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3
Preprint
March
1993

IMS
Institute of Mathematical Statistics
University of Copenhagen

ISSN 0902-8846

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Preprint 1993 No. 3

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN

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Abstract

This paper deals with the problem of testing for a unit root in the framework of the near-integrated process proposed by Phillips (1987b). The convergence of the near-integrated process to the Ornstein-Uhlenbeck process is a key to this paper. A sequence of Fourier type transformations $\tilde{Z}_k, k = 1, \ldots$ associated with the Karhunen-Loève expansion of the Brownian motion is considered. The likelihood functions of the family of the Ornstein-Uhlenbeck processes based on $\tilde{Z}^{(K)} = (\tilde{Z}_1, \ldots, \tilde{Z}_K), K = 1, \ldots$ and their approximations are derived. Two tests for a unit root against local alternatives are given as the discrete analogues of those for the Brownian motion against the Ornstein-Uhlenbeck process. Our tests are shown to be locally efficient in the sense that the asymptotic distribution under either of the null and the local alternative hypotheses is the same as that of the exact loglikelihood ratio statistic of the Gaussian AR(1) models. The consistency of the tests are also given.

Key Words: *Brownian Motion, Karhunen-Loève Expansion, Locally Efficient Test, Near-Integrated Process, Ornstein-Uhlenbeck Process, Unit Root*

1 Introduction

Consider a time series model generated by

$$y_t = \rho\, y_{t-1} + w_t \qquad (t = 1, 2, \ldots, T), \tag{1}$$

where $y_0 = 0$ and w_t is a zero mean error process satisfying suitable assumptions. The parameter ρ dominantly specifies the long-run behavior of the time series. This paper deals with testing for a unit root, that is $\rho = 1$, while the alternatives $|\rho| < 1$ and $|\rho| > 1$ mean stability and explosiveness, respectively.

The problem of testing for a unit root has been attracting a great deal of attention of many authors. Fuller (1976) brought the problem explicitly into the literature. Dickey & Fuller (1979,1981) examined the properties of the Gaussian maximum likelihood estimator of ρ and of the likelihood ratio statistics assuming the autoregressive models for y_t in (1). Said & Dickey (1981,1984), Solo (1984) and Hall (1989) are among those who proposed a variety of testing procedures assuming general classes of ARMA models for y_t and investigated their asymptotic behavior under the null hypothesis. Phillips (1987a) proposed a test procedure assuming a quite general class of stochastic processes for the error term.

Suppose, for a while, that w_t is a zero mean Gaussian white noise with $Ew_t^2 = \sigma_w^2$. The maximum likelihood estimator (MLE) of ρ is the least squares estimator (LSE) $\hat{\rho}$, and the loglikelihood ratio statistic λ for the hypothesis $\rho = \rho_0$ is approximately $\sum y_{t-1}^2(\hat{\rho} - \rho_0)^2/s^2$, the squared deviation of the LSE adjusted by the observed Fisher-Information, where $s^2 = T^{-1}\sum(y_t - \hat{\rho}y_{t-1})^2$. The asymptotic distribution of $\hat{\rho}$ (White (1958), Dickey & Fuller (1979)) and λ (Dickey & Fuller (1981)) under the null hypothesis $\rho = 1$ are ratios of some functionals of the standard Brownian motion $\{W(\tau), \tau \in [0,1]\}$: $T(\hat{\rho} - \rho) \xrightarrow{d} \int W(\tau)dW(\tau)/\int W(\tau)^2 d\tau$ and $\lambda \xrightarrow{d} \{\int W(\tau)dW(\tau)\}^2/\int W(\tau)^2 d\tau$, respectively, where and throughout this paper '$\xrightarrow{d}$' denotes the convergence in distribution, '$\xrightarrow{p}$' denotes the convergence in probability and integrals are over the interval $[0, 1]$ unless explicitly indicated. On the other hand, it is well known that $\{T/(1-\rho^2)\}^{1/2}(\hat{\rho}-\rho) \xrightarrow{d} N(0,1)$, if $|\rho| < 1$. When $|\rho| > 1$, it is also shown that the limiting distribution of $\hat{\rho}$ is a Cauchy distribution (White (1958)) and that of λ is $\chi^2(1)$ distribution (White (1958) and Anderson (1959)).

The gap of the asymptotic distributions of this kind is filled with the class of near-integrated processes, in which the parameter ρ is defined by $\rho = \exp(-\alpha/T)$ or by

$$\rho = 1 - \alpha/T, \tag{2}$$

which are equivalent in the limit. This class of models was considered by many authors, e. g. Ahtola & Tiao (1984), Chan & Wei (1987), Phillips(1987b), and Phillips & Perron (1988), mainly as local alternatives in the analysis of a unit root and has been useful in power studies.

Define the normalized near-integrated process $Y_T(\tau) = y_{[T\tau]}/T^{1/2}$, $\tau \in [0,1]$, where $[a]$ denotes the largest integer not exceeding a. Then it is a special case of Chan (1988) that $Y_T(\tau)/\sigma_w$ converges weakly to the Ornstein-Uhlenbeck process in $D[0,1]$, the space of functions x on $[0,1]$ that are right-continuous and have left-hand limits. The Ornstein-Uhlenbeck process is a one-parameter family of stochastic processes $\{X_\alpha(\tau), -\infty < \alpha < \infty, 0 \leq \tau \leq 1\}$ defined by the Wiener integral

$$X_\alpha(\tau) = \int_0^\tau e^{-\alpha(\tau-\tau')} dW(\tau') \tag{3}$$

and is known to be the solution to the stochastic differential equation

$$dX(\tau) + \alpha X(\tau) d\tau = dW(\tau), \tag{4}$$

with initial condition $X(0) = 0$.

Rewriting (1) with (2) as

$$\Delta y_t/T^{1/2} + \alpha(y_{t-1}/T^{1/2})(1/T) = w_t/T^{1/2}, \tag{5}$$

where Δ is the difference operator, we see that (4) is a continuous time parameter analogue of (1). Hence we could expect a close relation between the inference on the near-integrated process (1) with ρ defined by (2) and that on the Ornstein-Uhlenbeck process (3).

Indeed, it has been well established that the loglikelihood function of $X(\tau) = X_\alpha(\tau)$ with respect to the Wiener measure is given by

$$\alpha^2 \int X(\tau)^2 d\tau + 2\alpha \int X(\tau) dX(\tau), \tag{6}$$

where the integral in the second term is interpreted as the Ito's integral : $\int X(\tau)dX(\tau) = (X(1)^2 - 1)/2$. The MLE of α for the Ornstein-Uhlenbeck process is then given by $\tilde{\alpha}_c = -\int X(\tau)dX(\tau)/\int X(\tau)^2 d\tau$, while the MLE of α for (5) is the LSE $\hat{\alpha} = -T^{-1}\sum y_{t-1}\Delta y_t/(T^{-2}\sum y_{t-1}^2)$. Obviously, $\hat{\alpha}$ is the discrete time analogue of $\tilde{\alpha}_c$. We can also show that

$$\hat{\alpha} \xrightarrow{d} \hat{A}_\alpha = -\int X_\alpha(\tau)dX_\alpha(\tau)/\int X_\alpha(\tau)^2 d\tau$$

i. e. to the same distribution as $\tilde{\alpha}_c$. First the denominator of $\hat{\alpha}$ is expressed as $\int Y_T(\tau)^2 d\tau$ which is a continuous functional of $Y_T(\tau)$, and thus converges in distribution to its continuous time counterpart $\sigma_w^2 \int X_\alpha(\tau)^2 d\tau$, since $Y_T(\tau) \xrightarrow{d} \sigma_w X_\alpha(\tau)$ in $D[0,1]$ and hence the continuous mapping theorem in Billingsley (1968) applies. We can also show the convergence of the numerator as follows, though it is rather indirect. It is easy to see that twice the numerator of $\hat{\alpha}$ is

$$y_T^2/T - \sum(\Delta y_{t-1})^2/T, \tag{7}$$

of which the first term converges in distribution to $\sigma_w^2 X_\alpha(1)^2$ by the same argument as above and the second term converges to $\sigma_w^2 = Ew_t^2$ in probability by the law of large numbers. Therefore the numerator of $\hat{\alpha}$ converges in distribution to its continuous time counterpart too. Though we have only discussed the estimation of

α so far, it is clear that similar argument could be given to the likelihood ratio test statistics λ for a unit root to show that

$$\lambda \xrightarrow{d} \Lambda_\alpha = \frac{\{\int X_\alpha(\tau)dX_\alpha(\tau)\}^2}{\int X_\alpha(\tau)^2 d\tau}.$$

Perron (1991) argued this kind of similarity in detail.

It is worth noting that the asymptotic null distribution of each of the test statistics proposed so far is either of $\hat{A}_0$ and Λ_0 according as it is essentially an estimate of ρ or a likelihood ratio type statistic. Hence extensive analyses of these two distribution are given by Satchell (1984), Evans & Savin (1981) and others. There does not seem to have been enough power studies on previously proposed test procedures under local alternatives, especially in the parametric framework. However, it might well be conjectured that every reasonable inference has the same asymptotic performance as the likelihood based procedure under the simplest case. Therefore we say that an estimator of ρ or a test for unit root is locally efficient if it is asymptotically distributed as $\hat{A}_\alpha$ or Λ_α, respectively, for all finite α.

Now let us proceed further along this line by weakening the assumption for the error term. Phillips (1987b) proved under a set of fairly mild conditions on w_t, which are satisfied by the stationary ARMA models, a wide class of linear processes and others, that the normalized near-integrated process $Y_T(\tau)$ converges to $\sigma X_\alpha(\tau)$ for each $\tau \in [0,1]$, where the so-called long run variance $\sigma^2 = \lim_{T\to\infty} V(\sum_{t=1}^T w_t)/T$ is assumed to exist and be strictly positive. Later we show the weak convergence of $Y_T(\tau)$ in the space $D[0,1]$ by a slight modification of his proof. Hence arguments similar to what we have seen above for the case of Gaussian white noise error can be traced with σ_w^2 replaced by σ^2 except one point ; $\sum(\Delta y_t)^2/T$ in the alternative expression (7) of the numerator of $\hat{\alpha}$ converges in probability to $\sigma_0^2 = \lim_{T\to\infty} T^{-1}\sum Ew_t^2$ instead of σ^2. Phillips (1987a) and Phillips & Perron (1988) proposed to collect the difference by estimating σ^2 using the sample autocovariance functions.

In this paper we shall persist in the analogy between the near-integrated process and the Ornstein-Uhlenbeck process in order to construct locally efficient tests for a unit root. The basic idea is the following : Firstly, we pay attention to the fact that $Y_T(\tau)$ converges, irrespective of the probabilistic structure of the error term w_t, to the same limit which is specified by the parameter α and σ^2. This necessarily means that the *short range properties* of the data has little information on α and that inference on α would be made properly based only on the *long range properties* of the data at least in the asymptotic limit. That the behavior of the periodograms with near zero frequencies reflects the long range property of time series has been recognized and made use of in several papers in relation to co-integration analysis, see Phillips & Ouliaris (1988) and Phillips (1988) for example. However, the periodograms employed in all of these papers are the conventional ones which are based on the spectral theory of stationary processes. It is evident that the spectral theory is useful because of its property of simultaneous orthogonalization of the covariance matrices of all the stationary processes. Although, as is seen later, in the family of near-integrated processes this eminent property unfortunately does not hold, we could expect that the transformations associated with the orthogonal expansion of the Brownian motion known as the Karhunen-Loève expansion, e. g. Wong (1971), would still enjoy some good property for inference of the Ornstein-Uhlenbeck process and the near-integrated process. Secondly, the breakdown of the analogy comes from the irregularity comprised in the Ito's integral in (6) which hinders the direct application of the continuous mapping theorem. So if we have an efficient alternative estimator of α for the parameter of the Ornstein-Uhlenbeck process as rather a smooth functional of the sample path, then we might well expect that its discrete time analogue has the same asymptotic property as $\tilde{\alpha}_c$. From this point, the use of periodograms with near zero frequencies would also have a merit because they are rather *smooth* functions of $Y_T(\tau)$.

This idea is implemented in the rest of the paper as follows: Section 2 is preliminary, where we discuss convergence of near-integrated process to the Ornstein-Uhlenbeck process and the orthogonal expansion of the Ornstein-Uhlenbeck process. In section 3, we derive an approximate expression to its likelihood function in the frequency domain, that is, in terms of the transformations corresponding to the Karhunen-Loève expansion of the Brownian motion. Two test statistics for the Brownian motion against the Ornstein-Uhlenbeck process are given based on the approximate likelihood and their properties are discussed. Section 4 deals with tests for a unit root which are naturally suggested by the findings of the preceding section. We show local efficiency and consistency of our tests. In the final section, we give miscellaneous remarks.

2 Preliminaries

We consider the near-integrated process generated by (1) with ρ defined by (2). Following the framework employed by Phillips (1987b), we assume that the error terms w_t satisfy the following conditions with $\beta > 2$:

(i). $E(w_t) = 0$ for all $t > 0$ and $\sup_t E|w_t^{2\beta}| < \infty$.

(ii). $\sigma^2 = \lim_{t\to\infty} T^{-1}V(\sum_{t=1}^{T} w_t)$ exists and is strictly positive.

(iii). $\{w_t\}_1^\infty$ is strongly mixing with mixing coefficients γ_m satisfying $\sum_1^\infty \gamma_m^{1-2/\beta} < \infty$.

We show weak convergence of the near-integrated process to the Ornstein-Uhlenbeck process which is an extension of Lemma 1 (a) of Phillips (1987b) to function space. Let $W_T(\tau) = \sum_{t=1}^{[T\tau]} w_t/T^{1/2}, \tau \in [0,1]$, be the normalized sum of the error terms.

Lemma 1 *Suppose that $W_T(\cdot)$ converges weakly to the Brownian motion $\sigma W(\cdot)$ and $\sup_t E|w_t| < \infty$, then the process $Y_T(\tau) = y_{[T\tau]}/T^{1/2}, \tau \in [0,1]$, converges weakly to $\sigma X_\alpha(\cdot)$, the Ornstein-Uhlenbeck process.*

Herrndorf (1984) proved that under the above assumptions (i)-(iii) $W_T(\cdot)$ converges weakly to $\sigma W(\cdot)$. Hence this lemma implies weak convergence of $Y_T(\cdot)$ to $\sigma X_\alpha(\cdot)$, which is the result that our argument in this paper depends upon.

Proof: By definition of $Y_T(\tau)$, we have

$$\begin{aligned} Y_T(\tau) &= y_{[T\tau]}/T^{1/2} = \sum_{t=1}^{[T\tau]}(1-\frac{\alpha}{T})^{[T\tau]-t} w_t/T^{1/2} \\ &= \int_0^\tau e^{-\alpha(\tau-\tau')}dW_T(\tau') + \sum_{t\leq[T\tau]} c(\tau,t)w_t/T^{1/2}, \end{aligned} \tag{8}$$

where we put $c(\tau,t) = \{(1-\alpha/T)^{[T\tau]-t} - e^{-\alpha(\tau-t/T)}\}$. By integration by parts, we have

$$\int_0^\tau e^{-\alpha(\tau-\tau')}dW_T(\tau') = W_T(\tau) + \alpha\int_0^\tau e^{-\alpha(\tau-\tau')}W_T(\tau')d\tau'.$$

Since a function $f : D[0,1] \mapsto D[0,1]$ which is defined by $f(x)(\tau) = x(\tau) + \alpha\int_0^\tau e^{-\alpha(\tau-\tau')}x(\tau')d\tau'$, is continuous on the support of the Wiener measure $C[0,1] \subset D[0,1]$ and $W_T(\cdot) \xrightarrow{d} \sigma W(\cdot)$, the first term of the RHS of (8) converges weakly to

$$\sigma\{W(\tau) + \alpha\int_0^\tau e^{-\alpha(\tau-\tau')}W(\tau')d\tau'\},$$

which turns out to be $\sigma X_\alpha(t)$ again by integration by parts.

Thus it suffices to show that the remainder term, say $R(\tau)$, converges in probability to 0 in the function space $D[0,1]$. Since $c(\tau,t)$ is bounded by C/T uniformly in τ and t, where C is some constant, it holds that $|R(\tau)| \leq CT^{-3/2}\sum_{t=1}^T |w_t|$ for every $\tau \in [0,1]$. Taking the expectation we have

$$E\sup_\tau |R(\tau)| \leq C\sup_t E|w_t|/T^{1/2} \to 0,$$

which implies the desired statement.$\square$

Hereafter in this section we make a brief review of the orthogonal expansion of the Ornstein-Uhlenbeck process. By definition, it is obvious that $EX_\alpha(\tau) = 0$. Simple calculation using expression (3) leads to the following formula for the covariance function : $R_\alpha(\tau, \tau') = EX_\alpha(\tau)X_\alpha(\tau') = \alpha^{-1}\exp\{-\alpha(\tau \vee \tau')\}\sinh\{\alpha(\tau \wedge \tau')\}$, where $\tau \vee \tau'$ and $\tau \wedge \tau'$ are the maximum and the minimum of τ and τ', respectively. $X_0(\tau)$ is $W(\tau)$ the standard Brownian motion, and $\lim_{\alpha\to 0} R_\alpha(\tau, \tau') = \tau \wedge \tau'$, which is the familiar covariance function of the standard Brownian motion.

The following Karhunen-Loève expansion, see Loève (1978), of the Ornstein-Uhlenbeck process was derived by Chan (1988) : The integral equation with $R_\alpha(\tau, \tau')$ as the kernel function

$$\int R_\alpha(\tau, \tau')\psi(\tau')d\tau' = \lambda\psi(\tau) \tag{9}$$

reduces to the differential equation $\ddot{\psi}(\tau) + (1/\lambda - \alpha^2)\psi(\tau) = 0$, with boundary conditions $\psi(0) = 0$ and $\alpha\psi(1) + \dot{\psi}(1) = 0$ where a dot over the symbol of a function denotes differentiation with respect to τ. For $\alpha > -1$, the sequence of solutions to this equation is given by $\lambda_k = 1/(\alpha^2 + \omega_k^2)$ and $c_k \sin\omega_k\tau$, where ω_k, $k = 1, 2, \ldots$, are increasing positive solutions to $\tan\omega = -\omega/\alpha$ and c_k's are normalizing constants. Random variables $Z_k = \int X_\alpha(\tau)c_k \sin\omega_k\tau d\tau, k = 1, 2, \ldots$, are mutually independent and satisfy $EZ_k^2 = \lambda_k$ and $X_\alpha(\tau) = \sum_{k=1}^{\infty} Z_k c_k \sin\omega_k\tau$, where, and in the sequel of this paper, the infinite sum of random variables is understood as the limit in the mean square unless otherwise mentioned. When $\alpha \le -1$, an extra eigenvalue λ_0 and eigenfunction $\psi_0(\tau)$ of (9) are given by $\lambda_0 = 1/(\alpha^2 - \omega_0^2)$ and $\psi_0(\tau) = c_0 \sinh\omega_0\tau$ where ω_0 is the unique positive solution to $\tanh\omega = -\omega/\alpha$ if $\alpha < -1$ and by $\lambda_0 = 1$ and $\psi_0(\tau) = \sqrt{2}\tau$ if $\alpha = -1$ respectively.

3 Likelihood of Ornstein-Uhlenbeck processes

In this section, we shall derive a sequence of approximations to the likelihood function of the Ornstein-Uhlenbeck process in the frequency domain, which turns out to be equivalent to the exact one in the limit.

We wish that the orthogonal transformations discussed in the previous section would take the role of the conventional periodogram in the analysis of Ornstein-Uhlenbeck processes. This is in fact impossible since the transformations depend on the parameter α. Nevertheless, we fix a transformation corresponding to $\alpha = 0$, because it is sensible to consider the null case as special. We note that, when $\alpha = 0$, the eigenvalues of (9) have an explicit form $\lambda_k = C_k^{-2}$ where $C_k = \pi(k - 1/2)$ $k = 1, 2, \ldots$, and corresponding eigenfunctions are given by

$$\psi_k(t) = \sqrt{2} \sin C_k \tau.$$

Assume that an observation $X(\tau)$ on $\sigma X_\alpha(\tau)$ is given. According to the above eigenfunctions, we consider the following sequence of random variables :

$$\tilde{Z}_k = \int X(\tau)\sqrt{2} \sin C_k \tau d\tau, \qquad k = 1, 2, \ldots. \tag{10}$$

We make a brief remark on some properties of the above transformation. Let $g(\tau)$ be any continuous function defined on $[0, 1]$ satisfying $g(0) = 0$ and define an extension $g^*(\tau), \tau \in [-2, 2]$ of g by $g^*(\tau) = g(\tau), \tau \in [0, 1]$, $g^*(2 - t) = g(\tau), \tau \in (1, 2]$ and $g^*(\tau) = g^*(-\tau), \tau \in [-2, 0)$. Then by symmetry some of the Fourier coefficients vanish, and the Fourier series for $g^*(\tau)$ is formally given by

$$\sum_{k=1}^{\infty} c_k \sqrt{2} \sin C_k \tau,$$

where $c_k = \int_0^1 g(\tau)\sqrt{2} \sin C_k \tau$. This series converges if suitable additional condition is satisfied. Since $g(\tau) \in L^2[0, 1]$, the Parseval's equality

$$\int g(\tau)^2 d\tau = \sum_{k=1}^{\infty} c_k^2$$

holds.

We derive some properties of $\tilde{Z}_k$'s. Substituting (3) into (10), we obtain

$$\begin{aligned}\tilde{Z}_k &= \sigma \int \int_0^\tau e^{-\alpha(\tau-\tau')} dW(\tau')\sqrt{2}\sin C_k\tau d\tau \\ &= \sigma \int dW(\tau') \int_{\tau'}^1 e^{-\alpha(\tau-\tau')}\sqrt{2}\sin C_k\tau d\tau \\ &= \sigma \int f_k(\tau')dW(\tau'),\end{aligned}$$

$k = 1, 2, \ldots$, where we put

$$f_k(\tau) = \frac{\sqrt{2}}{\alpha^2 + C_k^2}\{C_k \cos C_k\tau + \alpha \sin C_k\tau - (-1)^{(k-1)}\alpha e^{-\alpha(1-\tau)}\}.$$

The following formula for the covariance $E\tilde{Z}_k\tilde{Z}_l = \sigma^2 \int f_k(\tau)f_l(\tau)d\tau$ is obtained by straightforward calculation :

$$\sigma^{-2}E\tilde{Z}_k\tilde{Z}_l = \frac{\delta_{kl}}{\alpha^2 + C_k^2} - (-1)^{k+l}\frac{\alpha(1+e^{-2\alpha})}{(\alpha^2 + C_k^2)(\alpha^2 + C_l^2)}, \tag{11}$$

where $k, l = 1, 2, \ldots$, and δ_{kl} is the conventional Kronecker's delta.

Next we show that

$$\sum_{k=1}^K \tilde{Z}_k^2 \xrightarrow{p} \sigma^2 \int X_\alpha(\tau)^2 d\tau \tag{12}$$

as $K \to \infty$ and that

$$\sigma X_\alpha(\tau) = \sum_{k=1}^\infty \tilde{Z}_k \sqrt{2}\sin C_k\tau. \tag{13}$$

By simple calculation we obtain

$$E\int X_\alpha(\tau)^2 d\tau = \int R_\alpha(\tau,\tau)d\tau = \frac{1}{4\alpha^2}(e^{-2\alpha} + 2\alpha - 1). \tag{14}$$

It is easy to obtain the following Fourier coefficients of the function $\sinh\alpha\tau$: $\int \sinh\alpha\tau\sqrt{2}\sin C_k\tau d\tau = \cosh\alpha\,(-1)^{k-1}\sqrt{2}\alpha/(\alpha^2+C_k^2)$. Since $\sinh\alpha\tau$ satisfies Dini's condition at $\tau = 1$, we have a convergent Fourier series $\sinh\alpha = \cosh\alpha \sum_{k=1}^\infty 2\alpha/(\alpha^2 + C_k^2)$. Thus we have obtained

$$\tanh\alpha = \sum_{k=1}^\infty \frac{2\alpha}{\alpha^2 + C_k^2}. \tag{15}$$

By the Parseval's equality we also have

$$\int \sinh^2 \alpha\tau d\tau = 2\cosh^2 \alpha \sum_{k=1}^{\infty} (\alpha^2 + C_k^2)^{-2}.$$

It is shown by straightforward manipulation using this identity, (11), (14) and (15) that

$$\sum_{k=1}^{\infty} E\tilde{Z}_k^2 = \sigma^2 \int EX_\alpha(\tau)^2 d\tau. \tag{16}$$

Now define

$$X^{(K)}(\tau) = \sum_{k=1}^{K} \tilde{Z}_k \sqrt{2} \sin C_k \tau,$$

then we easily obtain

$$\int \{\sigma X_\alpha(\tau) - X^{(K)}(\tau)\}^2 d\tau = \sigma^2 \int X_\alpha(t)^2 d\tau - \sum_{k=1}^{K} \tilde{Z}_k^2 \geq 0.$$

By virtue of (16) the expectation of this tends to zero as $K \to \infty$, which implies (12).

Inspecting the formula (11), it is obvious that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |E\tilde{Z}_k \tilde{Z}_l| < \infty. \tag{17}$$

From this and that $|\psi_k(\tau)| \leq \sqrt{2}$ we see that the sequence $X^{(K)}(\tau), K = 1, \ldots,$ is fundamental for all τ. Hence

$$X^{(\infty)}(\tau) = \lim_{K \to \infty} X^{(K)}(\tau).$$

exists. Since the bound (17) is independent of τ it is easy to see that the above convergence is uniform in $\tau \in [0, 1]$. Thus $X^{(\infty)}(\cdot)$ is continuous in mean square since it is the limit of uniformly convergent sequence of mean square continuous processes $X^{(K)}(\cdot)$. On the other hand, we have

$$\begin{aligned}
&\int E\{\sigma X_\alpha(\tau) - X^{(\infty)}(\tau)\}^2 d\tau \\
&= \int E[\{\sigma X_\alpha(\tau) - X^{(K)}(\tau)\} + \{X^{(K)}(\tau) - X^{(\infty)}(\tau)\}]^2 d\tau \\
&\leq 2[\int E\{\sigma X_\alpha(\tau) - X^{(K)}(\tau)\}^2 d\tau + \int E\{X^{(K)}(\tau) - X^{(\infty)}(\tau)\}^2 d\tau] \\
&\to 0.
\end{aligned}$$

Therefore, we have shown that

$$\int E\{\sigma X_\alpha(\tau) - X^{(\infty)}(\tau)\}^2 d\tau = 0.$$

Since the integrand of the above equation is nonnegative and $X_\alpha(\tau)$ and $X^{(\infty)}(\tau)$ are both mean square continuous, we have proved (13).

Let $\tilde{Z}^{(K)}$ denote the vector $(\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_K)'$ and $\sigma^2 G_K(\alpha)$ its covariance matrix, $K = 1, 2, \ldots$. As the Ornstein-Uhlenbeck process is by definition a zero mean Gaussian process, the distribution of $\tilde{Z}^{(K)}$ is multivariate normal with $E\tilde{Z}^{(K)} = 0$ and its loglikelihood function $\log f(\tilde{Z}^{(K)}; \alpha, \sigma^2)$ is given by

$$-\frac{K}{2}\log\sigma^2 - \frac{1}{2}\log|G_K(\alpha)| - \frac{1}{2\sigma^2}\tilde{Z}^{(K)\prime} G_K(\alpha)^{-1}\tilde{Z}^{(K)}. \tag{18}$$

In view of (13), we might consider (18) as an approximation to the exact loglikelihood (6). In the following, we derive explicit forms for $G_K(\alpha)^{-1}$ and $|G_K(\alpha)|$ and their limiting properties as $K \to \infty$. By the formula (11) for the covariances, we see that $G_K(\alpha)$ is expressed as $G_K(\alpha) = diag\{\mathbf{d}_{K0}\} - \alpha(1 + e^{-2\alpha})\ \mathbf{d}_{K1}(\alpha)\ \mathbf{d}'_{K1}(\alpha)$, where $\mathbf{d}_{K0}(\alpha)$ and $\mathbf{d}_{K1}(\alpha)$ are K-dimensional vectors with k-th entries $1/(\alpha^2 + C_k^2)$ and $(-1)^{k-1}/(\alpha^2 + C_k^2)$, respectively. Its inverse and determinant are easily obtained:

$$G_K(\alpha)^{-1} = diag\{\mathbf{d}_{K0}(\alpha)\}^{-1} + \alpha B_K(\alpha)\ \mathbf{j}_K\ \mathbf{j}'_K \tag{19}$$

and

$$|G_K(\alpha)| = |diag\{\mathbf{d}_{K0}(\alpha)\}|\ \{1 - \alpha(1 + e^{-2\alpha})\ \mathbf{j}'_K \mathbf{d}_{K1}(\alpha)\ \}, \tag{20}$$

where we put $\mathbf{j}_K = diag\{\mathbf{d}_{K0}(\alpha)\}^{-1}\mathbf{d}_{K1}(\alpha) = (1, -1, \ldots, (-1)^{K-1})'$ and

$$B_K(\alpha) = \frac{(1 + e^{-2\alpha})}{1 - \alpha(1 + e^{-2\alpha})\,\mathbf{j}_K'\mathbf{d}_{K1}(\alpha)}.$$

Thus we have obtained the following explicit form for (18) :

$$-\frac{K}{2}\log\sigma^2 + \frac{1}{2}[\sum \log(\alpha^2 + C_k^2) - \log\{1 - \alpha(1 + e^{-2\alpha})\sum 1/(\alpha^2 + C_k^2)\}]$$
$$-\frac{1}{2\sigma^2}[\sum_{k=1}^{K}(\alpha^2 + C_k^2)\tilde{Z}_k^2 + \frac{\alpha(1 + e^{-2\alpha})}{1 - \alpha(1 + e^{-2\alpha})\sum 1/(\alpha^2 + C_k^2)}\{\sum_{k=1}^{K}(-1)^k \tilde{Z}_k\}^2], \quad (21)$$

where the summation is over $k = 1$ to K.

The apparent complexity of the likelihood function can be considerably reduced in the limit $K \to \infty$ as follows: Integrating both side of (15) over the interval $[0, \alpha]$, the following formula for the logarithm of the infinite product is also derived :

$$\log\cosh\alpha = \sum_{k=1}^{\infty}\log(1 + \frac{\alpha^2}{C_k^2}). \quad (22)$$

Let us define $r_K(\alpha) = \sum_{k=K+1}^{\infty} 2\alpha/(\alpha^2 + C_k^2)$ the truncation error of (15). Then, noting that $\mathbf{j}_K'\mathbf{d}_{K1}(\alpha) = \sum_{k=1}^{K} 1/(\alpha^2 + C_k^2)$, it is easy to derive from (15) that $B_K(\alpha) = 2/\{1 + r_K(\alpha)\}$ and from (22) that $\log|G_K(\alpha)|/|G_K(0)| = -\alpha + R_K(\alpha) + \log\{1 + r_K(\alpha)\}$, where $R_K(\alpha) = \int_0^\alpha r_K(a)da$. Evaluating the integral of $1/(x^2 + \alpha^2)$, we obtain a bound $|r_K(\alpha)| < (2/\pi)\tan^{-1}\{|\alpha|/C_K\}$. From this it follows that $r_K(\alpha) \to 0$ as $\alpha/K \to 0$. We then have

$$\lim_{K\to\infty} B_K(\alpha) = 2 \quad (23)$$

and

$$\lim_{K\to\infty} \log\{|G_K(\alpha)|/|G_K(0)|\} = -\alpha. \quad (24)$$

We make a few remarks concerning what are obtained above. Firstly, we see from (21) that the triplet $(\sum_{k=1}^{K}\tilde{Z}_K^2, \sqrt{2}\sum_{k=1}^{K}(-1)^k\tilde{Z}_k, \sum_{k=1}^{K}C_k^2\tilde{Z}_k^2/K)$, which we denote by $(\tilde{U}_K, \tilde{V}_K, \tilde{S}_K)$, is sufficient for parameters (α, σ^2). The extra factors in the definition of $\tilde{V}_K$ and $\tilde{S}_K$ are for later convenience.

Secondly, note that (12) and (13) imply that

$$\tilde{U}_K \xrightarrow{p} \sigma^2 \int_0^1 X_\alpha(\tau)^2 d\tau \quad \text{and} \quad \lim_{K\to\infty} \tilde{V}_K = \sigma X_\alpha(1).$$

Using the fact that $\sum(\alpha^2 + C_k^2)^{-1}$ and $\sum(\alpha^2 + C_k^2)^{-2}$ are convergent, we can also show that $E(\tilde{S}_K) = \sigma^2 \mathrm{tr}\{G_K(0)^{-1}G_K(\alpha)\}/K = \sigma^2 + O(1/K)$ and $V(\tilde{S}_K) = 2\sigma^4 \mathrm{tr}\{G_K(0)^{-1}G_K(\alpha)\}^2/K^2 = O(1/K)$. Thus, as $K \to \infty$

$$(\tilde{U}_K, \tilde{V}_K^2, \tilde{S}_K) \xrightarrow{d} \sigma^2(\int X_\alpha(\tau)^2 d\tau, X_\alpha(1)^2, 1). \tag{25}$$

Thirdly, the equivalence of (18) to (6) in the limit $K \to \infty$ can be shown roughly as follows: Suppose that the convergence (25) is almost sure, then in the space l^2 the distribution of the sequence $\tilde{Z}^\infty = (\tilde{Z}_1, \tilde{Z}_2, \ldots)$ is concentrated on the set $\{(z_1, z_2, \ldots) \in l^2; \lim_{K\to\infty} \sum_1^K C_k^2 z_k^2/K = \sigma^2\}$. By (23), (24) and (25) it is easily seen that the likelihood ratio $\log\{f(\tilde{Z}^{(K)}, \alpha, \sigma^2)/f(\tilde{Z}^{(K)}, 0, \sigma^2)\}$ converges to (6) on this set . This gives the equivalence of inference based on $\tilde{Z}^{(\infty)}$ to that on the whole sample path.

Hereafter we propose two tests for $\alpha = 0$ based on $\tilde{Z}^{(K)}$ for finite K, which have the same property as Λ_α, the exact likelihood ratio test, in the limit $K \to \infty$. It is quite natural that our tests should be based only on the minimal sufficient statistics $(\tilde{U}_K, \tilde{V}_K^2, \tilde{S}_K)$.

The first one is naive. Let $\tilde{\nu}^{(1)}(u, v) = (v-1)^2/4u$. It is obvious from (25) that

$$\tilde{\nu}^{(1)}(\tilde{U}_K/\tilde{S}_K, \tilde{V}_K^2/\tilde{S}_K) \xrightarrow{d} \Lambda_\alpha$$

for all finite α as $K \to \infty$. Thus the test based on $\tilde{\nu}^{(1)}$ has asymptotically the same property as the exact test.

Secondly, we make more use of what we have seen above on the likelihood function of $\tilde{Z}^{(K)}$. The exact loglikelihood function (21) is highly complicated. However, in view of (19),(20),(23) and (24) it is appropriate to approximate the loglikelihood function (18) by $l_K(\alpha, \sigma; \tilde{U}_K, \tilde{V}_K, \tilde{S}_K)$, where

$$l_K(\alpha, \sigma; u, v, s) = -K\log\sigma + \alpha + \frac{1}{2}\log|G_K(0)| - \frac{1}{2\sigma^2}(Ks + \alpha^2 u + \alpha v^2).$$

For fixed α, $l_K(\alpha, \sigma; u, v, s)$ is maximized with respect to σ^2 at $s\hat{\sigma}_K^2(\alpha; u', v')$ where $u' = u/s$ and $v' = v^2/s$ and $\hat{\sigma}_K^2(\alpha; u', v') = 1 + (\alpha^2 u' + \alpha v')/K$. The maximizing procedure is then equivalent to maximizing the approximate loglikelihood ratio

$$\lambda_K(\alpha; u', v') = -K \log\{1 + \frac{1}{K}(\alpha^2 u' + \alpha v')\} + \alpha. \tag{26}$$

Note that $\lambda_K(\pm\infty; u', v') = \pm\infty$. This improper divergence of likelihood ratio to $+\infty$ reflects the fact that our approximation to the loglikelihood is valid only if α/K is small. However, we are interested in testing the hypothesis $\alpha = 0$. So we need to pay attention only to the behavior of the likelihood function in such a neighbourhood of zero that the null distribution of the MLE for α is almost concentrated in it. Therefore we consider

$$\tilde{\nu}_{K,R}^{(2)}(\tilde{U}_K/\tilde{S}_K, \tilde{V}_K^2/\tilde{S}_K),$$

where $\tilde{\nu}_{K,R}^{(2)}(u', v') = \max_{\alpha \in [-R,R]} \lambda_K(\alpha; u', v')$. The following Lemma gives the asymptotic distribution of this statistic:

Lemma 2 *Suppose that a sequence of random pairs (U_n, V_n) converges in distribution to $(\int X_\alpha(\tau)^2 d\tau, X_\alpha(1)^2)$ as $n \to \infty$. Let K_n and R_n be sequences such that* $\lim_{n\to\infty} K_n = \infty$, $\lim_{n\to\infty} R_n = \infty$ *and* $\lim_{n\to\infty} R_n/K_n = 0$. *Then*

$$\tilde{\nu}_{K_n,R_n}^{(2)}(U_n, V_n) \xrightarrow{d} \Lambda_\alpha.$$

Proof: Differentiating $\lambda_{K_n}(\alpha; U_n, V_n)$ with respect to α, we obtain

$$2\alpha U_n + V_n - \hat{\sigma}_{K_n}^2(\alpha; U_n, V_n) = 0 \tag{27}$$

This reduces to a quadratic equation $(U_n/K_n)\alpha^2 - 2(U_n - V_n/2K_n)\alpha - (V_n - 1) = 0$. Thus the local maximum of $\lambda_{K_n}(\alpha; U_n, V_n)$ is, if exists, attained at

$$\hat{\alpha}_{K_n}^{(2)} = (K_n - \frac{V_n}{2U_n})[1 - \sqrt{1 + \frac{V_n - 1}{K_n U_n \{1 - V_n/(2K_n U_n)\}^2}}\,]. \tag{28}$$

Note that $\lambda_{K_n}(\alpha; U_n, V_n)$ is maximized at either edge of $[-R_n, R_n]$ or at $\hat{\alpha}^{(2)}_{K_n} \in [-R_n, R_n]$. Since by the assumption U_n and V_n are of constant order, the second term in the square root symbol in RHS of the above expression is $O_p(1/K_n)$. This implies that $\lim_{n\to\infty} P(\hat{\alpha}^{(2)}_{K_n}$ is not real$) = 0$, that $\hat{\alpha}^{(2)}_{K_n} = -(V_n - 1)/(2U_n) + O_p(1/K_n)$ and that the other solution $\hat{\alpha}^{(+)}$ to (27) is $O_p(K_n)$. Thus we have that $P(\hat{\alpha}^{(2)}_{K_n} \in [-R_n, R_n]$ and $\hat{\alpha}^{(+)} > R_n) \to 1$ and thus $P(\tilde{\nu}^{(2)}_{K_n,R_n}(U_n, V_n) = \lambda(\hat{\alpha}^{(2)}_{K_n}; U_n, V_n)) \to 1$. Finally by Taylor expansion of $\log(1+x)$ around 0 we obtain

$$\lambda(\hat{\alpha}^{(2)}_{K_n}; U_n, V_n) = \frac{(V_n - 1)^2}{4U_n} + O_p(1/K_n).$$

This completes the proof.□

This Lemma implies that the test based on $\tilde{\nu}^{(2)}_{K,R_K}(\tilde{U}_K/\tilde{S}_K, \tilde{V}_K^2/\tilde{S}_K)$,where R_K satisfies the conditions $\lim_{K\to\infty} R_K = \infty$ and $\lim_{K\to\infty} R_K/K = 0$, has asymptotically the same property as the exact likelihood ratio test too.

4 Tests for a unit root

This section deals with the discrete time counterparts of the tests given in the previous section. Proofs of all the Lemmata in this section are given in Appendix.

We define a sequence of transformations

$$\tilde{z}_k = T^{-3/2} \sum_{t=1}^{T} \sqrt{2} y_t \sin C_k \frac{t - 1/2}{T} \qquad k = 1, 2, \ldots, T, \tag{29}$$

and statistics which are the discrete analogues of the sufficient statistics $(\tilde{U}_K, \tilde{V}_K, \tilde{S}_K)$ for the Ornstein-Uhlenbeck process

$$\begin{aligned} \tilde{u}_K &= \sum_{k=1}^{K} \tilde{z}_k^2 \\ \tilde{v}_K &= \sqrt{2} \sum_{k=1}^{K} (-1)^k \tilde{z}_k \end{aligned}$$

and

$$\tilde{s}_K = \sum_{k=1}^{K}(C_k\tilde{z}_k)^2/K,$$

where C_k's are frequencies defined in section 2 and $K = 1, 2, \ldots$. We note that C_k's are different from the frequencies of the conventional finite Fourier transformation which are $2k\pi/T, k = 0, 1, \ldots, [T/2]$.

By Lemma 1 and the note which follows the lemma, $Y_T(\cdot) \xrightarrow{d} \sigma X_\alpha(\cdot)$ so that it is easy to see that $\tilde{z}_k \xrightarrow{d} \tilde{Z}_k, k = 1, \ldots K$ jointly. Hence $(\tilde{u}_K, \tilde{v}_K, \tilde{s}_K) \xrightarrow{d} (\tilde{U}_K, \tilde{V}_K, \tilde{S}_K)$ for any fixed K. However, in order to achieve the local efficiency, it is clear that we need to let K go to infinity in such a way that convergence similar to (25) holds. The following lemma really shows how K should be increased with T :

Lemma 3 *Suppose that the error term w_t satisfies the assumption* (i)-(iii) *and let K_T be an non-decreasing sequence of positive integer such that for some $\epsilon > 0$*

(iv). $\lim_{T\to\infty} K_T/T^{1/2+\epsilon} = \infty$ *and* $\lim_{T\to\infty} K_T/T = 0$.

Then

$$(\tilde{u}_{K_T}, \tilde{v}^2_{K_T}, \tilde{s}_{K_T}) \xrightarrow{d} \sigma^2(\int X_\alpha(\tau)^2 dt, X_\alpha(1)^2, 1).$$

It is thus clear from Lemma 2 that the two tests for unit root associated with the following statistics are both locally efficient :

$$\nu^{(1)} = \tilde{\nu}^{(1)}(\tilde{u}_{K_T}/\tilde{s}_{K_T}, \tilde{v}_{K_T}/\tilde{s}_{K_T})$$

and

$$\nu^{(2)} = \tilde{\nu}^{(2)}_{K_T,R_T}(\tilde{u}_{K_T}/\tilde{s}_{K_T}, \tilde{v}_{K_T}/\tilde{s}_{K_T}),$$

where K_T satisfies (iv) and R_T satisfies $\lim_{T\to\infty} R_T = \infty$ and $\lim_{T\to\infty} R_T/K_T = 0$.

Though we are mainly interested in the local properties of our test procedures, the consistency is a minimum requirement for any test to be valid. To see what happens if y_t is stable let us examine briefly the limiting behavior of $\tilde{Z}_K$ in the limit $\alpha \to \infty$. We may argue from the formula for covariances among $\tilde{Z}_k$ that $\alpha(\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_K)$ is distributed nearly as $N(0, \sigma^2 I_K)$ if α is large. Thus $\tilde{V}_K^2/\tilde{S}_K$ is $O_p(1/K)$ and $\tilde{S}_K/(K\tilde{U}_K) \xrightarrow{p} \lim_{K\to\infty} \sum C_k^2/K^3 = \pi^2/3$. This fact suggests the following Lemma

Lemma 4 *Suppose that y_t is generated by (1) with fixed $|\rho| < 1$, and that $\{w_t\}$ and K_T are the same as in Lemma 3. Then it holds that*

$$(T^2/K_T)\tilde{u}_{K_T} \xrightarrow{p} (1-\rho)^{-2}\sigma^2,$$

$$(T^2/{K_T}^2)\tilde{s}_{K_T} \xrightarrow{p} (\pi^2/3)(1-\rho)^{-2}\sigma^2$$

and

$$E\{(T^2/K_T)\tilde{v}_{K_T}^2\} = O(1).$$

Put $u_T = \tilde{u}_{K_T}/\tilde{s}_{K_T}$ and $v_T = \tilde{v}_{K_T}^2/\tilde{s}_{K_T}$. The Lemma implies that

$$K_T u_T \xrightarrow{p} 3/\pi^2 \quad \text{and} \quad v_T = O_p(1/K_T). \tag{30}$$

Thus we have that $\nu^{(1)}/K_T = (v_T - 1)^2/(4K_T u_T) \xrightarrow{p} \pi^2/12$, which implies the consistency of the test based on $\nu^{(1)}$.
As for $\nu^{(2)}$, note that the proof to Lemma 2 gives how to maximize $\lambda_{K_T}(\alpha; u_T, v_T)$ on $[-R_T, R_T]$ given u_T and v_T. Also note that $\lambda_{K_T}(\alpha; u_T, v_T)$ is monotone increasing if (28) is not real. Indeed, it is easy to see from (30) that the quantity under the square root symbol in (28) converges to $1 - \pi^2/3 < 0$ and hence $\lim_{T\to\infty} P(\lambda_{K_T}(\alpha; u_T, v_T)$ is monotone increasing in $\alpha) = 1$. Therefore $\lim_{T\to\infty} P(\nu^{(2)} = \lambda_{K_T}(R_T; u_T, v_T)) = 1$. It is obvious that

$$\lambda_{K_T}(R_T; u_T, v_T)/R_T = 1 - (K_T/R_T)\log(1 + R_T^2 u_T/K_T + R_T v_T/K_T) \xrightarrow{p} 1,$$

since both u_T and v_T are $O_p(1/K_T)$. Thus we have proved the consistency of both of the two tests under stable alternatives.

Next we turn to the explosive case. The second term is dominant in (11) and the covariance matrix $G_K(\alpha)$ is almost of rank 1 in the limit $\alpha \to -\infty$. So the distribution of $\tilde{Z}^{(K)}$ is one dimensional, i. e. $\tilde{Z}^{(K)} = \tilde{Z}_1 \, \mathbf{j}_K$ in the limit $\alpha \to -\infty$ and hence $\tilde{V}_K^2 \sim 2K^2\tilde{Z}_1^2$, $\tilde{U}_K^2 \sim K\tilde{Z}_1^2$ and $\tilde{S}_K^2 \sim K^{-1}\sum_{k=1}^K C_k^2\tilde{Z}_1^2 \sim (\pi^2/3)K^2\tilde{Z}_1^2$. The following Lemma suggested by this fact really holds :

Lemma 5 *Under the same assumptions in Lemma 4 except that $\rho > 1$ instead of $|\rho| < 1$, it holds that*

$$(\tilde{v}_{K_T}^2/\tilde{s}_{K_T}, K_T\tilde{u}_{K_T}/\tilde{s}_{K_T}) \xrightarrow{p} (6/\pi^2, 3/\pi^2).$$

Put $u_T = \tilde{u}_{K_T}/\tilde{s}_{K_T}$ and $v_T = \tilde{v}_{K_T}^2/\tilde{s}_{K_T}$ again. By direct calculation this Lemma implies that $\nu^{(1)}/K_T \xrightarrow{p} (\pi^2-6)^2/12\pi^2$ so that the test provided by $\nu^{(1)}$ is consistent when $\rho > 1$. The Lemma also implies that $1 - v_T/(2K_Tu_T) \xrightarrow{p} 0$, $v_T - 1 \xrightarrow{p} 6/\pi^2 - 1 < 0$ and $K_Tu_T \xrightarrow{p} 3/\pi^2 > 0$. Hence through inspection of (28) we see that $\lim_{T\to\infty} P(\lambda_{K_T}(\alpha; u_T, v_T)$ is monotone increasing in $\alpha) = 1$. Since $v_T \xrightarrow{p} 6/\pi^2$ and $u_T = O_p(1/K_T)$, we have that

$$\begin{aligned}
\lambda_{K_T}(\alpha; u_T, v_T)/R_T &= 1 - (K_T/R_T)\log(1 + R_T^2u_T/K_T + R_Tv_T/K_T) \\
&\xrightarrow{p} 1 - 6/(\pi^2) > 0.
\end{aligned}$$

Thus the test associated with $\nu^{(2)}$ is also consistent under the explosive alternatives.

5 Discussions

Testing for unit roots is intended for the *pre-stage* of model fitting. We *test* unit roots only to see whether difference of data is needed or not. Because near-integrated processes cover a wide class of models, i. e. most of the models conceivable, the necessity of differencing can be determined before the full analysis of data. If the unit root hypothesis is accepted, the differenced data should be analyzed. If it is rejected, then we fit a stationary model or some other models which do not include the difference operation. The point is that once we get conclusion about differencing, we can apply any model to the differenced or undifferenced data according to the result of the test. The merit of the nonparametric approach like ours consists in this point. We note that, though estimators for α seems to be naturally suggested similarly to the test statistics, we should not or at least do not need to *estimate* α. The reason for this is clear from the above remark. Also note that the above argument concerning the necessity of differencing is limited to the case of the model fitting for univariate time series; in the context of co-intergration analysis of multiple time series the problem of differencing is more complicated.

In the parametric frameworks such as AR models, the likelihood-based approach is usually adopted and attention is paid automatically to the distribution of test statistics under the alternative hypothesis. Our tests are semi-parametric in the sense that we make use of the parametric family of the Ornstein-Uhlenbeck processes as that of the asymptotic distributions. Thus the alternative hypothesis was taken into account implicitly in the course of the construction of test statistics. As for the nonparametric test we should keep attention to the alternatives. The following *pre-filtering* method is an interesting counter example which shows the importance of consideration of the alternatives : Suppose the observed sequence y_t is generated by (1) with $\rho = 1$. Assume ARMA models for w_t, that is $w_t = \Theta(\mathcal{L})\epsilon_t$, where ϵ is a white noise sequence and $\Theta(\mathcal{L})$ is a rational function of the lag operator $\mathcal{L}$. We can obtain an estimate $\hat{\Theta}(\mathcal{L})$ based on the differenced data Δy_t by some method. Suppose for simplicity that we estimate it exactly. Then we obtain the residual series $\hat{\epsilon}_t = \hat{\Theta}(\mathcal{L})^{-1}\Delta y_t = \epsilon_t$. Define $z_t = \sum_{t'=1}^{t} \hat{\epsilon}_{t'}$. Since z_t is the cumulative process of an i. i. d. sequence, the test statistic calculated from it is distributed as Λ_0, the Dickey-Fuller distribution. This seems quite a good test procedure.

Next, let us consider the alternative case. Assume the same framework as above except that we assume that $\rho = 0$. We apply the above procedure to the data y_t. By assumption, $\Delta y_t = \Delta w_t = \Theta^*(\mathcal{L})\epsilon_t$ where $\Theta^*(\mathcal{L}) = (1-\mathcal{L})\Theta(\mathcal{L})$. Again, suppose that we estimate $\Theta^*(\mathcal{L})$ exactly by $\hat{\Theta}^*(\mathcal{L})$ and obtain the residual $\hat{\epsilon}_t = \hat{\Theta}^*(\mathcal{L})^{-1}\Delta y_t = \epsilon_t$. It is clear that the test based on $z_t = \sum_{t'=1}^{t} \hat{\epsilon}_{t'}$ is distributed as Λ_0 too. Thus this test has intrinsically no power, since the fitting of $\Theta^*(\mathcal{L})$ is designed to produce white noise as residual sequence whatever we start with.

Two tests have been proposed. Lemma 2 gives only the first order asymptotics of the test. So it does not explicitly imply any superiority of either of $\tilde{\nu}^{(1)}(\tilde{U}_K/\tilde{S}_K, \tilde{V}_K^2/\tilde{S}_K)$ and $\tilde{\nu}_K^{(2)}(\tilde{U}_K/\tilde{S}_K, \tilde{V}_K^2/\tilde{S}_K)$. However, since the latter is the likelihood ratio at least approximately, we could expect that it has good properties in some sense. The comparison of $\nu^{(1)}$ and $\nu^{(2)}$ is less conclusive. It turns out that $\nu^{(1)}$ is obtained by the maximizing $\lambda^{(1)} = -\{\alpha^2 u^2 + \alpha(v-1)\}$, where we put $u = \tilde{u}_{K_T}/\tilde{s}_{K_T}$ and $v = \tilde{v}_{K_T}^2/\tilde{s}_{K_T}$. Hence if (u, v) is distributed as $(\int X_\alpha(t)^2 dt, X_\alpha(1)^2)$, $\lambda^{(1)}$ is the exact loglikelihood ratio of (u, v) and $\nu^{(1)}$ is preferable. While if (u, v) is distributed as $(\tilde{U}_{K_T}/\tilde{S}_{K_T}, \tilde{V}_{K_T}^2/\tilde{S}_{K_T})$, $\nu^{(2)}$ may be recommended. Therefore the conclusion depends on which of the above two the true distribution of (u, v) is closer to. This requires inspection of the terms of order $1/K_T$ which we neglected in the proof of Lemma 3 and so far we have had no conclusion on this problem.

Acknowledgement

This work was done while the author was visiting the Institute of Mathematical Statistics, University of Copenhagen. He is grateful to Professor Søren Johansen for discussions, comments and suggestions for presentation.

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Appendix

We often deal with quantities which depend on the sample size T, the number of $\tilde{z}$'s K or K_T, indices $t = 1, \ldots, T$ and $k = 1, \ldots, K$ or K_T and so on. By expressions like $a_{kt} = O(1/k)$ we mean that $|a_{kt}|$ is bounded by R/k for some constant R uniformly in t, T and so on. We also denote by $O(1/k)$ terms bounded by R/k uniformly in t, T and so on.

Each $\tilde{z}_k$ is expressed as follows in terms of w_t :

$$\tilde{z}_k = T^{-3/2}\sqrt{2}\sum_{t=1}^{T}\sum_{\tau=1}^{t}\rho^{t-\tau}w_\tau \sin C_k \frac{t-1/2}{T} = \sum_{t=1}^{T} b_{kt}\tilde{w}_t,$$

where we put $\tilde{w}_t = w_t/T^{1/2}$ and

$$\begin{aligned} b_{kt} \quad = \quad & \frac{T^{-1}\sqrt{2}}{1+\rho^2-2\rho\cos C_k/T}\{(1+\rho)\sin\frac{C_k}{2T}\ \cos\frac{C_k}{T}(t-1) \\ & +(1-\rho)\cos\frac{C_k}{2T}\ \sin\frac{C_k}{T}(t-1) - (-1)^k(1-\rho)\cos\frac{C_k}{2T}\ \rho^{T-t+1}\}. \qquad \text{(a.1)} \end{aligned}$$

The following Lemma is useful in the course of proving the Lemmata in Section 4.

Lemma A.1 *Let $I_T \subset [1,T]$ be an interval and $|I_T|$ denote the number of integers in I_T. Then the following hold :*

(a). $$\mathrm{Var}(\sum_{t\in I_T} a_t\tilde{w}_t) = O(\max_{t\in I_T} a_t^2 |I_T|/T)$$

(b). $$|E\sum_{t,t'=1}^{T} c_{tt'}\tilde{w}_t\tilde{w}_{t'}| = O(\max_{1\le t,t'\le T}|c_{tt'}|)$$

(c). $$E|\sum_{t=1}^{T-m}(\tilde{w}_t\tilde{w}_{t+m} - E\tilde{w}_t\tilde{w}_{t+m})| = O\{(m^{1/2}\gamma_m^{1-2/\beta}+1)/T^{1/2}\}$$

(d). $$E|\sum_{1\le t\le s\le T, t+s=m}(\tilde{w}_t\tilde{w}_s - E\tilde{w}_t\tilde{w}_s)| = O(T^{-1/2}), \quad \text{where} \quad 2\le m\le 2T.$$

(e). $$E|\sum_{t,t'=1}^{T} c_{tt'}(\tilde{w}_t\tilde{w}_{t'} - E\tilde{w}_t\tilde{w}_{t'})| = O(T^{1/2}\max_{1\le t,t'\le T}|c_{tt'}|)$$

Proof : The following bounds for the moments are derived from Theorem 17.2.2 of Ibragimov and Linnik (1971) together with assumption (i) :

$$|Ew_tw_{t+m}| = O(\gamma_{|m|}^{1-2/\beta}) \tag{a.2}$$

and if $t_1 \leq t_2 \leq t_3 \leq t_4$,

$$|\mathrm{Cov}(w_{t_1}w_{t_2}, w_{t_3}w_{t_4})| = O(\gamma_{t_3-t_2}^{1-2/\beta}). \tag{a.3}$$

We note that, though the Theorem assumes strict stationarity for w_t, stationarity is not used in its proof so that (a.2) and (a.3) are true. Thus by assumption (iii) we have

$$\sum_{m=0}^{\infty} \sup_t |Ew_tw_{t+m}| < \infty \tag{a.4}$$

and

$$\sum_{m=0}^{\infty} \sup_{t_1\leq t_2, t_2+m=t_3\leq t_4} |\mathrm{Cov}(w_{t_1}w_{t_2}, w_{t_3}w_{t_4})| < \infty \tag{a.5}$$

From (a.4), (a) and (b) are easily derived.

(c): We evaluate

$$\sum_{1\leq t\leq t'\leq T}^{T} \mathrm{Cov}(w_tw_{t+m}, w_{t'}w_{t'+m}) = \sum_{t=1}^{T}\Big(\sum_{t'=t}^{t+m-1} + \sum_{t'=t+m}^{T}\Big) \mathrm{Cov}(w_tw_{t+m}, w_{t'}w_{t'+m})$$

The second part of the summation is bounded by

$$\sum_{t=1}^{T}\sum_{t'=t+m}^{T} |\mathrm{Cov}(w_tw_{t+m}, w_{t'}w_{t'+m})| = \sum_{t=1}^{T} O(1) = O(T)$$

by (a.5). As for the first part, we note that

$$\begin{aligned} Ew_tw_{t'}w_{t+m}w_{t'+m} &= \mathrm{Cov}(w_tw_{t'}, w_{t+m}w_{t'+m}) + Ew_tw_{t'}Ew_{t+m}w_{t'+m} \\ &= \mathrm{Cov}(w_tw_{t+m}, w_{t'}w_{t'+m}) + Ew_tw_{t+m}Ew_{t'}w_{t'+m} \end{aligned}$$

so that if $t \leq t' \leq t+m$ and $m \geq 0$,

$$|\mathrm{Cov}(w_tw_{t+m}, w_{t'}w_{t'+m})| \leq R\{\gamma_m^{2(1-2/\beta)} + \gamma_{(t+m)-t'}^{1-2/\beta} + \gamma_{t'-t}^{2(1-2/\beta)}\}$$

for some constant R. Hence

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{t'=t}^{t+m-1}|\mathrm{Cov}(w_t w_{t+m}, w_{t'} w_{t'+m})| \\
&\leq \; RT\{m\gamma_m^{2(1-2/\beta)} + \sum_{s'=0}^{m-1}\gamma_{s'}^{2(1-2/\beta)} + \sum_{s'=0}^{m-1}\gamma_{s'}^{1-2/\beta}\}.
\end{aligned}$$

Therefore by assumption (iii) we obtain

$$\mathrm{Var}(\sum_{t=1}^{T-m}\tilde{w}_t\tilde{w}_{t+m}) = O\{(m\gamma_m^{2(1-2\beta)}+1)/T\}.$$

By the relation between variance and mean absolute deviation, this implies (c).

(d): For simplicity of notation we assume without loss of generality that $m \leq T$. We evaluate $\mathrm{Var}(\sum_{1\leq t\leq [m/2]} w_t w_{m-t})$. It suffices to show that

$$\sum_{1\leq t\leq t'\leq [m/2]} \mathrm{Cov}(w_t w_{m-t}, w_{t'} w_{m-t'}) = O(T).$$

We have, similarly to the evaluation of the covariance in the proof of (c), that for $1 \leq t \leq t' \leq [m/2]$

$$\begin{aligned}
&|\mathrm{Cov}(w_t w_{m-t}, w_{t'} w_{m-t'})| \\
&\leq \; |\mathrm{Cov}(w_t w_{t'}, w_{m-t'} w_{m-t})| + |Ew_t w_{m-t} E w_{t'} w_{m-t'}| + |Ew_t w_{t'} E w_{m-t'} w_{m-t}| \\
&\leq \; R(\gamma_{m-2t'}^{1-2/\beta} + \gamma_{m-2t}^{1-2/\beta}\gamma_{m-2t'}^{1-2/\beta} + \gamma_{t'-t}^{2(1-2/\beta)})
\end{aligned}$$

, since $1 \leq t \leq t' \leq m-t' \leq m-t$, where R is some constant. Therefore by (a.4) and (a.5) we have

$$\begin{aligned}
&|\sum_{t=1}^{[m/2]}\sum_{t'=t}^{[m/2]} \mathrm{Cov}(w_t w_{m-t}, w_{t'} w_{m-t'})| \\
&\leq \; \sum_{t=1}^{[m/2]}\sum_{t'=t}^{[m/2]} R(\gamma_{m-2t'}^{1-2/\beta} + \gamma_{m-2t}^{1-2/\beta}\gamma_{m-2t'}^{1-2/\beta} + \gamma_{t'-t}^{2(1-2/\beta)}) \\
&\leq \; \sum_{t=1}^{[m/2]} R',
\end{aligned}$$

where R' is some constant. This implies desired result.

(e): By (a) and assumption (i) we have

$$\begin{aligned} E|\sum_{t,t'} c_{tt'}\tilde{w}_t\tilde{w}_{t'}| &\leq E\sum_t |\tilde{w}_t||\sum_{t'} c_{tt'}\tilde{w}_{t'}| \leq \sum_t |\tilde{w}_t|_2|\sum_{t'} c_{tt'}\tilde{w}_{t'}|_2 \\ &= O(\max_{t,t'}|c_{tt'}|T^{1/2}).\square \end{aligned}$$

Proof of Lemma 3 : Define $\phi(x) = x^{-3}(\cos x - 1 + x^2/2)$, then when $\rho = 1 - \alpha/T$ we have

$$T^2(1+\rho^2 - 2\rho\cos\frac{C_k}{T}) = \alpha^2 + C_k^2(1-\varphi_{kT}) \tag{a.6}$$

where $\varphi_{kT} = 2(1-\alpha/T)\phi(C_k/T)C_k/T + \alpha/T$, $k = 1, 2, \ldots$. Since $\lim_{\tau\to 0}\phi(\tau) = 0$, it holds that

$$\varphi_{kT} = O(k/T) \tag{a.7}$$

and hence $\lim_{T\to\infty}\sup_{k\leq K_T}|\varphi_{kT}| = 0$. Thus we have a bound for $|b_{kt}|, k \leq K_T$

$$|b_{kt}| \leq \frac{2\sqrt{2}}{C_k^2}\{2C_k + |\alpha|(1+\min(1, e^{-\alpha})\}$$

for sufficiently large T such that $\sup_{k\leq K_T}|\varphi_{kT}| < 1/2$ and $|1+\rho|/2 < 2$, that is

$$b_{kt} = O(1/k). \tag{a.8}$$

Convergence of $\tilde{u}_{K_T}$: Because $\tilde{u}_K \xrightarrow{d} \sigma^2\tilde{U}_K$ for all fixed K and $\tilde{U}_K \xrightarrow{d} \int X_\alpha(t)^2 dt$ by (25), it suffices to show that for any $\epsilon > 0$

$$\lim_{K\to\infty}\limsup_{T\to\infty} P(|\tilde{u}_{K_T} - \tilde{u}_K| > \epsilon) = 0,$$

owing to Theorem 4.2 of Billingsley (1968). In the course of proving the above statement, we first fix K so that we may assume that $K_T \geq K$. Since $|\sum_{k=K+1}^{K_T} b_{kt}b_{ks}| = O(1/K)$ by (a.8) and $\tilde{u}_{K_T} - \tilde{u}_K \geq 0$, we have

$$E|\tilde{u}_{K_T} - \tilde{u}_K| = E\sum_{t,t'}\sum_{k=K+1}^{K_T} b_{kt}b_{kt'}\tilde{w}_t\tilde{w}_{t'} = O(1/K)$$

because of (b) of Lemma A.1. Hence, letting K tend to infinity, we have the desired assertion.

Convergence of $\tilde{v}_{K_T}$: As in the case of $\tilde{u}_{K_T}$, we need only to show that

$$\lim_{K\to\infty}\limsup_{T\to\infty} P(|\tilde{v}_{K_T}-\tilde{v}_K|>\epsilon)=0$$

for any $\epsilon>0$. We define

$$\tilde{z}_k^* = \sum_{t=1}^{T} b_{kt}^*\tilde{w}_t \quad \text{where} \quad b_{kt}^* = \frac{\sqrt{2}}{C_k}\cos C_k(\frac{t-1}{T}).$$

Now we show that

$$|b_{kt}-b_{kt}^*| = O(1/k^2+1/T). \tag{a.9}$$

It is obvious from (a.6) that the second and the third term in (a.1) are $O(1/C_k^2) = O(1/k^2)$. The first term of (a.1) is

$$\begin{aligned}&\sqrt{2}(1+\rho)T\sin(C_k/2T)/\{T^2(1+\rho^2-2\rho\cos C_k/T)\}\\ &= \sqrt{2}C_k(1+\tilde{\varphi}_{kT})/\{\alpha^2+C_k^2(1+\varphi_{kT})\},\end{aligned}$$

where $\tilde{\varphi}_{kT}=(1-\alpha/2T)\tilde{\phi}(C_k/2T)-1$ and $\tilde{\phi}(\tau)=\sin\tau/\tau$. Since $\tilde{\phi}(\tau)$ is bounded and $\lim_{\tau\to 0}\tau^{-1}\{\tilde{\phi}(\tau)-1\}=0$, we have

$$\tilde{\varphi}_{kT}=O(k/T).$$

From this and (a.7) we see that the difference between the first term of (a.1) and b_{kt}^* is bounded by

$$\begin{aligned}&\sqrt{2}\{\alpha^2+C_k^2(|\varphi_{kT}|+|\tilde{\varphi}_{kT}|)\}/[C_k\{\alpha^2+C_k^2(1-\varphi_{kT}\}]\\ &= O[\{1+C_k^2(k/T)\}/C_k^3] = O(1/k^3+1/T).\end{aligned}$$

Therefore we have shown (a.9).

Define

$$\tilde{v}_K^* = \sqrt{2}\sum_{k=1}^{K}(-1)^k\tilde{z}_k^* = \sqrt{2}\sum_t B_{kt}^*\tilde{w}_t,$$

where we put $B^*_{Kt} = \sum_{k=1}^K (-1)^k b^*_{kt}$. Then by (a.9) and (a) of Lemma (A.1) we obtain that $\mathrm{Var}(\tilde{v}_K - \tilde{v}^*_K) = O\{(1/K + K/T)^2\}$ and $\mathrm{Var}(\tilde{v}_{K_T} - \tilde{v}^*_{K_T}) = O\{(1/K_T + K_T/T)^2\}$. Hence it suffices to show that

$$\lim_{K\to\infty} \limsup_{T\to\infty} P(|\tilde{v}^*_{K_T} - \tilde{v}^*_K| > \epsilon) = 0 \tag{a.10}$$

for any $\epsilon > 0$. Note that $\sin C_k(1-\tau) = (-1)^{k-1}\cos C_k\tau$. Put $\tau = (T-t+1)/T$ and $g_K(\tau) = \sum_{k=1}^K \sin C_k\tau/C_k - \sum_{k=1}^K \sin C_k/C_k$ then $B^*_{Kt} = g_K(\tau) + \sum_{k=1}^K \sin C_k/C_k$. Note that $\sum_{k=1}^K \sin C_k/C_k = \sum_{k=1}^K (-1)^{k-1}/C_k$ is a convergent series and is $O(1)$ and that

$$|\sum_{k=K+1}^{K_T} \frac{1}{C_k}\sin C_k| \le |\sum_{k=K+1}^{\infty} \frac{(-1)^{k-1}}{C_k}| + |\sum_{k=K_T+1}^{\infty} \frac{(-1)^{k-1}}{C_k}| = O(1/K).$$

We evaluate $g_K(\tau)$ as

$$\begin{aligned} g_K(\tau) &= \int_1^\tau \sum_{k=1}^K \cos C_k\tau' d\tau' = \int_1^\tau \frac{\sin \pi K\tau'}{2\sin(\pi/2)\tau'} d\tau' \\ &= [\frac{\cos\pi K\tau'}{2\pi K\sin(\pi/2)\tau'}]_\tau^1 + \int_\tau^1 \frac{\pi\cos\pi K\tau'\cos(\pi/2)\tau'}{4\pi K\{\sin(\pi/2)\tau'\}^2} d\tau', \end{aligned}$$

where the last equality is obtained by integration by part. Let $0 < \delta < 1$, then for $\delta \le \tau \le 1$ the RHS of the above equality is and thus $g_K(\tau)$ is bounded by $R/(K\delta^2)$ for some constant R, since $(2/\pi)\tau \le \sin\tau$, for $0 \le \tau \le \pi/2$.

Now let $I_1 = [1, T - TK^{-1/3}]$, $I_2 = (T - TK^{-1/3}, T - TK_T{}^{-1/3}]$ and $I_3 = (T - TK_T{}^{-1/3}, T]$. Note that we have a bound $B^*_{Kt} = O(\log K)$ for all t , since $b^*_{kt} = O(1/k)$. From the above bound for $g_K(\tau)$ we have, $B^*_{K_Tt} - B^*_{Kt} = g_{K_T}(\tau) - g_K(\tau) + \sum_{k=K+1}^{K_T} (-1)^k/C_k = O(K^{-1/3})$ in I_1, $B^*_{K_Tt} - B^*_{Kt} = g_{K_T}(\tau) - (B^*_{Kt} - \sum_{k=1}^{K_T} (-1)^k/C_k) = O(\log K)$ in I_2 and $B^*_{K_Tt} - B^*_{Kt} = O(\log K_T)$ in I_3. Therefore we obtain by (a) of Lemma A.1

$$\mathrm{Var}(\sum_{t\in I_1} (B^*_{K_Tt} - B^*_{Kt})\tilde{w}_t) = O(K^{-2/3}),$$

$$\mathrm{Var}(\sum_{t\in I_2} (B^*_{K_Tt} - B^*_{Kt})\tilde{w}_t) = O((\log K)^2/K^{1/3})$$

and

$$\mathrm{Var}(\sum_{t\in I_3}(B^*_{K_T t} - B^*_{Kt})\tilde{w}_t) = O\{(\log K_T)^2/{K_T}^{1/3}\}$$

since we assumed $K_T > K$. These together imply (a.10).

Convergence of $\tilde{s}_{K_T}$: The outline of the proof is as follows : $\tilde{s}_{K_T}$ is a quadratic form $\sum_{t,t'}(\sum_{k=1}^{K_T} b_{kt}b_{kt'}/K_T)\tilde{w}_t\tilde{w}_{t'}$. It is true that, roughly speaking, *far* off diagonal entries of this quadratic form vanish and nearly diagonal ones converge to unity as $T \to \infty$. Hence $E\tilde{s}_{K_T}$ is close to $\sum_{t,t'=1}^{T} E\tilde{w}_t\tilde{w}_{t'}$ which tends to σ^2, since the far off diagonal elements of the summand vanish too by the mixing assumption. Moreover because of the mixing condition, the law of large numbers holds for $\tilde{s}_{K_T}$ ((c) of Lemma A.1) so that it converges to its limit of expectation. Implementation of this idea is given in the following :

Define $\tilde{s}^*_K = K^{-1}\sum_{k=1}^{K}(C_k\tilde{z}^*_k)^2$, then

$$\tilde{s}_K - \tilde{s}^*_K = \sum_{t,t'} K^{-1}\sum_{k}^{K} C_k^2(b_{kt}b_{kt'} - b^*_{kt}b^*_{kt'})\tilde{w}_t\tilde{w}_{t'}.$$

Because of (a.8) and (a.9) we see that $C_k^2(b_{kt}b_{kt'} - b^*_{kt}b^*_{kt'}) = O(1/k + k/T)$ and thus ${K_T}^{-1}\sum_{k=1}^{K_T} C_k^2(b_{kt}b_{kt'} - b^*_{kt}b^*_{kt'}) = O(\log K_T/K_T + K_T/T)$. Hence by (e) of Lemma A.1 and the assumptions of Lemma 3, we obtain $\tilde{s}_{K_T} - \tilde{s}^*_{K_T} \xrightarrow{p} 0$ as $T \to \infty$. Therefore it suffices to show that $\tilde{s}^*_{K_T} \xrightarrow{p} \sigma^2$.

Put

$$\kappa_K(\tau) = K^{-1}\sum_{k=1}^{K}\cos C_k\tau = \frac{1}{K}\frac{\sin \pi K\tau}{2\sin \pi\tau/2}.$$

It is obvious that

$$|\kappa_K(\tau)| \le 1 \wedge 1/(K|\tau|) \tag{a.11}$$

and that

$$\mathrm{d}^p\kappa_K(\tau)/\mathrm{d}\tau^p \,|_{\tau=0} = 0 \quad \text{if} \quad p = 1, 3, \ldots. \tag{a.12}$$

It is also clear that

$$\begin{aligned} |\mathrm{d}^p\kappa_K(\tau)/\mathrm{d}\tau^p| &\leq |\mathrm{d}^p\kappa_K(\tau)/\mathrm{d}\tau^p\ |_{\tau=0}\ | \\ &= |(-1)^{p/2}K^{-1}\sum_{k=1}^{K} C_k^p| \leq (\pi K)^p/(p+1) = O(K^p), \\ &\quad \text{if} \quad p = 2, 4, \ldots. \end{aligned} \tag{a.13}$$

It is easy to see that

$$\tilde{s}_K^* = \sum_{t,t'} K^{-1} \sum_{k=1}^{K} 2\cos C_k(\frac{t-1}{T})\cos C_k(\frac{t'-1}{T})\tilde{w}_t\tilde{w}_{t'} = \tilde{s}_K^{*(1)} + \tilde{s}_K^{*(2)}$$

where $\tilde{s}_K^{*(1)} = \sum_{t,t'} \kappa_K\{(t-t')/T\}\tilde{w}_t\tilde{w}_{t'}$ and $\tilde{s}_K^{*(2)} = \sum_{t,t'} \kappa_K\{(t+t'-2)/T\}\tilde{w}_t\tilde{w}_{t'}$.

In order to obtain that $\tilde{s}_{K_T}^{*(1)} \xrightarrow{p} \sigma^2$, we show that

$$\lim_{T\to\infty} E\tilde{s}_{K_T}^{*(1)} = \lim_{T\to\infty} E\sum_{t,t'} \tilde{w}_t\tilde{w}_{t'} = \sigma^2 \tag{a.14}$$

and that

$$\lim_{T\to\infty} E|\tilde{s}_{K_T}^{*(1)} - E\tilde{s}_{K_T}^{*(1)}| = 0. \tag{a.15}$$

Let $\chi_M(t,t')$ be the indicator function for $|t-t'| \leq M$ and $M_T = (T/K_T)^{1/2}$. Then,

$$\begin{aligned} \tilde{s}_{K_T}^{*(1)} - \sum_{t,t'} \tilde{w}_t\tilde{w}_{t'} &= \sum_{t,t'}[\chi_{M_T}(t,t')\{\kappa_{K_T}((t-t')/T) - 1\} \\ &\quad +\{1-\chi_{M_T}(t,t')\}\{\kappa_{K_T}((t-t')/T) - 1\}]\tilde{w}_t\tilde{w}_{t'}. \end{aligned}$$

Since $\kappa_{K_T}(0) = 1$ and (a.12), we see that $\kappa_{K_T}(\tau) - 1 = \kappa''_{K_T}(\theta\tau)\tau^2/2$ for some $0 < \theta < 1$. Hence from (a.13) with $p = 2$ we obtain

$$\chi_{M_T}(t,t')\{\kappa_{K_T}((t-t')/T) - 1\} = O\{(K_T M_T/T)^2\} = O(K_T/T)$$

and by (b) of Lemma A.1

$$E\sum_{t,t'} \chi_{M_T}(t,t')\{\kappa_{K_T}((t-t')/T) - 1\}\tilde{w}_t\tilde{w}_{t'} = O(K_T/T).$$

We also obtain by (a.2) and (a.11) that

$$|E\sum_{t,t'}\{1-\chi_{M_T}(t,t')\}\{\kappa_{K_T}((t-t')/T)-1\}\tilde{w}_t\tilde{w}_{t'}| \le 2R\sum_{m=[M_T]}^{\infty}\gamma_m^{1-2/\beta}$$

for some constant R. Since this vanishes as $M_T \to \infty$, we have shown (a.14).

As for (a.15), we see by (a.11) and (c) of Lemma A.1 that

$$\begin{aligned}
E|\tilde{s}^{*(1)}_{K_T} - E\tilde{s}^{*(1)}_{KT}| &\le \sum_{m=-T+1}^{T-1}\sum_{t=1\vee(1-m)}^{T\wedge(T-m)} |\kappa_{K_T}(m/T)|E|\tilde{w}_t\tilde{w}_s - E\tilde{w}_t\tilde{w}_s| \\
&\le \sum_{m=1}^{T}|\kappa_{K_T}(m/T)|T^{-1/2}O(m^{1/2}\gamma_m^{1-2/\beta}+1)+O(T^{-1/2}) \\
&= \sum_{m=1}^{T}O(m^{1/2}\gamma_m^{1-2/\beta})/T^{1/2} + T^{1/2}/K_T\sum_{m=1}^{T}O(1/m)+O(T^{-1/2}).
\end{aligned}$$

The first term converges to zero, since by the summability of $\gamma_m^{1-2/\beta}$ we have

$$\sum_{m=1}^{T}(m/T)^{1/2}\gamma_m^{1-2/\beta} \to 0.$$

The second term is clearly $O(T^{1/2}\log T/K_T)$. Thus the RHS of above equality tends to zero as $T \to \infty$ by the assumption of the Lemma for K_T.

We turn to showing that $\tilde{s}^{*(2)}_{K_T} \xrightarrow{p} 0$. We note that $\cos C_k(2-\tau) = -\cos C_k\tau$ and hence $\kappa_K(2-\tau) = -\kappa_K(\tau)$. Since (a.4) implies $\sum_{t+s=m}|E\tilde{w}_t\tilde{w}_s| = O(T^{-1})$, we have by (a.11)

$$|E\tilde{s}^{*(2)}_{K_T}| \le 2\sum_{m=0}^{T}|\kappa_{K_T}(m/T)|O(T^{-1}) = O(\log T/K_T)$$

so that $\lim_{T\to\infty} E\tilde{s}^{*(2)}_{K_T} = 0$. We also have by (d) of Lemma A.1 and (a.11) that

$$\begin{aligned}
&E|\tilde{s}^{*(2)}_{K_T} - E\tilde{s}^{*(2)}_{K_T}| \\
&\le \sum_{m=0}^{2T-2}|\kappa_{K_T}(m/T)|\ E(|\sum_{t+s=m+2}(\tilde{w}_t\tilde{w}_s - E\tilde{w}_t\tilde{w}_s)|) \\
&\le 2\sum_{m=0}^{T-1}|\kappa_{K_T}(m/T)|O(T^{-1/2}) = O(T^{1/2}\log T/K_T).
\end{aligned}$$

Thus we have shown that $\tilde{s}^{*(2)}_{K_T} - E\tilde{s}^{*(2)}_{K_T} \xrightarrow{p} 0$. This completes the proof of Lemma 3.□

Proof of Lemma 4 :

Let $w_t^* = (1-\rho\mathcal{L})^{-1}w_t = \sum_{t'=0}^{t-1}\rho^{t'}w_{t-t'}$, where $\mathcal{L}$ is the lag operator. Since $|\rho| < 1$ is fixed, (1) is equivalently expressed as

$$y_t = w_t^*.$$

We show w_t^* satisfies some condtitions similar to assumptions (i)-(iii).

By Minkowski' inequality and assumption (i), we have

$$|w_t^*|_\beta \le (1-\rho)^{-1}\sup_t |w_t|_\beta. \tag{a.16}$$

It is easily seen that

$$\lim_{T\to\infty} T^{-1}\mathrm{Var}(\sum w_t^*) = (1-\rho)^{-2}\sigma^2. \tag{a.17}$$

Lemma A.2 *Define* $\tilde{\sigma}_{t,s} = Ew_t^*w_s^*$, $\tilde{\gamma}_m = \sup_t |\tilde{\sigma}_{t,t+m}|$ *and* $\tilde{\delta}_m = \sup_{t_1\le t_2, t_2+m=t_3\le t_4}$ $\mathrm{Cov}(w_{t_1}^*w_{t_2}^*, w_{t_3}^*w_{t_4}^*)$. *Then*

$$\sum_{m=0}^{\infty}\tilde{\gamma}_m < \infty. \tag{a.18}$$

and

$$\sum_{m=0}^{\infty}\tilde{\delta}_m < \infty. \tag{a.19}$$

Proof is given in the end of Appendix.

Note that the proof of Lemma A.1 was given essentially based on the summability of moments given by (a.4) and (a.5). Thus it is easily confirmed that the statesments of Lemma A.1 with w_t and $\gamma_m^{1-1/2\beta}$ in (c) replaced by w_t^* and $\tilde{\delta}_m$, respectively, hold by virtue of (a.18) and (a.19). We prove the Lemma by evaluation of moments somewhat similar to those in the proof of Lemma 3. Hereafter we write simply $w_t = w_t^*$ and $\tilde{w}_t = w_t^*/T^{1/2}$ in order to make the similarity clear.

Now we have

$$T\tilde{z}_k = \sqrt{2}\sum_{t=1}^{T}\sin\frac{C_k}{T}(t-\frac{1}{2})\tilde{w}_t.$$

It is easy to see that $\sum_{k=1}^{K}(-1)^k\sin C_k\tau = \sin\pi K(1-\tau)/\{2\sin C_k(1-\tau)/2\}$. Hence

$$T\tilde{v}_{K_T} = \sqrt{2}\sum_{t=1}^{T} v_{K_T t}\tilde{w}_{T-t+1},$$

where $v_{K_T t} = \{\sin\pi K_T(t-1/2)/T\}/\{2\sin\pi(t-1/2)/(2T)\}$. Note that

$$|v_{K_T t}| \leq K_T \wedge (2T/t). \tag{a.20}$$

We divide the summation into two parts as $\sum_{t=1}^{T} = \sum_{t=1}^{[T/K_T]} + \sum_{t=[T/K_T]+1}^{T}$. By (a) of Lemma A.1,

$$E(\sum_{t=1}^{[T/K_T]} v_{K_T t}\tilde{w}_{T-t+1})^2 = O(K_T).$$

The second part is evaluated using (a.18) and (a.20) as

$$\begin{aligned}
&\mathrm{Var}(\sum_{t=[T/K_T]+1}^{T} v_{K_T t}\tilde{w}_{T-t+1}) \\
\leq\ & \textstyle\sum_{u=[T/K_T]+1}^{T}\sum_{t\wedge t'=u}|v_{K_T t}||v_{K_T t'}||\mathrm{Cov}(\tilde{w}_{T-t+1},\tilde{w}_{T-t'+1})| \\
\leq\ & \textstyle\sum_{u=[T/K_T]+1}^{T}(T/u)^2 O(1/T) = O(K_T),
\end{aligned}$$

where the second inequality comes from (a.18). Thus we have proved that $E\tilde{v}_{K_T}^2 = O(K_T)$.

Proof of convergence $T^2\tilde{u}_{K_T}/K_T$ is quite similar to that of $\tilde{s}_{K_T}^*$ in Lemma 3 except that $\lim_{T\to\infty}T^{-1}E\sum_{t,t'}\tilde{w}_t w_{t'} = (1-\rho)^{-2}\sigma^2$ instead of (a.14), because

$$\begin{aligned}
T^2\tilde{u}_{K_T}/K_T &= 2\sum_{t,t'}K_T{}^{-1}\sum_{k=1}^{K_T}\sin\frac{C_k}{T}(t-\frac{1}{2})\sin\frac{C_k}{T}(t'-\frac{1}{2})\tilde{w}_t\tilde{w}_{t'} \\
&= \sum_{t,t'}\{\kappa_{K_T}((t-t')/T) - \kappa_{K_T}((t+t'-1)/T\}\tilde{w}_t\tilde{w}_{t'}.
\end{aligned}$$

Proof of convergence $T^2\tilde{s}_{K_T}/{K_T}^2$ is given similarly. We note that

$$\begin{aligned} T^2{K_T}^{-2}\tilde{s}_{K_T} &= \sum_{t,s} {K_T}^{-3}\sum_{k=1}^{K_T} C_k^2 \sin\frac{C_k}{T}(t-\frac{1}{2})\sin\frac{C_k}{T}(s-\frac{1}{2})\tilde{w}_t\tilde{w}_s \\ &= \tilde{s}^{**(1)}_{K_T} - \tilde{s}^{**(2)}_{K_T}, \end{aligned}$$

where we put $\tilde{s}^{**(1)}_K = \sum_{t,t'}(-K^{-2})\kappa''_K\{(t-t')/T\}\tilde{w}_t\tilde{w}_{t'}$, $\tilde{s}^{**(2)}_K = \sum_{t,t'}(-K^{-2})\kappa''_K\{(t+t'-1)/T\}\tilde{w}_t\tilde{w}_{t'}$. Therefore similar argument to that in the proof of convergence of $\tilde{s}_{K_T}$ in Lemma 3 could be traced with $\kappa_K(\tau)$ replaced by $-\kappa''_K(\tau)/K^2$. By direct calculation we have

$$\begin{aligned} \kappa''_K(\tau) &= -\{\pi^2K^2 - (\pi/2)^2 - \frac{(\pi\cos\pi\tau/2)^2}{2(2\sin\pi\tau/2)^2}\}\kappa_K(\tau) \\ &\quad -\{\pi^2\cos(\pi\tau/2)\cos(\pi K\tau)\}/(2\sin\pi\tau/2)^2. \end{aligned}$$

Thus instead of (a.11) the following bound is available :

$$|K^{-2}\kappa''_K(\tau)| \le R[\{1\wedge 1/(K\tau)\} + \{1\wedge 1/(K\tau)^2\} + \{1\wedge 1/(K\tau)^3\}] \qquad \text{(a.21)}$$

where R is some constant.

To see that

$$\lim_{T\to\infty} E\tilde{s}^{**(1)}_{K_T} = \lim_{T\to\infty}(\pi^2/3)E\sum_{t,t'}\tilde{w}_t w_{t'} = (\pi^2/3)(1-\rho)^{-2}\sigma^2,$$

we note that

$$\lim_{T\to\infty}\{-{K_T}^{-2}\kappa''_{K_T}(0)\} = \pi^2/3,$$

that by (a.13) with $p=4$ we have

$$\chi_{M_T}(t,t')\{-{K_T}^{-2}\kappa''_{K_T}((t-t')/T) - \pi^2/3\} = O\{(K_TM_T/T)^2\} = O(K_T/T)$$

and that

$$\begin{aligned} &|E\sum_{t,t'}\{1-\chi_{M_T}(t,t')\}\{-{K_T}^{-2}\kappa''_{K_T}((t-t')/T) - \pi^2/3\}\tilde{w}_t\tilde{w}_{t'}| \\ &\le O(\sum_{m=[M_T]}^{\infty}\tilde{\gamma}_m) \to 0, \end{aligned}$$

since ${K_T}^{-2}\kappa''_{K_T}((t-t')/T)$ is bounded. Then similar argument to that in the proof of (a.14) gives desired equality.

To show

$$\lim_{T\to\infty} E|\tilde{s}^{**(1)}_{K_T} - E\tilde{s}^{**(1)}_{K_T}| = 0,$$

we need to evaluate $\sum_{m=1}^{T} {K_T}^{-2}\kappa''_{K_T}(m/T)O(m^{1/2}\tilde{\delta}_m + 1)$. It is easily shown that $\sum_{m=0}^{T}\{1 \wedge (T/(mK_T))\} = O(T\log T/K_T)$. For $p > 1$ we have

$$\sum_{m=0}^{T}\{1 \wedge (T/(mK_T))^p\} = (\sum_{m\le T/K_T} 1) + (T/K_T)^p \sum_{m>T/K_T} 1/m^p = O(T/K_T).$$

This implies

$$\sum_{m=0}^{T} {K_T}^{-2}\kappa''_{K_T}(m/T)O(1/T^{1/2}) = O(T^{1/2}\log T/K_T). \tag{a.22}$$

Finally we have

$$\sum_{m=0}^{T} |{K_T}^{-2}\kappa''_{K_T}(m/T)|O(m^{1/2}\tilde{\delta}_m)/T^{1/2} \le R\sum_{m=0}^{\infty}(m/T)^{1/2}\tilde{\delta}_m \to 0.$$

Hence by virtue of (c) of Lemma A.1 we have that

$$E|\tilde{s}^{**(1)}_{K_T} - E\tilde{s}^{**(1)}_{KT}| \to 0.$$

The evaluation of $E\tilde{s}^{**(2)}_{K_T}$ and $E|\tilde{s}^{**(2)}_{K_T} - E\tilde{s}^{**(2)}_{K_T}|$ is quite similar to that in the proof of Lemma 3 using (a.22). This completes the proof of Lemma 4. □

Proof of Lemma 5 : Since $\rho > 1$ the third term is clearly dominant in (a.1), we see that for $k = 1, \ldots, K_T$

$$\tilde{z}_k \sim \frac{(\rho-1)T^{-1}\sqrt{2}\cos C_k/2T}{1+\rho^2-2\rho\cos C_k/T}(-1)^k\sum_{t=1}^{T}\rho^{T-t+1}\tilde{w}_t.$$

Because the coefficient of each $\tilde{z}_k$ converges to a common limit uniformly, the claims of the Lemma are obtained by simple calculation using the definition of $\tilde{v}_{K_T}, \tilde{u}_{K_T}$ and $\tilde{s}_{K_T}$.□

Proof of Lemma A.2 :

$$\begin{aligned}
|\tilde{\sigma}_{t,t+m}| &= |\sum_{t'=0}^{t-1}\sum_{s'=0}^{s-1}\rho^{t'+s'}Ew_{t-t'}w_{t+m-s'}| \\
&\le \sum_{t',s'\ge 0}|\rho|^{t'+s'}\gamma_{|m+(t'-s')|} \le \sum_{v=-\infty}^{\infty}\gamma_{|m+v|}|\rho|^{|v|}
\end{aligned}$$

Thus (a.18) is proved as

$$\begin{aligned}
\sum_{m=0}^{\infty}\tilde{\gamma}_{|m|} &\le (1-|\rho|)^{-1}\sum_{k=-\infty}^{\infty}\gamma_{|k|}\sum_{v\le k}|\rho|^{|v|} \\
&\le (1-|\rho|)^{-2}(1+|\rho|)\sum_{k=-\infty}^{\infty}\gamma_{|k|} < \infty.
\end{aligned}$$

Let $m > 0, t_1 \le t_2$ and $t_3 = t_2 + m \le t_4$. We evaluate

$$\mathrm{Cov}(w_{t_1}w_{t_2}, w_{t_3}w_{t_4}) = \sum_{I}\rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4}),$$

where and in the sequel we put $u_j = t_j - t'_j, 1 \le j \le 4$, for simplicity of notation. Let $I = \{(t'_1,t'_2,t'_3,t'_4); 0 \le t'_j < T, 1 \le j \le 4\}$, $I_0 = \{(t'_1,t'_2,t'_3,t'_4) \in I; \max(u_1,u_2) \ge \min(u_3,u_4)\}$ and $I_k = \{(t'_1,t'_2,t'_3,t'_4) \in I; \max(u_1,u_2) + k = \min(u_3,u_4)\}$, $k = 1,\ldots,T-1$. Obviously, $I = I_0 \cup I_1 \cup \cdots \cup I_{T-1}$. We denote $\{(t'_1,t'_2,t'_3,t'_4) \in I; u_1 \ge u_3)\}$ simply by $\{u_1 \ge u_3\}$ and so on. Note that

$$I_0 \subset \{u_1 \ge u_3\} \cup \{u_2 \ge u_3\} \cup \{u_1 \ge u_4\} \cup \{u_3 \ge u_4\}.$$

We have a bound for the summation over $\{u_1 \ge u_4\}$

$$\begin{aligned}
&|\sum_{u_1\ge u_4}\rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \\
&\le R\sum_{t'_1,t'_2,t'_3}|\rho|^{t'_1+t'_2+t'_3}\sum_{t'_4\ge t_4-t_1+t'_1}|\rho|^{t'_4} \\
&\le R(1-|\rho|)^{-3}\sum_{t'_1}|\rho|^{m+2t'_1} = R(1-|\rho|)^{-3}(1-|\rho|^2)^{-1}|\rho|^m
\end{aligned}$$

where R is a constant such that $\sup|\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \leq R$. It is obvious that we have the same bounds for $\sum_{u_1 \geq u_3} \rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})$ and others. Hence

$$|\sum_{I_0} \rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \leq 4R(1-|\rho|)^{-3}(1-|\rho|^2)^{-1}|\rho|^m$$

Next let $k > 0$, then

$$I_k \subset \{u_1 + k = u_3\} \cup \{u_2 + k = u_3\} \cup \{u_1 + k = u_4\} \cup \{u_3 + k = u_4\}.$$

Note by (a.3) there is a constant R' such that $\sup_{I_k}|\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \leq R'\gamma_k^{1-2/\beta}$. Thus we have a bound

$$\begin{aligned}
&|\sum_{\{u_1+k=u_4\}\cap I_k} \rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \\
&\leq R'\gamma_k^{1-2/\beta} \sum_{t'_1,t'_2,t'_3} |\rho|^{t'_1+t'_2+t'_3} \sum_{t'_4 \geq t_4 - t_1 + t'_1} |\rho|^{t'_4} \\
&\leq R'\gamma_k^{1-2/\beta}(1-|\rho|)^{-3}(1-|\rho|^2)^{-1}|\rho|^m
\end{aligned}$$

and so on. Hence

$$|\sum_{I_k} \rho^{t'_1+t'_2+t'_3+t'_4}\mathrm{Cov}(w_{u_1}w_{u_2}, w_{u_3}w_{u_4})| \leq 4R'\gamma_k^{1-2/\beta}(1-|\rho|)^{-3}(1-|\rho|^2)^{-1}|\rho|^m.$$

Finally we obtain

$$|\mathrm{Cov}(w_{t_1}w_{t_2}, w_{t_3}w_{t_4})| \leq 4(1-|\rho|)^{-3}(1-|\rho|^2)^{-1}(R + R'\sum_{k=1}^{\infty}\gamma_k^{1-2/\beta})|\rho|^m.$$

This immediately implies (a.19).□

Reference

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