Lemma 4.2 below, $\sum \varepsilon_t\varepsilon_{t-1}$ is asymptotically uncorrelated with $\sum S_{t-1}\varepsilon_t$ and $\sum S_{t-1}^2$. Indeed, as will be shown in theorem 5.2, this term is also $O(T^{-1})$.

Hence, we should have

$$TE\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right)\to A,\quad E\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}\left(\sum S_{t-1}\varepsilon_t\right)^2}{\sum S_{t-1}^2}\right)\to B,\quad TE\left(\frac{\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2}{\sum S_{t-1}^2}\right)\to C,$$

$$E\left(\frac{\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right)\to D,\quad \frac{1}{T}E\left(\frac{\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2\left(\sum S_{t-1}\varepsilon_t\right)^2}{\sum S_{t-1}^2}\right)\to E,\quad T^2E\left(\frac{\left(\sum S_{t-1}\varepsilon_t\right)^2}{\left(\sum S_{t-1}^2\right)^2}\right)\to F,$$

$$TE\left(\frac{\left(\sum S_{t-1}\varepsilon_t\right)^3}{\left(\sum S_{t-1}^2\right)^2}\right)\to G\quad\text{and}\quad E\left(\frac{\left(\sum S_{t-1}\varepsilon_t\right)^4}{\left(\sum S_{t-1}^2\right)^2}\right)\to H,$$

for some constants A-H, and so (3.1) yields

$$(3.2)\qquad R_2(0) = -2(A+B)+C+2(D+E)+F+2G+H+O(T^{-1}).$$

In the following, numerical values of these constants will be calculated. Our technique is based on the ideas outlined in Mikulski & Monsour, who calculate moments of the univariate Dickey-Fuller distribution.

4. The Mikulski & Monsour idea

To start with, consider the trivial equality

$$\frac{1}{x} = \int_0^\infty e^{-sx} ds.$$

Replacing x by $\sum X_{t-1}^2$, where X_t is defined by (1.1) with $\rho_2 = 0$, i.e. as an AR(1) process (for convenience, let $\sigma^2 = 1$). Taking expectation and using Fubini's theorem gives us

$$E\left(\frac{1}{\sum X_{t-1}^2}\right) = \int_0^\infty E\left(e^{-s\sum X_{t-1}^2}\right) ds = \int_0^\infty \varphi(\rho_1; s) ds, \tag{4.1}$$

where $\varphi(\rho_1; s) \stackrel{\text{def}}{=} E\left(e^{-s\sum X_{t-1}^2}\right)$ is the moment generating function (Laplace transform) of $\sum X_{t-1}^2$. On the other hand,

$$E\left(\frac{1}{\sum X_{t-1}^2}\right) = \int \ldots \int \frac{1}{\sum x_{t-1}^2} (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\sum (x_t - \rho_1 x_{t-1})^2} dx_1 \ldots dx_T. \tag{4.2}$$

Putting (4.1) equal to (4.2) and differentiating w.r.t. ρ_1, we have

$$\int_0^\infty \frac{\partial}{\partial \rho_1} \varphi(\rho_1; s) ds = \int \ldots \int \frac{\sum x_{t-1}(x_t - \rho_1 x_{t-1})}{\sum x_{t-1}^2} (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\sum (x_t - \rho_1 x_{t-1})^2} dx_1 \ldots dx_T = E\left(\frac{\sum X_{t-1}\varepsilon_t}{\sum X_{t-1}^2}\right),$$

and so, letting $\rho_1 \to 1$,

$$\int_0^\infty \frac{\partial}{\partial \rho_1} \varphi(1; s) ds = E\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right).$$

Finishing off by calculating $\frac{\partial}{\partial \rho_1}\varphi(1; s)$, this and similar arguments help Mikulski & Monsour to derive, among others, the results listed in the following theorem (the figures are obtained by employing numerical integration):

Theorem 4.1.

$$\lim_{T\to\infty} TE\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) = -\frac{1}{2}\int_0^\infty \frac{x}{\sqrt{\cosh x}} dx + 1 \approx -1.781, \tag{4.3}$$

$$\lim_{T\to\infty} E\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2}\right) = \int_0^\infty \left(\frac{x}{4\sqrt{\cosh x}} + \frac{3\sinh^2 x}{4x\cosh^{\frac{5}{2}} x}\right) dx - 1 \approx 1.142, \tag{4.4}$$

$$\lim_{T\to\infty} T^2 E\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{(\sum S_{t-1}^2)^2}\right) = \frac{1}{4}\int_0^\infty \left(\frac{x^3}{2\sqrt{\cosh x}} - \frac{3x}{\sqrt{\cosh x}}\right) dx + \frac{1}{2} \approx 13.286, \tag{4.5}$$

and

$$\text{(4.6)} \qquad \lim_{T\to\infty} E\left(\frac{(\sum S_{t-1}\varepsilon_t)^4}{(\sum S_{t-1}^2)^2}\right) = \int_0^\infty \left(\frac{x^3}{32\sqrt{\cosh x}} - \frac{x}{8\sqrt{\cosh x}} + \frac{105\sinh^4 x}{32x\cosh^{\frac{9}{2}} x}\right) dx - \frac{9}{4} \approx 3.522.$$

∎

Thanks to these results, we are spared from calculating F and G, the values of which are given by (4.5) and (4.6), respectively. Moreover, as a consequence of the following lemma, the theorem in effect also provides us with D and E.

Lemma 4.2. *$\sum \varepsilon_t\varepsilon_{t-1}$ is asymptotically uncorrelated with $\sum S_{t-1}\varepsilon_t$ and $\sum S_{t-1}^2$.*

∎

The lemma implies that $\sum \varepsilon_t\varepsilon_{t-1}$ is asymptotically uncorrelated with any smooth function of $\sum S_{t-1}\varepsilon_t$ and $\sum S_{t-1}^2$, and so

$$E\left(\frac{(\sum \varepsilon_t\varepsilon_{t-1})^2 \sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) \approx E\left(\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2\right) E\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right),$$

where $\approx$ means equality to the first order. But since $E\left((\sum \varepsilon_t\varepsilon_{t-1})^2\right) = T$, this means that the value of D is given by (4.3), and similarly we conclude that E is given by (4.4), leaving only A, B, C and G to be calculated. Furthermore, the calculation of C is simplified, since as above,

$$\text{(4.7)} \qquad TE\left(\frac{(\sum \varepsilon_t\varepsilon_{t-1})^2}{\sum S_{t-1}^2}\right) \approx T^2 E\left(\frac{1}{\sum S_{t-1}^2}\right).$$

Proof of Lemma 4.2: As is easily verified,

$$E\left(\sum \varepsilon_t\varepsilon_{t-1}\right) = 0, \quad E\left(\sum S_{t-1}\varepsilon_t\right) = 0, \quad E\left(\sum \varepsilon_t\varepsilon_{t-1} \sum S_{t-1}\varepsilon_t\right) = T, \quad E\left(\left(\sum \varepsilon_t\varepsilon_{t-1}\right)^2\right) = T$$

and

$$E\left(\left(\sum S_{t-1}\varepsilon_t\right)^2\right) = \frac{1}{2}T(T-1),$$

and so

$$Corr\left(\sum \varepsilon_t\varepsilon_{t-1}, \sum S_{t-1}\varepsilon_t\right) = O\left(T^{-\frac{1}{2}}\right),$$

which proves that $\sum \varepsilon_t\varepsilon_{t-1}$ and $\sum S_{t-1}\varepsilon_t$ are asymptotically uncorrelated. The fact that $\sum \varepsilon_t\varepsilon_{t-1}$ is also asymptotically uncorrelated with $\sum S_{t-1}^2$ is proved similarly.

∎

5. The calculation of A, B, C and G

Calculating the remaining terms A, B, C and G by generalising the Mikulski & Monsour procedure, we at first obtain the following lemma:

Lemma 5.1. *Let*

$$\varphi(\rho_1,\rho_2;s) \stackrel{\text{def}}{=} E\left(e^{-s\sum X_{t-1}^2}\right),$$

where X_t is the AR(2) process defined by (1.1). Then

$$TE\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) = T\int_0^\infty \left(T\varphi + \frac{\partial\varphi}{\partial\rho_1} + \frac{\partial^2\varphi}{\partial\rho_1^2} - \frac{\partial^2\varphi}{\partial\rho_1\partial\rho_2}\right) ds + o(1), \tag{5.1}$$

$$E\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2}\right) = \int_0^\infty \left(2T\frac{\partial\varphi}{\partial\rho_1} + 2\frac{\partial^2\varphi}{\partial\rho_1^2} + \frac{\partial^3\varphi}{\partial\rho_1^3} - \frac{\partial^3\varphi}{\partial\rho_1^2\partial\rho_2}\right) ds + 2 + o(1), \tag{5.2}$$

$$E\left(\frac{1}{\sum S_{t-1}^2}\right) = \int_0^\infty \varphi ds \quad \text{and} \tag{5.3}$$

$$E\left(\frac{(\sum S_{t-1}\varepsilon_t)^3}{(\sum S_{t-1}^2)^2}\right) = \int_0^\infty \left(s\frac{\partial^3\varphi}{\partial\rho_1^3} + 3\frac{\partial\varphi}{\partial\rho_1}\right) ds, \tag{5.4}$$

where φ and all its derivatives are calculated at $(\rho_1,\rho_2) = (1,0)$.

∎

Proof: With X_t defined by (1.1), we of course still get the equalities (4.1) and (4.2), and so

$$\int_0^\infty \varphi(\rho_1,\rho_2;s)ds = E\left(\frac{1}{\sum X_{t-1}^2}\right) = \int\cdots\int \frac{1}{\sum x_{t-1}^2} L(\rho_1,\rho_2)dx_1\ldots dx_T, \tag{5.5}$$

where

$$L(\rho_1,\rho_2) \stackrel{\text{def}}{=} (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\sum \varepsilon_t{}^2}, \quad \varepsilon_t \stackrel{\text{def}}{=} x_t - \rho_1 x_{t-1} - \rho_2 x_{t-2}.$$

Letting $\rho_1 \to 1$ and $\rho_2 \to 0$ in this equation gives us (5.3). Furthermore, succesive differentiation of $L(\rho_1,\rho_2)$ yields

$$\frac{\partial L}{\partial\rho_1} = \sum x_{t-1}\varepsilon_t L, \tag{5.6}$$

$$\frac{\partial^2 L}{\partial\rho_1^2} = \left(-\sum x_{t-1}^2 + \left(\sum x_{t-1}\varepsilon_t\right)^2\right) L, \tag{5.7}$$

$$\frac{\partial^2 L}{\partial\rho_1\partial\rho_2} = \left(-\sum x_{t-1}x_{t-2} + \sum x_{t-1}\varepsilon_t \sum x_{t-2}\varepsilon_t\right) L, \tag{5.8}$$

(5.9) $$\frac{\partial^3 L}{\partial \rho_1^3} = \left(-3\sum x_{t-1}^2 \sum x_{t-1}\varepsilon_t + \left(\sum x_{t-1}\varepsilon_t\right)^3\right) L \quad \text{and}$$

(5.10) $$\frac{\partial^3 L}{\partial \rho_1^2 \partial \rho_2} = \left(-\sum x_{t-1}^2 \sum x_{t-2}\varepsilon_t - 2\sum x_{t-1}x_{t-2}\sum x_{t-1}\varepsilon_t + \left(\sum x_{t-1}\varepsilon_t\right)^2 \sum x_{t-2}\varepsilon_t\right) L.$$

Now, combining (5.6)-(5.8) with (5.5) and letting $\rho_1 \to 1$ and $\rho_2 \to 0$ (throughout, the argument of φ and its derivatives is $(\rho_1, \rho_2) = (1,0)$),

$$\int_0^\infty \left(T\varphi + \frac{\partial \varphi}{\partial \rho_1} + \frac{\partial^2 \varphi}{\partial \rho_1^2} - \frac{\partial^2 \varphi}{\partial \rho_1 \partial \rho_2}\right) ds =$$

$$= E\left(\frac{1}{S_{t-1}^2}\left(T + \sum S_{t-1}\varepsilon_t - \sum S_{t-1}^2 + \left(\sum S_{t-1}\varepsilon_t\right)^2 + \sum S_{t-1}S_{t-2} - \sum S_{t-1}\varepsilon_t \sum S_{t-2}\varepsilon_t\right)\right).$$

But since

$$\sum S_{t-1}^2 - \sum S_{t-1}S_{t-2} = \sum S_{t-1}\varepsilon_{t-1} = \sum S_{t-2}\varepsilon_{t-1} + \sum {\varepsilon_{t-1}}^2,$$

implying

(5.11) $$T + \sum S_{t-1}\varepsilon_t - \sum S_{t-1}^2 + \sum S_{t-1}S_{t-2} = T - \sum {\varepsilon_{t-1}}^2 + \sum S_{t-1}\varepsilon_t - \sum S_{t-2}\varepsilon_{t-1} =$$

$$= O_p(1) + S_{T-1}\varepsilon_T = o_p(T),$$

(the notation $o_p(\cdot)$ has the obvious meaning), and since

$$\left(\sum S_{t-1}\varepsilon_t\right)^2 - \sum S_{t-1}\varepsilon_t \sum S_{t-2}\varepsilon_t = \sum S_{t-1}\varepsilon_t \sum \varepsilon_{t-1}\varepsilon_t,$$

(5.1) follows.

Likewise, (5.6), (5.7), (5.9) and (5.10) together with (5.5) imply

$$\int_0^\infty \left(2T\frac{\partial \varphi}{\partial \rho_1} + 2\frac{\partial^2 \varphi}{\partial \rho_1^2} + \frac{\partial \varphi^3}{\partial \rho_1^3} - \frac{\partial^3 \varphi}{\partial \rho_1^2 \partial \rho_2}\right) ds =$$

$$= E\left(\frac{1}{\sum S_{t-1}^2}\left(2T\sum S_{t-1}\varepsilon_t - 2\sum S_{t-1}^2 + 2\left(\sum S_{t-1}\varepsilon_t\right)^2 - 3\sum S_{t-1}^2 \sum S_{t-1}\varepsilon_t + \left(\sum S_{t-1}\varepsilon_t\right)^3 + \right.\right.$$

$$\left.\left. + \sum S_{t-1}^2 \sum S_{t-2}\varepsilon_t + 2\sum S_{t-1}S_{t-2}\sum S_{t-1}\varepsilon_t - \left(\sum S_{t-1}\varepsilon_t\right)^2 \sum S_{t-2}\varepsilon_t\right)\right) =$$

$$= -2 + E\left(\frac{1}{\sum S_{t-1}^2}\left(2T\sum S_{t-1}\varepsilon_t + 2\left(\sum S_{t-1}\varepsilon_t\right)^2 + \left(\sum S_{t-1}\varepsilon_t\right)^3 + 2\sum S_{t-1}S_{t-2}\sum S_{t-1}\varepsilon_t - \right.\right.$$

$$\left.\left. - \left(\sum S_{t-1}\varepsilon_t\right)^2 \sum S_{t-2}\varepsilon_t\right)\right),$$

cancelling terms of expectation zero. But

$$2T\sum S_{t-1}\varepsilon_t + 2\left(\sum S_{t-1}\varepsilon_t\right)^2 + 2\sum S_{t-1}S_{t-2}\sum S_{t-1}\varepsilon_t =$$

$$= 2\sum S_{t-1}\varepsilon_t\left(T + \sum S_{t-1}\varepsilon_t + \sum S_{t-1}S_{t-2}\right) = 2\sum S_{t-1}\varepsilon_t\left(\sum S_{t-1}^2 + o_p(T)\right),$$

where the last equality follows from (5.11), and so, dividing by $\sum S_{t-1}^2$ and taking expectation, we get an $o_p(1)$ term, since $E\sum S_{t-1}\varepsilon_t = 0$. Thus, the fact that

$$\left(\sum S_{t-1}\varepsilon_t\right)^3 - \left(\sum S_{t-1}\varepsilon_t\right)^2\sum S_{t-1}\varepsilon_t = \left(\sum S_{t-1}\varepsilon_t\right)^2\sum \varepsilon_{t-1}\varepsilon_t,$$

leads us to conclude (5.2).

It remains to verify (5.4). To this end, the equality

$$\frac{1}{x^2} = \int_0^\infty se^{-sx}ds$$

with $x = \sum X_{t-1}^2$ and X_t as before yields

$$\int_0^\infty s\varphi(\rho_1,\rho_2;s)ds = E\left(\frac{1}{\left(\sum X_{t-1}^2\right)^2}\right) = \int\ldots\int\frac{1}{\left(\sum x_{t-1}^2\right)^2}L(\rho_1,\rho_2)dx_1\ldots dx_T.$$

Hence, in the usual manner, (5.9) implies

$$\int_0^\infty s\frac{\partial^3\varphi}{\partial\rho_1^3}ds = -3E\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) + E\left(\frac{\left(\sum S_{t-1}\varepsilon_t\right)^3}{\left(\sum S_{t-1}^2\right)^2}\right).$$

But, because of (5.5) and (5.6),

$$\int_0^\infty \frac{\partial\varphi}{\partial\rho_1}ds = E\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right),$$

which gives (5.4), and we are done.

∎

The final step is to calculate first order approximations of φ and its derivatives at $(\rho_1,\rho_2) = (1,0)$, which we do by Taylor expansion around that point. Since this is a highly computationally involved task, we postpone the calculations to the appendix, and confine ourselves to giving the final results here. (Again, numerical integration is used to obtain the figures.)

Theorem 5.2

(5.12) $$\lim_{T\to\infty} TE\left(\frac{\sum\varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) =$$

$$= \frac{1}{2}\int_0^\infty x(\cosh x)^{-\frac{5}{2}}\left(\frac{1}{2}x\cosh x\sinh x - \frac{1}{2}\sinh^2 x + 1\right)dx + 1 \approx 5.563,$$

$$(5.13)\quad \lim_{T\to\infty} E\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2}\right) = \int_0^\infty x(\cosh x)^{-\frac{7}{2}}\Bigg(-\cosh x -$$

$$-\frac{19}{8}\frac{\sinh^3 x}{x} + \frac{3}{2}\cosh x\left(\frac{\sinh x}{x}\right)^2 - \frac{1}{8}x\cosh^2 x\sinh x + \frac{1}{2}\cosh x\sinh^2 x\Bigg)dx + \frac{9}{5} \approx -1.280,$$

$$(5.14)\quad \lim_{T\to\infty} T^2 E\left(\frac{1}{\sum S_{t-1}^2}\right) = \int_0^\infty x(\cosh x)^{-\frac{1}{2}}dx \approx 5.563 \qquad \textit{and}$$

$$(5.15)\quad \lim_{T\to\infty}\frac{1}{T}E\left(\frac{(\sum S_{t-1}\varepsilon_t)^3}{(\sum S_{t-1}^2)^2}\right) = -\frac{1}{2}\int_0^\infty x^3(\cosh x)^{-\frac{7}{2}}\Bigg(\frac{1}{8}\cosh^3 x - \frac{9}{8}\cosh^2 x\frac{\sinh x}{x} +$$

$$+\frac{39}{8}\cosh x\left(\frac{\sinh x}{x}\right)^2 - \frac{15}{8}\left(\frac{\sinh x}{x}\right)^3 + \frac{3}{2}\frac{\cosh x}{x^2}\Bigg)dx + \frac{3}{2} \approx -5.643.$$

∎

Proof: See the appendix.

∎

We now have access to approximate values of all the constants *A*-*G*, which are $A \approx 5.563$ (from (5.12)), $B \approx -1.280$ ((5.13)), $C \approx 5.563$ ((5.14) and (4.7)), $D \approx -1.781$ ((4.3)), $E \approx 1.142$ ((4.4)), $F \approx 13.286$ ((4.5)), $G \approx -5.643$ ((5.15)) and $H \approx 3.522$ ((4.6)), and so (3.2) yields

$$(5.16)\qquad R_2(0) \approx 1.241.$$

6. The AR(1) correction

Our final task will be to calculate the AR(1) Bartlett correction (cf (2.3) and (2.4)). To this end, since $T(\sum \varepsilon_t{}^2)^{-1} = 1 + o_p(1)$, the "main term" EZ is already given by (4.4), but to find the rest term $\frac{R_1}{T}$ we need to be a little more careful. Generalizing the Mikulski & Monsour idea (cf chapter 4), we have

$$\frac{1}{xy} = \int_0^\infty \int_0^\infty e^{-sx-uy} ds du,$$

and so, replacing x and y by $\sum X_{t-1}^2$ and $\sum(\Delta X_t)^2$ respectively, where X_t is defined by (1.1) with $\rho_2 = 0$ (becoming AR(1)) and $\Delta X_t = X_t - X_{t-1}$, and taking expectations, we get

$$E\left(\frac{1}{\sum X_{t-1}^2 \sum(\Delta X_t)^2}\right) = \int_0^\infty \int_0^\infty E\left(e^{-s\sum X_{t-1}^2 - u\sum(\Delta X_t)^2}\right) ds du = \int_0^\infty \int_0^\infty \varphi(\rho_1; s, u) ds du, \tag{6.1}$$

where $\varphi(\rho_1; s, u) \stackrel{\text{def}}{=} E\left(e^{-s\sum X_{t-1}^2 - u\sum(\Delta X_t)^2}\right)$ is the m.g.f. of the pair $(\sum X_{t-1}^2, \sum(\Delta X_t)^2)$. On the other hand,

$$E\left(\frac{1}{\sum X_{t-1}^2 \sum(\Delta X_t)^2}\right) = \int \cdots \int \frac{1}{\sum x_{t-1}^2 \sum(\Delta x_t)^2} (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\sum(x_t - \rho_1 x_{t-1})^2} dx_1 \ldots dx_T,$$

and so, differentiating two times w.r.t. ρ_1, we have in view of (6.1)

$$\int_0^\infty \int_0^\infty \frac{\partial^2}{\partial \rho_1^2} \varphi(1; s, u) ds du = E\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2 \sum \varepsilon_t{}^2}\right) - E\left(\frac{1}{\sum \varepsilon_t{}^2}\right).$$

However, since $\sum \varepsilon_t{}^2$ is χ^2-distributed with T degrees of freedom, it follows that $E\left((\sum \varepsilon_t{}^2)^{-1}\right) = \frac{1}{T-2} = \frac{1}{T} + \frac{2}{T^2} + O\left(\frac{1}{T^3}\right)$, and so

$$TE\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2 \sum \varepsilon_t{}^2}\right) = T\int_0^\infty \int_0^\infty \frac{\partial^2}{\partial \rho_1^2} \varphi(1; s, u) ds du + 1 + \frac{2}{T} + O\left(\frac{1}{T^2}\right). \tag{6.2}$$

In the appendix we show (cf (4.4))

Theorem 6.1.

$$\lim_{T\to\infty} TE\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2 \sum \varepsilon_t{}^2}\right) = \int_0^\infty x(\cosh x)^{-\frac{5}{2}} \left(\frac{1}{4}\cosh^2 x + \frac{3}{4}\left(\frac{\sinh x}{x}\right)^2\right) dx - 1+ \tag{6.3}$$

$$+\frac{1}{T}\left(\frac{1}{4}\int_0^\infty x(\cosh x)^{-\frac{5}{2}} \left(\cosh^2 x + 1 - \frac{3}{4} x \sinh x \cosh x + \frac{15}{4}\frac{\sinh^3 x}{x \cosh x}\right) dx - 1\right) + O\left(\frac{1}{T^2}\right) \approx$$

$$\approx 1.142 - \frac{2.151}{T}.$$

∎

As before, the figures are obtained by numerical integration.

7. Comparison with simulations

In table 1 below, the corrections

$$EZ_T \approx EZ - \frac{R_1}{T} \approx 1.142 - \frac{2.151}{T}, \quad ETM_T \approx EZ_T - \frac{R_2(0)}{T} \approx EZ_T + \frac{1.241}{T}$$

and

$$ETM_T \approx EZ - \frac{R_1 + R_2(0)}{T} \approx 1.142 + \frac{-2.151 + 1.241}{T} = 1.142 - \frac{0.910}{T}$$

are compared with simulated values of EZ_T and ETM_T for $\gamma = 0$, respectively. The first two of these corrections are seen to be fairly accurate, whereas the third one performs less satisfactory, probably due to simulation errors and/or an unfortunate adding of higher order error terms. In the simulations, we used 1,000,000 replications, which gave us a standard error of about $1 \cdot 10^{-3}$.

Table 1: *Corrected and simulated expectations compared.*

Columns:

1. *Simulated values of EZ_T (the AR(1) statistic).*
2. *Corrected values of EZ_T through $EZ_T \approx 1.142 - \frac{2.151}{T}$.*
3. *Simulated values of ETM_T (the AR(2) statistic).*
4. *Corrected values of ETM_T through $ETM_T \approx EZ_T + \frac{1.241}{T}$ (EZ_T:s from column 1).*
5. *Corrected values of ETM_T through $ETM_T \approx 1.142 - \frac{0.910}{T}$.*

T	1.	2.	3.	4.	5.
10	0.999	0.927	1.096	1.123	1.051
20	1.063	1.034	1.116	1.125	1.097
30	1.088	1.070	1.124	1.129	1.112
40	1.098	1.088	1.126	1.129	1.119
50	1.109	1.099	1.132	1.134	1.124
60	1.114	1.106	1.133	1.135	1.127
80	1.119	1.115	1.134	1.135	1.131
100	1.125	1.120	1.137	1.137	1.133
200	1.133	1.131	1.138	1.139	1.138

8. Concluding remarks

The practical use of the results in this paper is the following: Suppose you want to test for a unit root of an AR(1) or AR(2) process, but that you only have access to a table of the asymptotic distribution of the test statistic. Then, it is clearly improper to use this table directly. However, with the aid of the corrected expectations derived in this paper, the asymptotic table is easily modified to a table which gives a good approximation to the distribution of the AR(1) or AR(2) test statistic, in the manner described in section 2. In the AR(2) case, we noted studying figure 1 that this would be a fairly accurate approximation as long as the parameter γ is sufficiently small and/or T is not too small. In other cases, we would need the improved approximation $R(\gamma) \approx R(0) + \gamma R'(0)$ instead of $R(\gamma) \approx R(0)$, i.e. we need to calculate $R'(0)$. However, we belive that this calculation is rather similar to the calculation of $R(0)$, and so this is an issue that we hope to investigate further.

Another interesting question to ask is whether our analytic method to find the corrections could be applicable to the perhaps more interesting multivariate case, where the unit root test carries over to a test of cointegration. Hopefully, we will get back to this problem in forthcoming papers.

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Appendix: Omitted proofs

Proof of theorem 5.2: With lemma 5.1 as a starting point, we are going to prove theorem 5.2 by at first calculating a Taylor expansion of $\varphi(\rho_1, \rho_2; s)$, the moment generating function of $\sum X_{t-1}^2$, where X_t is defined by (1.1) with $\sigma^2 = 1$, around $(\rho_1, \rho_2) = (1, 0)$. To this end, we will need the representation of φ given in the following lemma:

Lemma A.1.

$$\tilde{\varphi}(\theta, \rho; s) \overset{\text{def}}{=} \varphi(1-\theta, \rho; s) = \frac{1}{\sqrt{\det P}}, \quad P = P_0 + h. \tag{A.1}$$

Here, P_0 is the $T \times T$ matrix

$$P_0 = \begin{pmatrix} \alpha & -1 & 0 & 0 & & & \\ -1 & \alpha & -1 & 0 & & & \\ 0 & -1 & \alpha & -1 & & & \\ & & & & \cdots & & \\ & & & & -1 & \alpha & -1 \\ & & & & 0 & -1 & 1 \end{pmatrix}, \quad \alpha = 2(1+s) \tag{A.2}$$

and

$$h = \theta h_{1,0} + \rho h_{0,1} + \theta^2 h_{2,0} + \theta\rho h_{1,1} + \rho^2 h_{0,2}, \tag{A.3}$$

where the $h_{i,j}$:s, $i, j = 1, 2$, are $T \times T$ matrices given by

$$h_{1,0} \overset{\text{def}}{=} \begin{pmatrix} -2 & 1 & 0 & 0 & & & \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & & & \\ & & & & \cdots & & \\ & & & & 1 & -2 & 1 \\ & & & & 0 & 1 & 0 \end{pmatrix}, \tag{A.4}$$

$$h_{0,1} \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & & & & & \\ 1 & 0 & 1 & -1 & 0 & 0 & & & & & \\ -1 & 1 & 0 & 1 & -1 & 0 & & & & & \\ 0 & -1 & 1 & 0 & 1 & -1 & & & & & \\ & & & & & & \cdots & & & & \\ & & & & & & -1 & 1 & 0 & 1 & -1 \\ & & & & & & 0 & -1 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \tag{A.5}$$

$$h_{2,0} \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & \cdots & & & \\ & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix}, \tag{A.6}$$

$$(A.7) \qquad h_{1,1} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 & & & & \\ -1 & 0 & -1 & 0 & & & & \\ 0 & -1 & 0 & -1 & & & & \\ & & & & \cdots & & & \\ & & & & & -1 & 0 & -1 & 0 \\ & & & & & 0 & -1 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and}$$

$$(A.8) \qquad h_{0,2} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & \cdots & & & \\ & & & & 1 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix}.$$

∎

Proof: Using (1.1),

$$E\left(e^{-s\sum X_{t-1}^2}\right) = \int \cdots \int (2\pi)^{-\frac{T}{2}} e^{-s\sum x_{t-1}^2 - \frac{1}{2}\sum (x_t-\rho_1 x_{t-1}-\rho_2 x_{t-2})^2} dx_1 \dots dx_T =$$

$$= \int \cdots \int (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\underline{x}' P \underline{x}} dx_1 \dots dx_T = \frac{1}{\sqrt{\det P}},$$

where $\underline{x}' = (x_1, \dots, x_T)$, and since

$$\sum_{t=1}^{T} (x_t - \rho_1 x_{t-1} - \rho_2 x_{t-2})^2 =$$

$$= (1+\rho_1^2+\rho_2^2)\sum_{t=1}^{T-2} x_t^2 + (1+\rho_1^2)x_{T-1}^2 + x_T^2 - 2\rho_1 \sum_{t=2}^{T} x_t x_{t-1} - 2\rho_2 \sum_{t=3}^{T} x_t x_{t-2} - 2\rho_1\rho_2 \sum_{t=2}^{T-1} x_t x_{t-1},$$

$$P = \begin{pmatrix} a & b & c & 0 & 0 & 0 & & & & & \\ b & a & b & c & 0 & 0 & & & & & \\ c & b & a & b & c & 0 & & & & & \\ 0 & c & b & a & b & c & & & & & \\ & & & & & & \cdots & & & & \\ & & & & & & & c & b & a & b & c \\ & & & & & & & 0 & c & b & d & e \\ & & & & & & & 0 & 0 & c & e & 1 \end{pmatrix},$$

where $a \stackrel{\text{def}}{=} 1 + \rho_1^2 - \rho_2^2 + 2s$, $b = \rho_1\rho_2 - \rho_1$, $c = -\rho_2$, $d = 1 + \rho_1^2 + 2s$ and $e = -\rho_1$. Substituting $\rho_1 = 1 - \theta$ and $\rho_2 = \rho$, the lemma follows.

∎

The next lemma fits the results of lemma 5.1 to the present context.

Lemma A.2

$$(A.9)\qquad TE\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) = \frac{1}{T}\int_0^\infty xg_1(x)dx + o(1),$$

$$(A.10)\qquad E\left(\frac{\sum \varepsilon_t\varepsilon_{t-1}(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2}\right) = \frac{1}{T^2}\int_0^\infty xg_2(x)dx + 2 + o(1),$$

$$(A.11)\qquad T^2E\left(\frac{1}{\sum S_{t-1}^2}\right) = \int_0^\infty xg_3(x)dx + o(1) \qquad \text{and}$$

$$(A.12)\qquad TE\left(\frac{(\sum \varepsilon_t\varepsilon_{t-1})^3}{(\sum S_{t-1}^2)^2}\right) = \frac{1}{2T^3}\int_0^\infty x^3g_4(x)dx + \frac{3}{T}\int_0^\infty xg_5(x)dx + o(1),$$

where, letting $a_{ij} \stackrel{\text{def}}{=} \text{tr}(P_0^{-1}h_{i,j})$, $a_{ij\times kl} \stackrel{\text{def}}{=} \text{tr}\left((P_0^{-1}h_{i,j})(P_0^{-1}h_{k,l})\right)$ *and* $a_{ij\times kl\times mn} \stackrel{\text{def}}{=} \text{tr}\left((P_0^{-1}h_{i,j})(P_0^{-1}h_{k,l})(P_0^{-1}h_{m,n})\right)$,

$$(A.13)\quad g_1(x) = (\det P_0)^{-\frac{1}{2}}\left(T + \frac{1}{2}a_{10} + \frac{1}{4}a_{10}(a_{10}+a_{01}) - \frac{1}{2}(2a_{20}+a_{11}) + \frac{1}{2}(a_{10\times 10}+a_{10\times 01})\right),$$

$$(A.14)\quad g_2(x) = (\det P_0)^{-\frac{1}{2}}\Bigg(Ta_{10} - 2\left(a_{20} - \frac{1}{4}a_{10}^2 - \frac{1}{2}a_{10\times 10}\right) - \frac{1}{2}a_{10}(2a_{20}+a_{10}) -$$
$$-\frac{1}{2}\left(a_{20} - \frac{1}{4}a_{10}^2 - \frac{1}{2}a_{10\times 10}\right)(a_{10}+a_{01}) - (2a_{10\times 20}+a_{10\times 11}) - (a_{10\times 20}+a_{01\times 20}) +$$
$$+\frac{1}{2}a_{10}(a_{10\times 10}+a_{10\times 01}) + (a_{10\times 10\times 10}+a_{10\times 10\times 01})\Bigg),$$

$$(A.15)\quad g_3(x) = (\det P_0)^{-\frac{1}{2}},$$

$$(A.16)\quad g_4(x) = (\det P_0)^{-\frac{1}{2}}\left(\frac{1}{8}a_{10}^3 - \frac{3}{2}a_{10}a_{20} - 3a_{10\times 20} + \frac{3}{4}a_{10}a_{10\times 10} + a_{10\times 10\times 10}\right) \qquad \text{and}$$

$$(A.17)\quad g_5(x) = \frac{1}{2}(\det P_0)^{-\frac{1}{2}}a_{10}.$$

∎

The a_{ij}:s dependency on x will be explained below.

Proof: Recall that

$$\tilde{\varphi}(\theta,\rho;s) = \int \ldots \int (2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\underline{x}' P \underline{x}} dx_1 \ldots dx_T.$$

With $P = P_0 + h$, we may rewrite this formula as

$$\tilde{\varphi}(\theta,\rho;s) = \frac{1}{\sqrt{\det P_0}} \int \dots \int e^{-\frac{1}{2}\underline{x}'h\underline{x}} \sqrt{\det P_0}(2\pi)^{-\frac{T}{2}} e^{-\frac{1}{2}\underline{x}'P_0\underline{x}} dx_1 \dots dx_T = \frac{1}{\sqrt{\det P_0}} E\left(e^{-\frac{1}{2}\underline{x}'h\underline{x}}\right),$$

taking expectation w.r.t. a T-variate normal distribution with covariance matrix P_0^{-1}. Taylor expansion now yields

$$(A.18) \qquad \tilde{\varphi}(\theta,\rho;s) = \frac{1}{\sqrt{\det P_0}} \left(1 - \frac{1}{2}E\left(\underline{X}'h\underline{X}\right) + \frac{1}{8}E\left(\left(\underline{X}'h\underline{X}\right)^2\right) - \frac{1}{48}E\left(\left(\underline{X}'h\underline{X}\right)^3\right) + \dots\right),$$

where $\underline{X}' \stackrel{\text{def}}{=} (X_1,\dots,X_T)$ with X_t as in (1.1). The r.h.s. of this equation involves moments of the Wishart distribution, which are calculated by Magnus (1978) to be (with $\underline{Y} \stackrel{\text{def}}{=} P_0^{\frac{1}{2}}\underline{X} \sim N_T(0,I)$, we have $\underline{X}'h\underline{X} = \underline{Y}'(P_0^{-1}h)\underline{Y}$)

$$(A.19) \qquad E\left(\underline{X}'h\underline{X}\right) = \text{tr}(P_0^{-1}h),$$

$$(A.20) \qquad E\left(\left(\underline{X}'h\underline{X}\right)^2\right) = \text{tr}^2(P_0^{-1}h) + 2\text{tr}\left((P_0^{-1}h)^2\right) \qquad \text{and}$$

$$(A.21) \qquad E\left(\left(\underline{X}'h\underline{X}\right)^3\right) = \text{tr}^3(P_0^{-1}h) + 6\text{tr}(P_0^{-1}h)\text{tr}\left((P_0^{-1}h)^2\right) + 8\text{tr}\left((P_0^{-1}h)^3\right).$$

But, from (A.3),

$$P_0^{-1}h = \theta P_0^{-1}h_{1,0} + \rho P_0^{-1}h_{0,1} + \theta^2 P_0^{-1}h_{2,0} + \theta\rho P_0^{-1}h_{1,1} + \rho^2 P_0^{-1}h_{0,2},$$

and so, plugging in into (A.18)-(A.21), collecting terms and using Taylor's formula,

$$\frac{\partial\tilde{\varphi}}{\partial\theta} = -\frac{1}{2}a_{10}\tilde{\varphi},$$

$$\frac{\partial^2\tilde{\varphi}}{\partial\theta^2} = \left(-a_{20} + \frac{1}{4}a_{10}^2 + \frac{1}{2}a_{10\times10}\right)\tilde{\varphi},$$

$$\frac{\partial^2\tilde{\varphi}}{\partial\theta\partial\rho} = \left(-\frac{1}{2}a_{11} + \frac{1}{4}a_{10}a_{01} + \frac{1}{2}a_{10\times01}\right)\tilde{\varphi},$$

$$\frac{\partial^3\tilde{\varphi}}{\partial\theta^3} = \left(-\frac{1}{8}a_{10}^3 + \frac{3}{2}a_{10}a_{20} + 3a_{10\times20} - \frac{3}{4}a_{10}a_{10\times10} - a_{10\times10\times10}\right)\tilde{\varphi} \qquad \text{and}$$

$$\frac{\partial^3\tilde{\varphi}}{\partial\theta^2\partial\rho} = \left(-\frac{1}{8}a_{10}^2a_{01} + \frac{1}{2}a_{10}a_{11} + \frac{1}{2}a_{01}a_{20} + a_{10\times11} + a_{01\times20} - \frac{1}{2}a_{10}a_{10\times01} - \frac{1}{4}a_{01}a_{10\times10} - a_{10\times10\times01}\right)\tilde{\varphi},$$

where $\tilde{\varphi}$ and its derivatives are taken at $(\theta,\rho) = (0,0)$. Hence, since $\rho_1 = 1-\theta$ and $\rho_2 = \rho$, (5.1) yields

$$TE\left(\frac{\sum\varepsilon_t\varepsilon_{t-1}\sum S_{t-1}\varepsilon_t}{\sum S_{t-1}^2}\right) = T\int_0^\infty h(s)ds + o(1),$$

where

$$h(s) = (\det P_0)^{-\frac{1}{2}} \left(T + \frac{1}{2}a_{10} + \frac{1}{4}a_{10}(a_{10} + a_{01}) - \frac{1}{2}(2a_{20} + a_{11}) + \frac{1}{2}(a_{10\times 10} + a_{10\times 01}) \right),$$

However, since $\frac{1}{T^2}\sum_{t=1}^{T} X_{t-1}^2$ converges to a random variable with a non-degenerate distribution function as $T \to \infty$, it is natural to put $s^* = sT^2$ and define

$$\varphi^*(s^*) \stackrel{\text{def}}{=} E\left(e^{-s^* \frac{1}{T^2}\sum X_{t-1}^2}\right) = \varphi(s).$$

Letting $h^*(s^*)$ correspond to $h(s)$, we have for an arbitrary $\delta > 0$

$$T\int_0^\infty h(s)ds = \frac{1}{T}\int_0^\infty h^*(s^*)ds^* = \frac{1}{T}\int_0^{T^\delta} h^*(s^*)ds^* + o(1),$$

and so

$$T\int_0^\infty h(s)ds = T\int_0^{T^{-2+\delta}} h(s)ds + o(1).$$

Hence, since

$$\sigma \stackrel{\text{def}}{=} \sqrt{1 - \frac{4}{\alpha^2}} = \sqrt{1 - \frac{1}{(1+s)^2}} \Longleftrightarrow s = \frac{1}{\sqrt{1-\sigma^2}} - 1,$$

(cf Lemma A.3) the substitution $x = \sigma T$ implies

$$s = \frac{x^2}{2T^2} + o\left(\frac{1}{T^2}\right) \Rightarrow ds = \frac{xdx}{T^2} + o\left(\frac{1}{T^2}\right),$$

for $x = o(T)$ i.e. $\delta < 2$, which yields (A.9) and (A.13). (In effect, $s < T^{-2+\delta}$ implies $x \leq O(T^\delta)$, but since δ is arbitrary we may from now on assume $x = O(1)$, i.e. $s = \frac{x^2}{2T^2} + O\left(\frac{1}{T^4}\right)$ etc.) The rest of the results follow similarly. (Note that, by definition, the a_{ij}:s etc. are functions of s. Hence, they become functions of x after the substitution.)

∎

As we see from lemma A.2, we also need to calculate P_0^{-1} explicitly.

Lemma A.3. *Denoting an arbitrary element of the $T \times T$ matrix P_0^{-1} by a_{ij}, we have*

$$(A.22) \qquad a_{ij} = \begin{cases} \dfrac{1}{\det P_0} D^*_{i-1} D_{T-j}, & i \leq j, \\ \\ a_{ji}, & j < i, \end{cases}$$

where

$$(A.23) \quad D_k = \begin{cases} 1, & k = 0, \\ \\ \left(\dfrac{\alpha}{2}\right)^{k-1} \left(\dfrac{(1+\sigma)^{k-1} + (1-\sigma)^{k-1}}{2} + \left(1 - \dfrac{2}{\alpha}\right) \dfrac{(1+\sigma)^{k-1} - (1-\sigma)^{k-1}}{2\sigma}\right), & k \geq 1, \end{cases}$$

$$(A.24) \quad D^*_k = \begin{cases} 1, & k = 0, \\ \\ \left(\dfrac{\alpha}{2}\right)^{k-1} \left(\alpha \dfrac{(1+\sigma)^{k-1} + (1-\sigma)^{k-1}}{2} + \left(\alpha - \dfrac{2}{\alpha}\right) \dfrac{(1+\sigma)^{k-1} - (1-\sigma)^{k-1}}{2\sigma}\right), & k \geq 1 \end{cases}$$

and

$$\sigma = \sqrt{1 - \frac{4}{\alpha^2}}.$$

∎

Proof: Letting D_k be the determinant of the $k \times k$ lower right corner of P_0 and D^*_k the determinant of the $k \times k$ upper left corner, it follows that

$$a_{ij} = \frac{D^*_{i-1} D_{T-j}}{D_T},$$

where of course $D_T = \det P_0$, adopting the conventions $D^*_0 = D_0 = 1$. Expanding P_0 by the first row, we obtain the difference equation

$$D_T = \alpha D_{T-1} - D_{T-2},$$

with initial conditions $D_1 = 1$ and $D_2 = \alpha - 1$. From this, (A.23) follows.

For D^*_T we get the same difference equation, but here $D^*_1 = \alpha$ and $D^*_2 = \alpha^2 - 1$, implying (A.24), and we are done.

∎

Proof of Theorem 5.2: Our remaining task is the formidable one of deriving (5.12)-(5.15) out of (A.13)-(A.17). We start this project with the calculation of $\det P_0$, and to this end, (A.23) yields

$$\det P_0 = D_T = \left(\frac{\alpha}{2}\right)^{T-1}\left(\frac{(1+\sigma)^{T-1}+(1-\sigma)^{T-1}}{2} + \left(1-\frac{2}{\alpha}\right)\frac{(1+\sigma)^{T-1}-(1-\sigma)^{T-1}}{2\sigma}\right),$$

where $\alpha = 2(1+s)$. Now, substituting $x = \sigma T$, $\frac{\alpha}{2} = 1 + O\left(\frac{1}{T^2}\right)$, implying $\left(\frac{\alpha}{2}\right)^{T-1} = 1 + O\left(\frac{1}{T}\right)$, $1 - \frac{2}{\alpha} = \frac{x^2}{T^2} + O\left(\frac{1}{T^4}\right)$ and, due to the binomial theorem,

$$\frac{(1+\sigma)^{T-1}+(1-\sigma)^{T-1}}{2} = \cosh x + O\left(\frac{1}{T}\right)$$

and

$$\frac{(1+\sigma)^{T-1}-(1-\sigma)^{T-1}}{2\sigma} = T\frac{\sinh x}{x} + O(1).$$

Hence,

$$\det P_0 = D_T = \cosh x + O\left(\frac{1}{T}\right), \tag{A.25}$$

and furthermore,

$$D_T^* = T\frac{\sinh x}{x} + O(1). \tag{A.26}$$

In the calculations below, we will also need approximations of terms like D_i, where $1 \le i \le T$. Substituting $x = \sigma T$ and $y = \frac{i}{T}$, we get as above

$$D_i = \cosh(xy) + O\left(\frac{1}{T}\right)$$

and

$$D_i^* = T\frac{\sinh(xy)}{x} + O(1).$$

Moreover, introducing the notation

$$\Delta D_i \overset{\text{def}}{=} D_i - D_{i-1}, \quad \Delta^2 D_i \overset{\text{def}}{=} \Delta D_i - \Delta D_{i-1} = D_i - 2D_{i-1} + D_{i-2},$$

Taylor expansion yields

$$\Delta D_i = \frac{1}{T}\left(\frac{d}{dy}\cosh(xy) + O\left(\frac{1}{T}\right)\right) = \frac{1}{T}x\sinh(xy) + O\left(\frac{1}{T^2}\right),$$

and similarly

$$\Delta^2 D_i = \frac{1}{T^2}x^2\cosh(xy) + O\left(\frac{1}{T^3}\right),$$

$$\Delta D_i^* = \cosh(xy) + O\left(\frac{1}{T}\right) \qquad \text{and}$$

$$\Delta^2 D_i^* = \frac{1}{T}x\sinh(xy) + O\left(\frac{1}{T^2}\right).$$

In the following, this approximation technique will turn out to be useful.

We now start calculating $g_1(x)$, and in view of (A.13), a_{10} is the first term to tackle. To this end, note that from (A.22)

$$(A.27)\qquad \det P_0 \cdot P_0^{-1} = \begin{pmatrix} D_{T-1} & \dots & & & D_{T-i} & & & \dots & 1 \\ & & & & \dots & & & & \\ \dots & & & & D^*_{k-1}D_{T-i} & & & & \dots \\ & & & & \dots & & & & \\ D_{T-i} & \dots & D^*_{k-1}D_{T-i} & \dots & D^*_{i-1}D_{T-i} & \dots & D^*_{i-1}D_{T-l} & \dots & D^*_{i-1} \\ & & & & \dots & & & & \\ \dots & & & & D^*_{i-1}D_{T-l} & & & & \dots \\ & & & & \dots & & & & \\ 1 & \dots & & & D^*_{i-1} & & & \dots & D^*_{T-1} \end{pmatrix}.$$

(The indices i, k and l are to be thought of as running from 1 to T, from 1 to $i-1$ and from $i+1$ to T, respectively.) Now, letting $D_0 = D_0^* = 0$, (A.4) and (A.27) imply

$$(A.28)\qquad \det P_0 \cdot P_0^{-1} h_{1,0} =$$

$$= \begin{pmatrix} -2D_{T-1}+D_{T-2} & \dots & & & \Delta^2 D_{T-i+1} & & & \dots & 1 \\ & & & & \dots & & & & \\ \dots & & & & D^*_{k-1}\Delta^2 D_{T-i+1} & & & & \dots \\ & & & & \dots & & & & \\ \Delta^2 D^*_1 D_{T-i} & \dots & \Delta^2 D^*_k D_{T-i} & \dots & a & \dots & D^*_{i-1}\Delta^2 D_{T-l+1} & \dots & D^*_{i-1} \\ & & & & \dots & & & & \\ \dots & & & & \Delta^2 D^*_i D_{T-l} & & & & \dots \\ & & & & \dots & & & & \\ & & & & & & & -\Delta D^*_{T-2} & D^*_{T-2} \\ \Delta^2 D^*_1 & \dots & & & \Delta^2 D^*_i & & \dots & \Delta^2 D^*_{T-1} & D^*_{T-2} \end{pmatrix},$$

where

$$a = D^*_{i-2}D_{T-i} - 2D^*_{i-1}D_{T-i} + D^*_{i-1}D_{T-i-1} = D^*_{i-1}\Delta^2 D_{T-i+1} - D^*_{i-2}\Delta D_{T-i+1} - \Delta D^*_{i-1}D_{T-i+1}.$$

Hence,

$$(A.29)\quad \det P_0 \cdot a_{10} = \mathrm{tr}(\det P_0 \cdot P_0^{-1} h_{1,0}) =$$

$$= -2D_{T-1} + D_{T-2} + \sum_{i=2}^{T-2} \left(D^*_{i-1}\Delta^2 D_{T-i+1} - D^*_{i-2}\Delta D_{T-i+1} - \Delta D^*_{i-1}D_{T-i+1}\right) - \Delta D^*_{T-2} + D^*_{T-2}.$$

Approximating as above (with $T-i$ instead of i, we get $1-y$ instead of y), and replacing the sum by an integral (rendering a factor of T in front), we get

$$(A.30)\qquad \det P_0 \cdot a_{10} =$$

$$= T\int_0^1 \left(T\frac{\sinh(xy)}{x}\frac{1}{T^2}x^2\cosh(x(1-y)) - T\frac{\sinh(xy)}{x}\frac{1}{T}x\sinh(x(1-y)) - \cosh(xy)\cosh(x(1-y))\right)dy +$$

$$+ T\frac{\sinh x}{x} + O(1) =$$

$$= T\left(-\int_0^1 \Big(\sinh(xy)\sinh(x(1-y)) + \cosh(xy)\cosh(x(1-y))\Big)\,dy + \frac{\sinh x}{x}\right) + O(1) =$$

$$= -T\left(\cosh x - \frac{\sinh x}{x}\right) + O(1).$$

After this, we need

$$\det P_0(a_{10} + a_{01}) = \operatorname{tr}\Big(\det P_0 \cdot P_0^{-1}(h_{1,0} + h_{0,1})\Big).$$

However, it follows from (A.4) and (A.5) that

$$h_{1,0} + h_{0,1} = \left(\begin{array}{cccccccccccc} -2 & 2 & -1 & 0 & 0 & 0 & & & & & & \\ 2 & -2 & 2 & -1 & 0 & 0 & & & & & & \\ -1 & 2 & -2 & 2 & -1 & 0 & & & & & & \\ 0 & -1 & 2 & -2 & 2 & -1 & & & & & & \\ & & & & & & \ldots & & & & & \\ & & & & & & & -1 & 2 & -2 & 2 & -1 \\ & & & & & & & 0 & -1 & 2 & -2 & 1 \\ & & & & & & & 0 & 0 & -1 & 1 & 0 \end{array}\right),$$

and so (A.27) yields

$$(A.31) \qquad \det P_0 \cdot P_0^{-1}(h_{1,0} + h_{0,1}) =$$

$$\left(\begin{array}{cccccccccccccc} a & & & & & & & & & & & & & \\ & b & & & & & & & & & & & & \\ & & & & & & \ldots & & & & & & & \\ & & \ldots & c & \ldots & d & e & f & \ldots & g & \ldots & & & \\ & & & & & & \ldots & & & & & & & \\ & & & & & & & & & & & & -\Delta^2 D^*_{T-2} & \Delta D^*_{T-2} \\ & & & & & & & & & & & & -\Delta^2 D^*_{T-2} + \Delta D^*_{T-1} & \Delta D^*_{T-2} \end{array}\right),$$

where

$$\begin{aligned} a &= -2\Delta D_{T-1} - D_{T-3}, \\ b &= 2D_{T-1} - D^*_1(D_{T-2} + \Delta^2 D_{T-2}), \\ c &= -(\Delta^2 D^*_{k-1} + \Delta^2 D^*_{k+1})D_{T-i}, \\ d &= -\Delta^2 D^*_{i-2}D_{T-i} + \Delta D^*_{i-1}D_{T-i} + D^*_{i-1}\Delta D_{T-i}, \\ e &= -\Delta^2 D^*_{i-1}D_{T-i} - D^*_{i-1}\Delta^2 D_{T-i}, \\ f &= \Delta D^*_{i-1}D_{T-i} + D^*_{i-1}\Delta^2 D_{T-i} + D^*_{i-1}\Delta D_{T-i-1} \qquad \text{and} \\ g &= -D^*_{i-1}(\Delta^2 D_{T-l+1} + \Delta^2 D_{T-l-1}). \end{aligned}$$

(Here, i runs from 3 to $T-2$, k runs from 1 to $i-2$ and l runs from $i+2$ to T.) Now, using the same technique as before, we conclude (remember that $D^*_1 = \alpha = 2 + O\left(\frac{1}{T^2}\right)$)

$$(A.32) \quad \det P_0(a_{10} + a_{01}) = -D_{T-3} - \sum_{i=3}^{T-2}(\Delta^2 D^*_{i-1}D_{T-i} + D^*_{i-1}\Delta^2 D_{T-i}) + \Delta D^*_{T-2} + O\left(\frac{1}{T}\right) =$$

$$= -2x\int_0^1 \sinh(xy)\cosh(x(1-y))dy + O\left(\frac{1}{T}\right) =$$

$$= -x \int_0^1 \Big(\sinh x + \sinh(x(2y-1))\Big) dy + O\left(\frac{1}{T}\right) = -x \sinh x + O\left(\frac{1}{T}\right).$$

Our next task is to calculate

$$\det P_0 (2a_{20} + a_{11}) = \operatorname{tr}\Big(\det P_0 \cdot P_0^{-1}(2h_{2,0} + h_{1,1})\Big).$$

Now, observe that from (A.4), (A.6) and (A.7),

$$2h_{2,0} + h_{1,1} = -h_{1,0} + \delta, \tag{A.33}$$

where

$$\delta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & \cdots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix},$$

and so

$$\det P_0 (2a_{20} + a_{11}) = -\operatorname{tr}(\det P_0 \cdot P_0^{-1} h_{1,0}) + \operatorname{tr}(\det P_0 \cdot P_0^{-1} \delta).$$

The first of these terms is known from (A.30), and for the second one (A.27) yields, since $D_1 = 1$,

$$\det P_0 \cdot P_0^{-1} \delta = \begin{pmatrix} 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & D_1^* & D_1^* \\ & \dots & & \dots & \\ 0 & \dots & 0 & D_{T-2}^* & D_{T-2}^* \\ 0 & \dots & 0 & D_{T-1}^* & D_{T-2}^* \end{pmatrix}, \tag{A.34}$$

and so

$$\operatorname{tr}(\det P_0 \cdot P_0^{-1} \delta) = 2D_{T-2}^* = 2T \frac{\sinh x}{x} + O(1),$$

which together with (A.30) implies

$$\det P_0 (2a_{20} + a_{11}) = T\left(\cosh x + \frac{\sinh x}{x}\right) + O(1). \tag{A.35}$$

To complete the calculation of $g_1(x)$, it follows from (A.28) and (A.31) that

$$(\det P_0)^2 (a_{10\times 10} + a_{10\times 01}) = \operatorname{tr}\Big(\left(\det P_0 \cdot P_0^{-1}(h_{1,0} + h_{0,1})\right)\left(\det P_0 \cdot P_0^{-1} h_{1,0}\right)\Big) =$$

$$= S - \sum_{i=1}^{T-3} \Big((\Delta^2 D_{i-1}^* + \Delta^2 D_{i+1}^*) D_{i+1}^* + D_{T-2}^* (\Delta D_{T-1}^* + \Delta D_{T-2}^*) \Big) + O(1),$$

where

$$\begin{aligned} S \stackrel{\text{def}}{=} \sum_{i=3}^{T-2} \Big(& -D_{T-i} \Delta^2 D_{T-i+1} \sum_{k=1}^{i-2} D_{k-1}^* (\Delta^2 D_{k-1}^* + \Delta^2 D_{k+1}^*) + \\ & + (-\Delta^2 D_{i-2}^* D_{T-i} + \Delta D_{i-1}^* D_{T-i} + D_{i-1}^* \Delta D_{T-i}) D_{i-2}^* \Delta^2 D_{T-i+1} + \\ & + (\Delta D_{i-1}^* D_{T-i} + D_{i-1}^* \Delta^2 D_{T-i} + D_{i-1}^* \Delta D_{T-i-1}) \Delta^2 D_i^* D_{T-i-1} - \\ & - D_{i-1}^* \Delta^2 D_i^* \sum_{l=i+2}^{T} D_{T-l} (\Delta^2 D_{T-l+1} + \Delta^2 D_{T-l-1}) \Big) = O(1), \end{aligned}$$

due to our usual approximation arguments. Thus,

$$(A.36)\qquad (\det P_0)^2(a_{10\times 10}+a_{10\times 01}) =$$

$$= -\sum_{i=1}^{T-3}\Big((\Delta^2 D^*_{i-1}+\Delta^2 D^*_{i+1})D^*_{i+1} + D^*_{T-2}(\Delta D^*_{T-1}+\Delta D^*_{T-2})\Big) + O(1) =$$

$$= T\left(-2\int_0^1 \sinh^2(xy)dy + 2\cosh x\frac{\sinh x}{x}\right) + O(1) =$$

$$= T\left(-\int_0^1 (\cosh(2xy)-1)dy + 2\cosh x\frac{\sinh x}{x}\right) + O(1) = T\left(\cosh x\frac{\sinh x}{x}+1\right) + O(1).$$

Now, inserting (A.30), (A.32), (A.35) and (A.36) into (A.13),

$$g_1(x) = T(\cosh x)^{-\frac{5}{2}}\left(\cosh^2 x - \frac{1}{2}\cosh x\left(\cosh x - \frac{\sinh x}{x}\right) + \frac{1}{4}x\sinh x\left(\cosh x - \frac{\sinh x}{x}\right) -\right.$$

$$\left. - \frac{1}{2}\cosh x\left(\cosh x + \frac{\sinh x}{x}\right) + \frac{1}{2}\left(\cosh x\frac{\sinh x}{x}+1\right)\right) + O(1) =$$

$$= \frac{T}{2}(\cosh x)^{-\frac{5}{2}}\left(\cosh x\frac{\sinh x}{x} + \frac{1}{2}x\cosh x\sinh x - \frac{1}{2}\sinh^2 x + 1\right),$$

which, in view of (A.9), since

$$\int_0^\infty (\cosh x)^{-\frac{3}{2}}\sinh x dx = 2$$

implies (5.12).

As for $g_2(x)$, we note from (A.14) that, in addition to the terms already calculated, we have to look at a_{20}, $a_{10\times10}$, $2a_{10\times20}+a_{10\times11}$, $a_{10\times20}+a_{01\times20}$ and $a_{10\times10\times10}+a_{10\times10\times01}$. To start with, it follows from (A.6) and (A.27) that

$$(A.37)\qquad \det P_0\cdot P_0^{-1}h_{2,0}=$$

$$=\begin{pmatrix} D_{T-1} & \dots & & & D_{T-i} & & & \dots & 1 & 0\\ & & & & \dots & & & & & \\ \dots & & & & D^*_{k-1}D_{T-i} & & & \dots & \dots & \\ & & & & \dots & & & & & \\ D_{T-i} & \dots & D^*_{k-1}D_{T-i} & \dots & D^*_{i-1}D_{T-i} & \dots & D^*_{i-1}D_{T-l} & \dots & D^*_{i-2} & 0\\ & & & & \dots & & & & & \\ \dots & & & & D^*_{i-1}D_{T-l} & & & \dots & \dots & \\ & & & & \dots & & & & & \\ & & & & & & & & D^*_{T-2} & 0\\ 1 & \dots & & & D^*_{i-1} & & & \dots & D^*_{T-2} & 0 \end{pmatrix}.$$

(As before, i runs from 1 to T, k runs from 1 to $i-1$ and l runs from $i+1$ to T.) Hence,

$$(A.38)\ \det P_0\cdot a_{20}=\mathrm{tr}(\det P_0\cdot P_0^{-1}h_{2,0})=\sum_{i=1}^{T-1}D^*_{i-1}D_{T-i}=\frac{T^2}{x}\int_0^1\sinh(xy)\cosh(x(1-y))dy+O(T)=$$

$$=\frac{T^2}{2x}\int_0^1\Big(\sinh x+\sinh(x(2y-1))\Big)dy+O(T)=\frac{T^2}{2}\frac{\sinh x}{x}+O(T).$$

Moreover, (A.28) yields

$$(A.39)\qquad (\det P_0\cdot P_0^{-1}h_{1,0})^2=\begin{pmatrix} O(1) & \dots & O(1) & D^*_{T-2}+O(1)\\ & & & \dots\\ \dots & & \dots & D^*_{i-1}D^*_{T-2}+O(1)\\ & & & \dots\\ & & & {D^*_{T-2}}^2+O(1)\\ O(1) & \dots & O(1) & {D^*_{T-2}}^2+O(1)\end{pmatrix},$$

implying

$$(A.40)\qquad (\det P_0)^2a_{10\times10}=\mathrm{tr}\left((\det P_0\cdot P_0^{-1}h_{1,0})^2\right)={D^*_{T-2}}^2+O(T)=T^2\left(\frac{\sinh x}{x}\right)^2+O(T).$$

As for $2a_{10\times20}+a_{10\times11}$, it follows from (A.33) that

$$(\det P_0)^2(2a_{10\times20}+a_{10\times11})=\mathrm{tr}\Big((\det P_0\cdot P_0^{-1}h_{1,0})\left(\det P_0\cdot P_0^{-1}(2h_{2,0}+h_{1,1})\right)\Big)=$$

$$=-\mathrm{tr}\left((\det P_0\cdot P_0^{-1}h_{1,0})^2\right)+\mathrm{tr}\left((\det P_0\cdot P_0^{-1}h_{1,0})(\det P_0\cdot P_0^{-1}\delta)\right).$$

The first of these terms is given by (A.40), and by (A.28) and (A.34) the second one is

$$\mathrm{tr}\left((\det P_0\cdot P_0^{-1}h_{1,0})(\det P_0\cdot P_0^{-1}\delta)\right)=-\Delta D^*_{T-3}D^*_{T-2}+D^*_{T-2}D^*_{T-1}+\Delta^2D^*_{T-1}D^*_{T-2}+{D^*_{T-2}}^2=$$

$$=D^*_{T-2}(D^*_{T-1}+D^*_{T-2})+O(T)=2T^2\left(\frac{\sinh x}{x}\right)^2+O(T),$$

and so, by (A.40),

$$(A.41)\qquad (\det P_0)^2(2a_{10\times 20}+a_{10\times 11}) = T^2\left(\frac{\sinh x}{x}\right)^2 + O(T).$$

Moreover, (A.31) and (A.37) imply

$$(A.42)\quad (\det P_0)^2(a_{10\times 20}+a_{01\times 20}) = \mathrm{tr}\Big(\big(\det P_0\cdot P_0^{-1}(h_{1,0}+h_{0,1})\big)\big(\det P_0\cdot P_0^{-1}h_{2,0}\big)\Big) =$$

$$= \sum_{i=1}^{T-2}\Big(-D_{T-i}^2\sum_{k=1}^{i-2}(\Delta^2 D_{k-1}^* + \Delta^2 D_{k+1}^*)D_{k-1}^* + (\Delta D_{i-1}^* D_{T-i} + D_{i-1}^*\Delta D_{T-i})D_{i-2}^* D_{T-i}+$$

$$+ (\Delta D_{i-1}^* D_{T-i} + D_{i-1}^*\Delta D_{T-i-1})D_{i-1}^* D_{T-i-1} -$$

$$- {D_{i-1}^*}^2\sum_{l=i+2}^{T}(\Delta^2 D_{T-l+1} + \Delta^2 D_{T-l-1})D_{T-l} + O(1)\Big) + O(T) =$$

$$= 2T^2\int_0^1\Big(-\cosh^2(x(1-y))\int_0^y \sinh^2(xz)dz + \cosh(xy)\cosh(x(1-y))+$$

$$+\sinh(xy)\sinh(x(1-y)) - \sinh^2(xy)\int_y^1\cosh^2(x(1-z))dz\Big) + O(T) =$$

$$= \frac{T^2}{2}\left(\cosh x\frac{\sinh x}{x}+1\right)+O(T).$$

(To obtain the second equality, we put $x=\sigma T$, y=$\frac{i}{T}$ and $z=\frac{k}{T}$ or $\frac{l}{T}$ and argue in the usual manner. How to get the third equality is in principle trivial, at least for a formula manipulating computer program.) Also, as follows from (A.31) and (A.39),

$$(A.43)\quad (\det P_0)^3(a_{10\times 10\times 01}+a_{10\times 10\times 01}) = \mathrm{tr}\Big(\big(\det P_0\cdot P_0^{-1}(h_{1,0}+h_{0,1})\big)\big(\det P_0\cdot P_0^{-1}h_{1,0}\big)^2\Big) =$$

$$= -D_{T-2}^*\sum_{i=1}^{T-2}(\Delta^2 D_{i-1}^* + \Delta^2 D_{i+1}^*)D_{i-1}^* + (\Delta D_{T-1}^* + \Delta D_{T-2}^*){D_{T-2}^*}^2 + O(T) =$$

$$= 2T^2\frac{\sinh x}{x}\left(-\int_0^1 \sinh^2(xy)dy + \cosh x\frac{\sinh x}{x}\right) + O(T) = T^2\frac{\sinh x}{x}\left(\cosh x\frac{\sinh x}{x}+1\right)+O(T).$$

Hence, since by (A.38), (A.30) and (A.40),

$$(A.44)\qquad (\det P_0)^2\left(a_{20}-\frac{1}{4}a_{10}^2-\frac{1}{2}a_{10\times 10}\right) =$$

$$= T^2\left(\frac{1}{2}\cosh x\frac{\sinh x}{x} - \frac{1}{4}\left(\cosh x - \frac{\sinh x}{x}\right)^2 - \frac{1}{2}\left(\frac{\sinh x}{x}\right)^2\right) + O(T) =$$

$$= T^2\left(-\frac{1}{4}\cosh^2 x + \cosh x\frac{\sinh x}{x} - \frac{3}{4}\left(\frac{\sinh x}{x}\right)^2\right) + O(T),$$

(A.30), (A.35), (A.32), (A.41), (A.42), (A.36) and (A.43) plugged in into (A.14) gives us, after simplification

$$g_2(x) = T^2(\cosh x)^{-\frac{7}{2}}\Big(-\cosh x - \frac{1}{2}\frac{\sinh x}{x} - \frac{19}{8}\frac{\sinh^3 x}{x} + \frac{3}{2}\cosh x\left(\frac{\sinh x}{x}\right)^2 - \frac{1}{8}x\cosh^2 x\sinh x +$$

$$+\frac{1}{2}\cosh x\sinh^2 x\Big) + O(T),$$

and so, considering (A.10), and the fact that $\int_0^\infty (\cosh x)^{-\frac{7}{2}}\sinh x dx = \frac{2}{5}$, (5.13) is proved.

The derivation of (5.14) is immedeate from (A.11), (A.15) and the fact that $\det P_0 = \cosh x + O\left(\frac{1}{T}\right)$. It remains to derive (5.15), i.e. to calculate $g_4(x)$ and $g_5(x)$. As for $g_4(x)$, (A.16) hints that we will need the "new" terms $a_{10\times 20}$ and $a_{10\times 10\times 10}$. For the former, (A.28) and (A.37) imply

$$(A.45)\qquad (\det P_0)^2 a_{10\times 20} = \operatorname{tr}\Big((\det P_0\cdot P_0^{-1}h_{2,0})(\det P_0\cdot P_0^{-1}h_{1,0})\Big) = \sum_{i=1}^{T-1} {D^*_{i-1}}^2 + O(T) =$$

$$= \frac{T^3}{x^2}\int_0^1 \sinh^2(xy)dy = \frac{T^3}{2x^2}\left(\cosh x\frac{\sinh x}{x} - 1\right).$$

Considering the latter, (A.28) and (A.39) give

$$(A.46)\qquad (\det P_0)^3 a_{10\times 10\times 10} = {D^*_{T-2}}^3 + O(T^2) = T^3\left(\frac{\sinh x}{x}\right)^3 + O(T^2).$$

Now, inserting (A.30), (A.38), (A.45), (A.40) and (A.46) in (A.16) and simplifying, we get

$$(A.47)\quad g_4(x) = T^3(\cosh x)^{-\frac{7}{2}}\Big(-\frac{1}{8}\cosh^3 x + \frac{9}{8}\cosh^2 x\frac{\sinh x}{x} - \frac{15}{8}\cosh x\left(\frac{\sinh x}{x}\right)^2 + \frac{15}{8}\left(\frac{\sinh x}{x}\right)^3 +$$

$$+\frac{3}{2}\frac{\cosh x}{x^2} - \frac{3}{2}\cosh^2 x\frac{\sinh x}{x^3}\Big) + O(T^2).$$

Furthermore, (A.17) and (A.30) imply

$$g_5(x) = -\frac{T}{2}(\cosh x)^{-\frac{3}{2}}\left(\cosh x - \frac{\sinh x}{x}\right) + O(1),$$

which, since $\int_0^\infty (\cosh x)^{-\frac{3}{2}}\sinh x dx = 2$, together with (A.47) and (A.12) gives (5.15), and we are done.

∎

Proof of Theorem 6.1: In order to prove Theorem 6.1, we may use many of the results in the proof of Theorem 5.2. In view of (6.2), our task is to calculate $\frac{\partial^2}{\partial \rho_1^2}\varphi(1;s,u)$, where $\varphi(\rho_1;s,u) = E\left(e^{-s\sum X_{t-1}^2 - u\sum(\Delta X_t)^2}\right)$ and X_t is a process defined through (1.1) with $\rho_2 = 0$, i.e. an AR(1) process. Now, as in the proof of Lemma A.1, it follows that

$$\varphi(\rho_1;s,u) = \int \ldots \int (2\pi)^{-\frac{T}{2}} e^{-s\sum x_{t-1}^2 - u\sum(x_t - x_{t-1})^2 - \frac{1}{2}\sum(x_t - \rho_1 x_{t-1})^2} dx_1 \ldots dx_T = \frac{1}{\sqrt{\det \bar{P}}},$$

with

$$\bar{P} \stackrel{\text{def}}{=} (1+2u)P_0 + \theta h_{1,0} + \theta^2 h_{2,0},$$

$\theta = 1 - \rho_1$, and $h_{1,0}$ and $h_{2,0}$ as before. So is also P_0, but with s replaced by $\bar{s} \stackrel{\text{def}}{=} \frac{s}{1+2u}$. Furthermore, applying (A.18) with $\bar{P}_0 \stackrel{\text{def}}{=} (1+2u)P_0$ instead of P_0 and $\tilde{\varphi}(\theta;s) = \varphi(1-\theta;s)$, we get

$$\frac{\partial^2 \tilde{\varphi}}{\partial \theta^2} = \left(-\bar{a}_{20} + \frac{1}{4}\bar{a}_{10}^2 + \frac{1}{2}\bar{a}_{10\times 10}\right)\bar{\varphi},$$

where

$$\bar{a}_{20} \stackrel{\text{def}}{=} \operatorname{tr}(\bar{P}_0^{-1}h_{2,0}) = (1+2u)^{-1}a_{20}, \quad \bar{a}_{10} \stackrel{\text{def}}{=} \operatorname{tr}(\bar{P}_0^{-1}h_{1,0}) = (1+2u)^{-1}a_{10},$$

$$\bar{a}_{10\times 10} \stackrel{\text{def}}{=} \operatorname{tr}\left((\bar{P}_0^{-1}h_{1,0})^2\right) = (1+2u)^{-2}a_{10\times 10}$$

and

$$\bar{\varphi} = \frac{1}{\sqrt{\det \bar{P}_0}} = (1+2u)^{-\frac{T}{2}}\frac{1}{\sqrt{\det P_0}} = (1+2u)^{-\frac{T}{2}}\tilde{\varphi},$$

evaluating $\tilde{\varphi}$ and $\bar{\varphi}$ at $\theta = 0$. Hence,

$$\frac{\partial^2 \tilde{\varphi}}{\partial \theta^2} = (1+2u)^{-\frac{T}{2}-2}\left(-(1+2u)a_{20} + \frac{1}{4}a_{10}^2 - \frac{1}{2}a_{10\times 10}\right)\tilde{\varphi}.$$

Now, as in the proof of Lemma A.2, the substitution $x = \sigma T$, where

$$\sigma = \sqrt{1 - \frac{1}{(1+\bar{s})^2}} \Rightarrow ds = (1+2u)d\bar{s} = (1+2u)\frac{xdx}{T^2} + O\left(\frac{1}{T^4}\right),$$

together with (6.2) yields

$$(A.48) \qquad TE\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2 \sum {\varepsilon_t}^2}\right) = \frac{1}{T}\int_0^\infty\int_0^\infty (1+2u)^{-\frac{T}{2}-1} x g(x,u)dxdu + 1 + \frac{2}{T} + O\left(\frac{1}{T^2}\right)$$

where

$$(A.49) \qquad g(x,u) = (\det P_0)^{-\frac{1}{2}}\left(-(1+2u)a_{20} + \frac{1}{4}a_{10}^2 - \frac{1}{2}a_{10\times 10}\right).$$

(Observe that a_{20}, a_{10}^2 and $a_{10\times 10}$ are all $O(T^2)$.) To obtain the corrected expectation, we will need to approximate $g(x,u)$, i.e. $\det P_0$, a_{10}, a_{20} and $a_{10\times 10}$, to the second order! (Since $\frac{T}{\sum {\varepsilon_t}^2} = 1 + o_p(1)$, the leading term of (A.48) is given by (4.4).)

In the sequel, we will have repeated use of the formulae

$$(A.50)\ \frac{(1+\sigma)^{T-k}+(1-\sigma)^{T-k}}{2}=1+\binom{T-k}{2}\left(\frac{x}{T}\right)^2+\binom{T-k}{4}\left(\frac{x}{T}\right)^4+\binom{T-k}{6}\left(\frac{x}{T}\right)^6+\ldots=$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots-\frac{1}{T}\left(\frac{2k+1}{2!}x^2+\frac{4k+6}{4!}x^4+\frac{6k+15}{6!}x^6+\ldots\right)+O\left(\frac{1}{T^2}\right)=$$

$$=\cosh x-\frac{1}{2T}(2kx\sinh x+x^2\cosh x)+O\left(\frac{1}{T^2}\right)$$

and

$$(A.51)\qquad \frac{(1+\sigma)^{T-k}-(1-\sigma)^{T-k}}{2}=\binom{T-k}{1}\frac{x}{T}+\binom{T-k}{3}\left(\frac{x}{T}\right)^3+\binom{T-k}{5}\left(\frac{x}{T}\right)^5+\ldots=$$

$$=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots-\frac{1}{T}\left(kx+\frac{3k+3}{3!}x^3+\frac{5k+10}{5!}x^5+\ldots\right)+O\left(\frac{1}{T^2}\right)=$$

$$=\sinh x-\frac{1}{2T}(2kx\cosh x+x^2\sinh x)+O\left(\frac{1}{T^2}\right).$$

We will also need a second order approximation of $\det P_0$. But (as before, $\sigma=xT$)

$$\left(\frac{\alpha}{2}\right)^{T-1}=1+\frac{x^2}{2T}+O\left(\frac{1}{T^2}\right)\quad\text{and}\quad 1-\frac{2}{\alpha}=\frac{x^2}{2T^2}+O\left(\frac{1}{T^4}\right),$$

and so (A.23), (A.50) and (A.51) yield

$$(A.52)\ \ \det P_0=D_T=\left(1+\frac{x^2}{2T}\right)\left(\cosh x-\frac{1}{2T}(2x\sinh x+x^2\cosh x)+\frac{1}{2T}x\sinh x\right)+O\left(\frac{1}{T^2}\right)=$$

$$=\cosh x-\frac{1}{2T}x\sinh x+O\left(\frac{1}{T^2}\right).$$

To compute a_{10}, let us take a close look at (A.29). We know that

$$(A.53)\qquad D_{T-1}=\cosh x+O\left(\frac{1}{T}\right)=D_{T-2}+O\left(\frac{1}{T}\right)=\Delta D^*_{T-2}+O\left(\frac{1}{T}\right),$$

and in the usual manner

$$(A.54)\qquad \sum_{i=2}^{T-2}D^*_{i-1}\Delta^2 D_{T-i+1}=x\int_0^1\sinh(xy)\cosh(x(1-y))dy+O\left(\frac{1}{T}\right)=\frac{1}{2}x\sinh x+O\left(\frac{1}{T}\right).$$

Calculating D^*_{T-2} (which is $O(T)$), we have to find a second order approximation of (A.24) for $k=T-2$. To this end, we note that

$$\left(\frac{\alpha}{2}\right)^{T-3}=1+\frac{x^2}{2T}+O\left(\frac{1}{T^2}\right),\quad \alpha=2+O\left(\frac{1}{T^2}\right)\quad\text{and}\quad \alpha-\frac{2}{\alpha}=1+O\left(\frac{1}{T^2}\right).$$

Hence, (A.50) and (A.51) yield, inserting into (A.24),

$$(A.55) \qquad D^*_{T-2} = \left(1+\frac{x^2}{2T}\right)\left(2\cosh x + \frac{T}{x}\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right)\right) + O\left(\frac{1}{T}\right) =$$

$$= T\frac{\sinh x}{x} - \cosh x + O\left(\frac{1}{T}\right).$$

To complete the calculation of a_{10}, we need $\sum_{i=2}^{T-2} D^*_{i-2}\Delta D_{T-i+1}$ and $\sum_{i=2}^{T-2} \Delta D^*_{i-1} D_{T-i+1}$. (Since these sums are $O(T)$, the usual integral approximation technique does not suffice for our present purposes.) In order to evaluate the former sum, note that from (A.23)

$$\Delta D_k = D_k - D_{k-1} = \left(\frac{\alpha}{2}\right)^{k-2}\frac{1}{\alpha\sigma}\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)^2(1+\sigma)^{k-2} - \left(\frac{\alpha}{2}(1-\sigma)-1\right)^2(1-\sigma)^{k-2}\right).$$

Hence, since by (A.24)

$$D^*_k = \left(\frac{\alpha}{2}\right)^{k-1}\frac{1}{2\sigma}\left(\left((\alpha(1+\sigma)-\frac{2}{\alpha}\right)(1+\sigma)^{k-1} - \left(\alpha(1-\sigma)-\frac{2}{\alpha}\right)(1-\sigma)^{k-1}\right),$$

we have

$$D^*_{i-2}\Delta D_{T-i+1} = \left(\frac{\alpha}{2}\right)^{T-4}\frac{1}{2\alpha\sigma^2}\left(\left((\alpha(1+\sigma)-\frac{2}{\alpha}\right)(1+\sigma)^{i-3} - \left(\alpha(1-\sigma)-\frac{2}{\alpha}\right)(1-\sigma)^{i-3}\right)\cdot$$

$$\cdot\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)^2(1+\sigma)^{T-i-1} - \left(\frac{\alpha}{2}(1-\sigma)-1\right)^2(1-\sigma)^{T-i-1}\right).$$

Approximating in the usual manner,

$$(A.56) \qquad \alpha(1\pm\sigma) - \frac{2}{\alpha} = 1 \pm 2\frac{x}{T} + O\left(\frac{1}{T^2}\right)$$

and

$$(A.57) \qquad \frac{\alpha}{2}(1\pm\sigma) - 1 = \pm\frac{x}{T} + \frac{x^2}{2T^2} + O\left(\frac{1}{T^3}\right) = \pm\frac{x}{T}\left(1 \pm \frac{x}{2T} + O\left(\frac{1}{T^2}\right)\right).$$

It follows that

$$D^*_{i-2}\Delta D_{T-i+1} = \frac{1}{4}\left(1+\frac{x^2}{2T}\right)\left(\left(1+3\frac{x}{T}\right)(1+\sigma)^{T-4} + \left(1-3\frac{x}{T}\right)(1-\sigma)^{T-4} -\right.$$

$$\left. - \left(1+\frac{x}{T}\right)(1+\sigma)^{i-3}(1-\sigma)^{T-i-1} - \left(1-\frac{x}{T}\right)(1-\sigma)^{i-3}(1+\sigma)^{T-i-1}\right) + O\left(\frac{1}{T^2}\right).$$

Now, (A.51) implies

$$\sum_{i=2}^{T-2}(1+\sigma)^{i-3}(1-\sigma)^{T-i-1} = (1-\sigma)^{T-1}(1+\sigma)^{-3}\sum_{i=2}^{T-2}\left(\frac{1+\sigma}{1-\sigma}\right)^i =$$

$$= (1-\sigma)^{T-1}(1+\sigma)^{-3}\left(\frac{1+\sigma}{1-\sigma}\right)^2\frac{1-\left(\frac{1+\sigma}{1-\sigma}\right)^{T-3}}{1-\frac{1+\sigma}{1-\sigma}} = \frac{1-\sigma}{1+\sigma}\frac{(1+\sigma)^{T-3}-(1-\sigma)^{T-3}}{2\sigma} =$$

$$= \frac{T}{x}\left(1-2\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right),$$

and likewise

$$\sum_{i=2}^{T-2}(1-\sigma)^{i-3}(1+\sigma)^{T-i-1} = \frac{T}{x}\left(1+2\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right).$$

This, together with (A.50) and (A.51), yields

$$(A.58)\ \sum_{i=2}^{T-2} D^*_{i-2}\Delta D_{T-i+1} = \frac{1}{2}\left(1+\frac{x^2}{2T}\right)\left((T-3)\left(\cosh x - \frac{1}{2T}(8x\sinh x + x^2\cosh x) + \frac{3x}{T}\sinh x\right) -\right.$$

$$\left. - \frac{T}{x}\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right)\right) + O\left(\frac{1}{T}\right) =$$

$$= \frac{T}{2}\left(\cosh x - \frac{\sinh x}{x} - \frac{1}{T}x\sinh x\right) + O\left(\frac{1}{T}\right).$$

The calculation of $\sum_{i=2}^{T-2}\Delta D^*_{i-1}D_{T-i+1}$ very much follows the same lines. Indeed, (A.24) implies

$$\Delta D^*_k = D^*_k - D^*_{k-1} = \left(\frac{\alpha}{2}\right)^{k-2}\frac{1}{2\sigma}\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)\left(\alpha(1+\sigma)-\frac{2}{\alpha}\right)(1+\sigma)^{k-2}-\right.$$

$$\left. - \left(\frac{\alpha}{2}(1-\sigma)-1\right)\left(\alpha(1-\sigma)-\frac{2}{\alpha}\right)(1-\sigma)^{k-2}\right),$$

and by (A.23),

$$D_k = \left(\frac{\alpha}{2}\right)^{k-1}\frac{1}{\alpha\sigma}\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)(1+\sigma)^{k-1} - \left(\frac{\alpha}{2}(1-\sigma)-1\right)(1-\sigma)^{k-1}\right).$$

Thus,

$$\Delta D^*_{i-1}D_{T-i+1} = \left(\frac{\alpha}{2}\right)^{T-3}\frac{1}{2\alpha\sigma^2}\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)\left(\alpha(1+\sigma)-\frac{2}{\alpha}\right)(1+\sigma)^{i-3}-\right.$$

$$\left. - \left(\frac{\alpha}{2}(1-\sigma)-1\right)\left(\alpha(1-\sigma)-\frac{2}{\alpha}\right)(1-\sigma)^{i-3}\right)\cdot$$

$$\cdot\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)(1+\sigma)^{T-i} - \left(\frac{\alpha}{2}(1-\sigma)-1\right)(1-\sigma)^{T-i}\right),$$

and so, by (A.56) and (A.57),

$$\Delta D^*_{i-1}D_{T-i+1} = \frac{1}{4}\left(1+\frac{x^2}{2T}\right)\left(\left(1+3\frac{x}{T}\right)(1+\sigma)^{T-4} + \left(1-3\frac{x}{T}\right)(1-\sigma)^{T-4}+\right.$$

$$\left. + \left(1+2\frac{x}{T}\right)(1+\sigma)^{i-3}(1-\sigma)^{T-i} + \left(1-2\frac{x}{T}\right)(1-\sigma)^{i-3}(1+\sigma)^{T-i}\right) + O\left(\frac{1}{T^2}\right).$$

Now, using (A.51),

$$\sum_{i=2}^{T-2}(1+\sigma)^{i-3}(1-\sigma)^{T-i} = \frac{(1-\sigma)^2}{1+\sigma}\frac{(1+\sigma)^{T-3}-(1-\sigma)^{T-3}}{2\sigma} =$$

$$= \frac{T}{x}\left(1-3\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right)$$

and

$$\sum_{i=2}^{T-2}(1-\sigma)^{i-3}(1+\sigma)^{T-i} = \frac{T}{x}\left(1+3\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right).$$

Hence, by this, (A.50) and (A.51),

$$(A.59)\ \sum_{i=2}^{T-2}\Delta D^*_{i-1}D_{T-i+1} = \frac{1}{2}\left(1+\frac{x^2}{2T}\right)\left((T-3)\left(\cosh x - \frac{1}{2T}(6x\sinh x + x^2\cosh x) + \frac{3x}{T}\sinh x\right) + \right.$$

$$\left. + \frac{T}{x}\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x)\right)\right) + O\left(\frac{1}{T}\right) =$$

$$= \frac{T}{2}\left(\cosh x + \frac{\sinh x}{x} - \frac{6}{T}\cosh x\right) + O\left(\frac{1}{T}\right),$$

and, inserting (A.53), (A.54), (A.55), (A.58) and (A.59) in (A.29),

$$(A.60)\ \ \det P_0\cdot a_{10} = -2\cosh x + \frac{1}{2}x\sinh x + T\frac{\sinh x}{x} - \cosh x - \frac{T}{2}\left(\cosh x - \frac{\sinh x}{x}\right) + \frac{1}{2}x\sinh x -$$

$$-\frac{T}{2}\left(\cosh x + \frac{\sinh x}{x}\right) + 3\cosh x + O\left(\frac{1}{T}\right) =$$

$$= -T\left(\cosh x - \frac{\sinh x}{x}\right) + x\sinh x + O\left(\frac{1}{T}\right),$$

generalizing (A.30).

As for a_{20} we will, in view of (A.38), need a second order approximation of $\sum_{i=1}^{T-1} D^*_{i-1}D_{T-i}$. But from (A.23), (A.24), (A.56) and (A.57),

$$D^*_{i-1}D_{T-i} = \left(\frac{\alpha}{2}\right)^{T-3}\frac{1}{2\alpha\sigma^2}\left(\left(\alpha(1+\sigma)-\frac{2}{\alpha}\right)(1+\sigma)^{i-2} - \left(\alpha(1-\sigma)-\frac{2}{\alpha}\right)(1-\sigma)^{i-2}\right)\cdot$$

$$\cdot\left(\left(\frac{\alpha}{2}(1+\sigma)-1\right)(1+\sigma)^{T-i-1} - \left(\frac{\alpha}{2}(1-\sigma)-1\right)(1-\sigma)^{T-i-1}\right) =$$

$$= \frac{T}{4x}\left(1+\frac{x^2}{2T}\right)\left(\left(1+\frac{5}{2}\frac{x}{T}\right)(1+\sigma)^{T-3} - \left(1-\frac{5}{2}\frac{x}{T}\right)(1-\sigma)^{T-3} + \right.$$

$$\left. + \left(1+\frac{3}{2}\frac{x}{T}\right)(1+\sigma)^{i-2}(1-\sigma)^{T-i-1} - \left(1-\frac{3}{2}\frac{x}{T}\right)(1-\sigma)^{i-2}(1+\sigma)^{T-i-1}\right) + O\left(\frac{1}{T^2}\right).$$

But, since by (A.51),

$$\sum_{i=1}^{T-1}(1+\sigma)^{i-2}(1-\sigma)^{T-i-1} = \frac{1}{1+\sigma}\frac{(1+\sigma)^{T-1}-(1-\sigma)^{T-1}}{2\sigma} =$$

$$= \frac{T}{x}\left(1-\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(2x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right)$$

and

$$\sum_{i=1}^{T-1}(1-\sigma)^{i-2}(1+\sigma)^{T-i-1} = \frac{T}{x}\left(1+\frac{x}{T}\right)\left(\sinh x - \frac{1}{2T}(2x\cosh x + x^2\sinh x)\right) + O\left(\frac{1}{T}\right),$$

we have from (A.38), (A.50) and (A.51) that

$$(A.61)\quad \det P_0 \cdot a_{20} = \sum_{i=1}^{T-1} D_{i-1}^* D_{T-i} =$$

$$= \frac{T}{2x}\left(1+\frac{x^2}{2T}\right)\left((T-1)\left(\sinh x - \frac{1}{2T}(6x\cosh x + x^2\sinh x) + \frac{5}{2T}x\cosh x\right) + \frac{1}{2}\sinh x\right) + O(1) =$$

$$= \frac{T^2}{2}\frac{\sinh x}{x} - \frac{T}{4}\left(\cosh x + \frac{\sinh x}{x}\right) + O(1).$$

It remains to deal with $a_{10\times 10}$, and to this end it follows from (A.28) that

$$(\det P_0)^2 a_{10\times 10} = \mathrm{tr}\left((\det P_0 \cdot P_0^{-1}h_{1,0})^2\right) =$$

$$= \sum_{i=2}^{T-2}(D_{i-2}^*\Delta D_{T-i+1} + \Delta D_{i-1}^* D_{T-i+1})^2 + \sum_{i=1}^{T-1}\Delta^2 D_i^* D_{i-1}^* + {D_{T-2}^*}^2 + O(1).$$

Here, as usual

$$\sum_{i=2}^{T-2}(D_{i-2}^*\Delta D_{T-i+1} + \Delta D_{i-1}^* D_{T-i+1})^2 = T\int_0^1\Big(\sinh(xy)\sinh(x(1-y)) + \cosh(xy)\cosh(x(1-y))\Big)^2 dy =$$

$$= T\cosh^2 x + O(1)$$

and

$$\sum_{i=1}^{T-1}\Delta D_i^* D_{i-1}^* = T\int_0^1 \sinh^2(xy)dy = \frac{T}{2}\left(\frac{\sinh x}{x}\cosh x - 1\right) + O(1),$$

which together with (A.55) yields

$$(A.62)\qquad (\det P_0)^2 a_{10\times 10} = T\cosh^2 x + \frac{T}{2}\left(\frac{\sinh x}{x}\cosh x - 1\right) + \left(T\frac{\sinh x}{x} - \cosh x\right)^2 + O(1) =$$

$$= T^2\left(\frac{\sinh x}{x}\right)^2 + T\left(\cosh^2 x - \frac{3}{2}\frac{\sinh x}{x}\cosh x - \frac{1}{2}\right) + O(1).$$

Now, plugging in (A.52), (A.60), (A.61) and (A.62) into (A.49), Taylor expanding and simplifying, we get

$$g(x,u) = \left(\cosh x - \frac{1}{2T} x \sinh x\right)^{-\frac{5}{2}} \cdot$$

$$\cdot \left(-(1+2u)\left(\cosh x - \frac{1}{2T} x \sinh x\right)\left(\frac{T^2}{2}\frac{\sinh x}{x} - \frac{T}{4}\left(\cosh x + \frac{\sinh x}{x}\right)\right) + \right.$$

$$+ \frac{1}{4}\left(-T\left(\cosh x - \frac{\sinh x}{x}\right)^2 + x \sinh x\right)^2 +$$

$$\left. + \frac{T^2}{2}\left(\frac{\sinh x}{x}\right)^2 + \frac{T}{2}\left(\cosh^2 x - \frac{3}{2}\frac{\sinh x}{x}\cosh x - \frac{1}{2}\right)\right) + O(1) =$$

$$= T^2 (\cosh x)^{-\frac{5}{2}} \left(g_{11}(x) + u g_{12}(x) + \frac{1}{T}(g_{21}(x) + u g_{22}(x))\right) + O(1),$$

where

$$g_{11}(x) \stackrel{\text{def}}{=} \frac{1}{4}\cosh^2 x - \frac{\sinh x}{x}\cosh x + \frac{3}{4}\left(\frac{\sinh x}{x}\right)^2,$$

$$g_{12}(x) \stackrel{\text{def}}{=} -\frac{\sinh x}{x}\cosh x,$$

$$g_{21}(x) \stackrel{\text{def}}{=} \frac{1}{4}\cosh^2 x + \frac{1}{4} - \frac{3}{16} x \sinh x \cosh x + \frac{15}{16}\frac{\sinh^3 x}{x \cosh x} - \frac{1}{2}\frac{\sinh x}{x}\cosh x \quad \text{and}$$

$$g_{22}(x) \stackrel{\text{def}}{=} -\frac{1}{4}\cosh^2 x + \frac{3}{4} + \frac{1}{2}\frac{\sinh x}{x}\cosh x.$$

However, since

$$\int_0^\infty (1+2u)^{-\frac{T}{2}-1} du = \frac{1}{T} \quad \text{and} \quad \int_0^\infty u(1+2u)^{-\frac{T}{2}-1} du = \frac{1}{T(T-2)} = \frac{1}{T^2} + O\left(\frac{1}{T^3}\right),$$

(A.48) implies

$$(A.63) \qquad TE\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum S_{t-1}^2 \sum \varepsilon_t{}^2}\right) = \int_0^\infty x (\cosh x)^{-\frac{5}{2}} \left(g_{11}(x) + \frac{1}{T} g_2(x)\right) dx + 1 + \frac{2}{T} + O\left(\frac{1}{T^2}\right),$$

where

$$g_2(x) = g_{12}(x) + g_{21}(x) = \frac{1}{4}\cosh^2 x + \frac{1}{4} - \frac{3}{16} x \sinh x \cosh x + \frac{15}{16}\frac{\sinh^3 x}{x \cosh x} - \frac{3}{2}\frac{\sinh x}{x}\cosh x.$$

Hence, since

$$\int_0^\infty (\cosh x)^{-\frac{3}{2}} \sinh x dx = 2,$$

we obtain (6.3) from (A.63), which completes the proof.

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No. 1 Jacobsen, Martin: Weak Convergence of Autoregressive Processes.

No. 2 Larsson, Rolf: Bartlett Corrections for Unit Root Test Statistics.