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Homogeneous Gaussian Diffusions in Finite Dimensions

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Summary

This survey of the theory of Gaussian homogeneous diffusions in finite dimensions (GHD's) contains sections on the structure of the transition probabilities, the construction of GHD's as solutions to stochastic differential equations, a characterization of GHD's with nonsingular transition probabilities, a description of the smooth (at least one time differentiable) components of a GHD, a discussion of coordinatewise, possibly timedependent, affine transformations that carry one GHD into another, a characterization of stationary GHD's and finally, conditions for equivalence between the measures on the space of continuous paths (in finite time) induced by two different GHD's

AMS 1980 subject classifications. 60G15, 60J60

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0. Introduction

In 1944 Doob [4] published a paper, "The Elementary Gaussian Processes", where he presented a classification of all strictly stationary Gaussian processes $X = (X_t)$ which are timehomogeneous Markov. While his results deal with processes in discrete as well as in continuous time, we shall here discuss only the continuous time diffusion case. For that he showed, assuming X to have mean zero, that each X is (after a linear transformation) the direct product of stationary Gaussian processes of certain types, which he labelled $M(0)$, $M(1)$, $M(e^{i\theta})$ and M. Of these $M(0)$ is the trivial null process and the $M(1)$, $M(e^{i\theta})$-processes are deterministic in the sense that they are completely determined by their value at time 0, while the members of the class M were shown by Doob to have the representation

$$X_t = \int_0^t e^{(t-s)B} D dW_s + e^{tB} X_0 \tag{0.1}$$

as stochastic integrals with respect to a multidimensional Brownian motion W, where B is a square matrix with characteristic roots that all have strictly negative real parts and where D is symmetric and positive semidefinite.

The deterministic components of type $M(0)$, $M(1)$, $M(e^{i\theta})$ are all smooth in the sense that they are, as functions of time t, in fact differentiable infinitely often. Doob noticed however that even components of type M-processes may have sample paths that are at least one time continuously differentiable. This lead him to define the concept of a real-valued Gaussian process Y which is Markov of order N: the pathwise derivatives $Y^{(1)}, Y^{(2)}, \ldots, Y^{(N-1)}$ must all exist, and for all, $s \leq t$,

$$E(Y_t | (Y_u)_{u \leq s}) = E(Y_t | Y_s, Y_s^{(1)}, \ldots, Y_s^{(N-1)})$$

In the literature this property is later referred to as Y being N'th order Markov in the restricted sense.

Some years later, Lévy [12] developed what he called a general theory of Gaussian random functions, focusing on representations of the form

$$Y_t = \int_0^t F(t,s) dW_s$$

for a real valued Gaussian proces Y, where F is non-random and W is standard one-dimensional Brownian motion. Obviously Y has mean zero , but otherwise need neither be stationary nor Markov. However, in particular Lévy determined the structure of the representation for a Y which is N'th order Markov in the general sense, that is, a process Y such that $E(Y_t | (Y_u)_{u \leq s})$ for $s < t$ depends only on Y_s and $N-1$ other random variables measurable on the pre-s σ-algebra for Y.

We have highlighted some of the contents of these early papers by Doob and Lévy, because obviously, they inspired much of the work that was done over the next several years on Gaussian processes with (at least) some Markov like properties. While Lévy himself [13] went on to study representations of non-Gaussian processes, Hida [7] continued Lévy's original approach concentrating on the Lévy representation for a N'th order (general sense) Markov, stationary Gaussian process, and also connecting Levy's work to that of Doob. Some more recent references on this are Mandrekar [15] and Okabe [18]. The latter paper in particular, apart from discussing further results on N'th order Markov, stationary Gaussian processes, also contains a section on a derived class of N-dimensional, timehomogeneous Gaussian diffusions (see the end of Section 4 below). Finally, a much more recent reference continuing Doob's work, is Gzyl [6]. As we shall see below (Section 5), there appears to be a flaw in his main result, but once this is remedied, a nice product representation of a certain class of n-dimensional, homogeneous Gaussian diffusions results.

Although reciprocal processes (another class of generalized Markov processes) will not be discussed here, it is appropriate to list some references. Reciprocal stationary Gaussian processes have been studied by Jamison [10] (one dimension), Miroshin [17] (two dimensions) and very recently, by Carmichael et al. [2], (several dimensions).

The purpose of the present paper is to give an overview of the theory of timehomogeneous Gaussian diffusions in n dimensions. Of the earlier work mentioned above, that concerning stationary Gaussian diffusions is most closely related to what follows below. However, in principle we study all timehomogeneous Gaussian diffusions, whether they can be made stationary or not - with, of course, Brownian motion the most celebrated non-stationary member of this extended class. It should be stressed also, that the overlap with Doob's work [4] is through his class M only: we do not discuss diffusions with deterministic components.

It is quite amazing that there is hardly any literature on the class of timehomogeneous Gaussian diffusions as such, and even in the textbooks they are rarely mentioned and then mostly as examples. Undoubtedly, quite a few of the results that are presented below are known, but it would seem, either as part of the folklore or contained in scattered references by authors with highly varying backgrounds.

The present paper is organized as follows: in Section 1 we introduce the class of (A,B,C)- diffusions with the property that the transition probabilities are Gaussian and form a one-parameter transition semigroup. The three parameters determining the diffusion are a n-vector A (the constant drift coefficient), a $n \times n$ matrix B (the linear drift coefficient) and a symmetric, positive semidefinite $n \times n$ matrix C (the infinitesimal covariance matrix). Various functional equations and explicit expressions for the conditional expectations and covariances determining the transition probabilities, are found.

Section 2 discusses the construction of (A,B,C) - diffusions as solutions to stochastic differential equations of the form.

$$(0.2) \qquad dX_t = (A+BX_t)dt + DW_t,$$

where W is a multidimensional Brownian motion and D satisfies $DD^T = C$. We give a direct proof that the solution is a Gaussian diffusion with transitions as described in Section 1, and also discuss the explicit solution to (0.2) (which is the expression (0.1) in the case $A = 0$). A Lévy-type characterization of (A,B,C) - diffusions is included as well.

In Section 3 we determine which (A,B,C) - diffusions have transition probabilities with nonsingular covariances. It is shown that all the transitions are nonsingular if and only if the infinitesimal generator for the transition semigroup is hypoelliptic, a condition satisfied in particular if C is nonsingular. We call an (A,B,C) - diffusion with this property nonsingular, but point out, that even though the finitedimensional (Gaussian) distributions are nonsingular, a nonsingular (A,B,C) - diffusion with C singular, has singular paths in the sense that some of the components of the process are completely determined by others.

The remainder of the paper is concerned with nonsingular (A,B,C) - diffusions. In Section 4 we show that the members of this class for which C is singular, may be characterized by the property that there exists a nontrivial linear combination of the components with continuously differentiable sample paths. We also determine the linear combinations which are differentiable a specified number of times.

It is natural to try to reduce the class of all nonsingular (A,B,C) - diffusions to a few standard forms. This we attempt in Section 5, where we call two diffusions affinely equivalent if there exist affine mappings with, in general, time dependent coefficients, that transform one diffusion into the other. This permits the introduction of what we call rotating Brownian motion and rotating Ornstein-Uhlenbeck processes. In turn these processes (in two dimensions) appear as building blocks in our version of Gzyl's [6] representation theorem for (A,B,C) - diffusions with a normal linear drift coefficient B.

Section 6 is devoted to stationarity, and in particular it is shown that a nonsingular (A,B,C) - diffusion has a strictly stationary version if and only if the real parts of the characteristic roots for B are all strictly negative.

The final Section 7 briefly discusses equivalence between measures induced by nonsingular (A,B,C) - diffusions on the space of $\mathbb{R}^n$- valued continuous paths on a finite time interval. If C is nonsingular, the main result follows easily from standard theory, but we present also an expression for a Radon-Nikodym derivative valid for pairs of nonsingular (A,B,C)- diffusions with the necessarily same singular C.

1. (A,B,C) - diffusions

Let $X = (X_t)_{t \geq 0}$ denote a $\mathbb{R}^n$-valued stochastic process. We shall call X a Gaussian homogeneous diffusion (GHD) provided X is Gaussian and time-homogeneous Markov with continuous sample paths.

Suppose X is a GHD and let $t_0 \geq 0$, $t > 0$. Since X is Gaussian and Markov, there exists a version of the regular conditional distribution of X_{t_0+t} given X_{t_0}, of the form

$$(1.1) \qquad N(\alpha(t_0, t_0 + t) + \beta(t_0, t_0 + t)X_{t_0}, \Sigma(t_0, t_0 + t)).$$

Here $N(\xi,\Gamma)$ denotes the Gaussian law on $\mathbb{R}^n$ with expectation ξ and covariance Γ.

Now suppose in addition, that for every $t_0 > 0$, X_{t_0} given X_0 is nonsingular Gaussian (the conditional covariance is nonsingular). Combining (1.1) with the fact that X is timehomogeneous, it is seen that there exist functions α, β, Σ such that for all $t_0 \geq 0$, $t > 0$ and Lebesgue almost all $x \in \mathbb{R}^n$, the conditional distribution of X_{t_0+t} given $X_{t_0} = x$ is

$$(1.2) \qquad N(\alpha(t) + \beta(t)x, \Sigma(t))$$

Notational convention Vectors, such as $\alpha(t)$, x are viewed as $n \times 1$ column martrices, while $\beta(t)$, $\Sigma(t)$ are $n \times n$ matrices. □

Let $s > 0$. Using (1.2) and computing $E(X_{t_0+s+t} | X_{t_0} = x)$ by conditioning first on X_{t_0+s} it is seen that for almost all x,

$$\alpha(t + s) + \beta(t + s)x = \alpha(t) + \beta(t)(\alpha(s) + \beta(s)x)$$

and consequently

$$\alpha(t + s) = \alpha(t) + \beta(t)\alpha(s) \tag{1.3}$$

$$\beta(t + s) = \beta(t)\beta(s). \tag{1.4}$$

Considering instead the conditional covariances, one obtains

$$\Sigma(t + s) = \Sigma(t) + \beta(t)\Sigma(s)\beta^T(t), \tag{1.5}$$

where T denotes the transpose.

Because X is continuous and Gaussian, for almost all x,

$$t \to E(X_{t_0+t} | X_{t_0} = x), \quad t \to Var(X_{t_0+t} | X_{t_0} = x)$$

are both continuous. It follows that the functions α, β, Σ are continuous on $\mathbb{R}_+ = [0,\infty)$ with boundary values

$$\alpha(0) := \lim_{t \downarrow 0} \alpha(t) = 0, \tag{1.6}$$

$$\beta(0) := \lim_{t \downarrow 0} \beta(t) = I, \tag{1.7}$$

$$\Sigma(0) := \lim_{t \downarrow 0} \Sigma(t) = 0, \tag{1.8}$$

where $I = I_n$ denotes the $n \times n$ identity matrix.

It was assumed above, that the law af X_{t_0} given X_0 should be non-singular Gaussian for all $t_0 > 0$, and some such assumption is needed to arrive at (1.3) - (1.5): in general, for a given $t_0 > 0$, the expression (1.2) for the conditional distribution of X_{t_0+t} given $X_{t_0} = x$ will hold for almost all x with respect to the distribution of X_{t_0}, so for instance (1.2) need only be valid for all x in the affine subspace of $\mathbb{R}^n$ supporting that distribution. And since the affine support may depend on t_0, see Example 1.23 below, there may not be a universal choice of α, β, Σ such that (1.2) describes all the transition probabilities.

Suppose now that α, β, Σ are continuous functions that satisfy (1.3) -.(1.4), but allow for the $\Sigma(t)$ to be positive semidefinite, i.e. not necessarily nonsingular. The arguments leading to these equations also show, that if $p_t(x, \cdot)$ denotes the $N(\alpha(t) + \beta(t)x, \Sigma(t))$ distribution, the p_t form a one-parameter transition semigroup and hence, given any Gaussian distribution μ on $\mathbb{R}^n$, there exists a GHD X with transition probabilities p_t and initial distribution μ. (The path continuity of X follows from Theorem 2.1 below).

In the sequel we shall (Example 1.23 excepted) exclusively treat GHD's with transition probabilities of the form $N(\alpha(t) + \beta(t)x, \Sigma(t))$ as above. The resulting class of GHD's in $\mathbb{R}^n$ is denoted by $\mathcal{C}_n$. As the preceding discussion shows, any n-dimensional GHD such that for all $t_0 > 0$, X_{t_0} given X_0 follows a nonsingular Gaussian distribution, belongs to $\mathcal{C}_n$, but not all GHD's are in $\mathcal{C}_n$.

The first task is to obtain a parametrization of the class $\mathcal{C}_n$ by solving the functional equations (1.3) - (1.5).

<u>1.9 Proposition</u> The continous functions α,β,Σ are solutions to the functional equations (1.3) - (1.5) subject to the boundary conditions (1.6) - (1.8) and with all $\Sigma(t)$ positive semidefinite, if and only if

$$\beta(t) = e^{tB} \tag{1.10}$$

$$\alpha(t) = \int_0^t e^{sB}A\, ds, \tag{1.11}$$

$$\Sigma(t) = \int_0^t e^{sB}Ce^{sB^T} ds \tag{1.12}$$

for some $n \times 1$ column vector A, some $n \times n$ matrix B and some symmetric positive semidefinite $n \times n$ matrix C.

<u>Notation</u> $e^D = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k$ for any $n \times n$ matrix D.

<u>Sketch of proof</u> If α,β,Σ are continuous solutions to (1.3) - (1.5) satisfying the boundary conditions (1.6) - (1.8), (1.10) follows directly from (1.4). One next shows that the limits

$$\lim_{t\downarrow 0} \frac{1}{t}\,\alpha(t) = A, \quad \lim_{t\downarrow 0} \frac{1}{t}\,\Sigma(t) \;= C$$

exist with, evidently, C positive semidefinite. From (1.3) and (1.5) it then follows that α,Σ are differentiable with

(1.13) $$\alpha'(t) = \beta(t)A,$$

(1.14) $$\Sigma'(t) = \beta(t)C\beta^T(t)$$

and (1.11), (1.12) follow using (1.6), (1.8).

If conversely α,β,Σ are given by (1.10) - (1.12) one easily verifies (1.3) - (1.5) directly and (1.6) - (1.8) hold trivially. Also, if C is positive semidefinite, so is the integrand in (1.12) for every s, and hence, so is $\Sigma(t)$. □

From now on, call a GHD from the class $\mathscr{C}_n$ with transition probabilities of the form (1.2) and α,β,Σ given by (1.10) - (1.12) an <u>(A,B,C) - diffusion</u>. Note that

$$A = \alpha'(0),\ B = \beta'(0),\ C = \Sigma'(0).$$

Instead of (1.11), (1.12), one may use the series expansions

(1.15) $$\alpha(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} B^{k-1} A,$$

(1.16) $$\Sigma(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{j=0}^{k-1} \binom{k-1}{j} B^j\ C\ (B^T)^{k-1-j}.$$

We shall list some other formulae that are consequences of (1.10) - (1.12). For instance, (1.11) implies

(1.17) $$B\alpha(t) = (e^{tB} - I)A,$$

in particular, if B is nonsingular

$$(1.18) \qquad \alpha(t) = (e^{tB} - I)B^{-1}A.$$

A compact expression for $\Sigma(t)$ is available only in rather special cases, e.g. if $C = cI$ with $c \geq 0$ and if $BB^T = B^TB$ with $B + B^T$ nonsingular,

$$\Sigma(t) = c(e^{t(B+B^T)} - I)(B + B^T)^{-1}.$$

Finally note that always

$$(1.19) \qquad \Sigma'(t) = C + B\Sigma(t) + \Sigma(t)B^T,$$

indeed, by partial integration in (1.12),

$$B\Sigma(t) = \int_0^t Be^{sB}\, C\, e^{sB^T} ds$$

$$= e^{tB}Ce^{tB^T} - C - \Sigma(t)B^T$$

and (1.19) follows using (1.14).

The formulae above all relate to the functions α, β and Σ. It may be useful to have also an expression for the cross covariances: if X is an (A,B,C) - diffusion with $X_0 \sim N(\mu, \Gamma)$, we have

$$EX_t = \alpha(t) + \beta(t)\mu,$$

$$\text{Var } X_t = \Sigma(t) + \beta(t)\Gamma\beta^T(t),$$

and an easy calculation gives

$$(1.20) \qquad E(X_s - EX_s)(X_{s+t} - EX_{s+t})^T = (\text{Var } X_s)\beta^T(t)$$

$$= \Sigma(s)\beta^T(t) + \beta(s)\Gamma\beta^T(s + t).$$

1.21 Example (See Mehr and Mc Fadden [16]). For $n = 1$, $A = a$, $B = b$ and $C = c > 0$ are constants and

$$\alpha(t) = \frac{1}{b}(e^{tb} - 1)a, \quad \beta(t) = e^{tb},$$

$$\Sigma(t) = \frac{c}{2b}(e^{2tb} - 1).$$

For $b = 0$ we obtain one-dimensional Brownian motion with constant drift a and variance parameter c. For $c > 0$ and $b < 0$ $(b > 0)$, the (a,b,c) - diffusion is the one - dimensional Ornstein-Uhlenbeck process in its recurrent (transient) form. □

1.22 Example BM(n), the n - dimensional standard Brownian motion, is the $(0,0,I)$ - diffusion. □

1.23 Example This example of a twodimensional GHD not in $\mathscr{C}_2$, is due to

Tue Tjur (personal communication).

Let Y be a one dimensional (a,b,c) - diffusion (example 1.21) with $a = 0$, let $\theta > 0$ and define $X = (X_1, X_2)$ by

$$X_{1,t} = Y_t \cos \theta t,$$

$$X_{2,t} = Y_t \sin \theta t,$$

which is obviously a continuous Gaussian process. One may check that X is timehomogeneous Markov with transition probabilities $p_t(x,\cdot)$, which for $t > 0$, $x \neq 0$ are Gaussian with affine support the line through the origin and x, rotated counter clockwise through the angle θt, and where the position on the line is determined by the transitions for Y. (The assumption $a = 0$ is in fact needed for the argument to work: with $a = 0$ the transitions for X from $X \neq 0$ are the same whether x is identified with a positive or negative value y of Y).

The fact that the subspace supporting $p_t(x,\cdot)$ depends on t, shows that X is not in $\mathcal{C}_2$, cf. Theorem 3.2 (ii) below.

Although there is no natural candidate for the transitions from $x = 0$, since $p_t(\tilde{x},\{0\}) = 0$ for $t > 0$ and $\tilde{x} \neq 0$ it is easy to define $p_t(0,\cdot)$ so as to make the p_t an exact transition semigroup, i.e.

$$p_{t+s}(x,\cdot) = \int p_t(x,dx')p_s(x',\cdot)$$

for all $s,t \geq 0$, $x \in \mathbb{R}^2$. However, it is then also seen that the example is pathological in the sense that X is not strong Markov (another reason why it cannot be $\mathscr{C}_2$): the strong Markov property fails at the non-degenerate stopping time

$$\tau = \inf \{t : X_t = 0\},$$

since conditionally on the pre-τ behaviour of X within $(\tau < \infty)$, the law of $X_{\tau+t}$ is supported by a line which cannot be recovered from the information $X_\tau = 0$ alone. □

1.24 Example It is easily verified that an (A,B,C) - diffusion has independent and, necessarily, stationary increments iff $B = 0$, in which case

$$\alpha(t) = tA, \quad \beta(t) = I, \quad \Sigma(t) = tC. \qquad \square$$

2. (A,B,C) - diffusions as solutions to stochastic differential equations

Let $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ be a filtered probability space satisfying the "usual conditions". Also let W be a BM(n')- process (standard n'- dimensional Brownian motion) on this space, in particular W is adapted and $W_{t+s} - W_s$ is independent of $\mathcal{F}_s$.

Call a $\mathbb{R}^n$- valued process X an (A,B,C) - diffusion with respect to the filtration $(\mathcal{F}_t)$, if X is an adapted (A,B,C) - diffusion such that for any $t_0 \geq 0$, $t > 0$, conditionally on $\mathcal{F}_{t_0}$, X_{t_0+t} is $N(\alpha(t) + \beta(t)X_{t_0}, \Sigma(t))$- distributed.

Given a n-column vector A, a $n \times n$-matrix B and a $n \times n$ positive semidefinite matrix $C \neq 0$, let D be a $n \times n'$-matrix such that $DD^T = C$. We shall show

2.1. Theorem Given a $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued Gaussian random variable X_0, the stochastic differential equation

$$dX = (A+BX)dt + DdW \tag{2.2}$$

with initial value X_0 has a unique strong solution, and this solution is an (A,B,C) - diffusion with respect to the filtration $(\mathcal{F}_t)$.

Remarks That (2.2) has a unique strong solution which is a GHD is of course known, see e.g. Section IV.8 of Ikeda and Watanabe [9]. The argument presented there exploits the explicit solution (see (2.6) below), but here we find it useful to give a proof that permits a direct identification of the transition probabilities.

Proof Let $(.,.)$ denote the inner product on $\mathbb{R}^n$, let α, Σ be given by (1.11), (1.12) and recall that $\beta(t) = e^{tB}$ with inverse e^{-tB}. That X, the solution to (2.2), is an (A,B,C) - diffusion will follow if we show that for any $u \in \mathbb{R}^n$,

$$Z_t = \exp\{i(u, e^{-tB}(X_t-\alpha(t))) + \frac{1}{2} u^T e^{-tB}\Sigma(t)e^{-tB^T} u\}$$

is an adapted, complex - valued martingale: from $E(Z_{s+t}|\mathcal{F}_s) = Z_s$, replacing u by $e^{(s+t)B^T}v$ and using (1.3), (1.5), it follows that

$$E(\exp(i(v,X_{s+t}))|\mathcal{F}_s)$$

(2.3)

$$= \exp(i(v,\alpha(t) + \beta(t)X_s) - \frac{1}{2} v^T\Sigma(t)v),$$

i.e. conditionally on $\mathcal{F}_s$, X_{s+t} is Gaussian with mean $\alpha(t) + \beta(t)X_s$ and covariance $\Sigma(t)$, as desired.

We now show that Z is a local martingale. Since Z is uniformly bounded on compact intervals, it is then automatically a martingale, and the proof will be complete.

But by Ito's formula and (2.2),

$$\begin{aligned} dZ_t = Z_t\{&i(u,e^{-tB}[B\alpha(t) - \alpha'(t) + A])dt \\ &+ \frac{1}{2} u^T e^{-tB}(-B\Sigma(t) + \Sigma'(t) - \Sigma(t)B^T)e^{-tB^T} u\, dt \\ &+ i(u, e^{tB} D\, dW_t) - \frac{1}{2} u^T e^{-tB} d[X,X]_t\, e^{-tB^T} u\}, \end{aligned}$$

and since

$$d[X,X]_t = C\,dt,$$

it is seen that this reduces to

$$dZ_t = Z_t i(u, e^{-tB} D\, dW_t),$$

which renders Z a local martingale, provided we verify that

$$B\alpha(t) - \alpha'(t) + A = 0, \tag{2.4}$$

$$- B\Sigma(t) + \Sigma'(t) - \Sigma(t)B^T - C = 0. \tag{2.5}$$

But (2.5) is just (1.19), and (2.4) follows from (1.13) and (1.17). □

The SDE (2.2) is of course so simple that it may be solved explicitly. Expressions like (2.6) below were used already by Doob [4], Section 4 in his study of stationary, Gaussian Markov processes in continuous time.

Defining

$$Y = X - DW,$$

it is seen that the paths for Y satisfy the ordinary differential equation.

$$dY_t = (A+BDW_t)dt + BY_t\, dt,$$

and hence

$$X_t = e^{tB}X_0 + \int_0^t e^{(t-s)B}(A+BDW_s)ds + DW_t$$

(2.6)

$$= e^{tB}X_0 + \int_0^t e^{sB}Ads + e^{tB}\int_0^t e^{-sB}DdW_s.$$

The next result is a simple generalization of Lévy's characterization of Brownian motion as a cntinuous local martingale with quadratic variation $[W,W]_t = tI$.

2.7 <u>Theorem</u> Let X be an adapted, continuous process on $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ and suppose that X_0 is Gaussian. Then X is an (A,B,C) - diffusion if and only if the process U, where

(2.8) $$U_t = X_t - X_0 - tA - \int_0^t BX_s ds,$$

is a continuous local martingale with $[U,U]_t = tC$. Furthermore, if X is an (A,B,C) - diffusion, U is automatically a martingale.

<u>Remark</u> Note that if C is nonsingular and D is $n \times n$ with $DD^T = C$, then U is a continuous local martingale iff $D^{-1}U$ is a BM(n) process.

□

<u>Proof</u> If X solves (2.2), it is immediate that $d[X,X]_t = C\,dt$ and that U is a continuous, local martingale. If X is any (A,B,C) -

diffusion, it is still true that $d[X,X]_t = C\,dt$, this being a distributional property of the diffusion, and then also $[U,U]_t = tC$ by (2.8). We show that U is a martingale by direct computation, using the regular conditional probability given $\mathcal{F}_s$ on the σ-algebra $\sigma(X_u)_{u\geq s}$, obtained because X is Markov: for $s,t \geq 0$,

$$E(U_{s+t}|\mathcal{F}_s) = E(X_{s+t}|\mathcal{F}_s) - X_0 - (s+t)A - \int_0^s BX_u du - E(\int_s^{s+t} BX_u du|\mathcal{F}_s)$$

$$= \alpha(t) + \beta(t)X_s - X_0 - (s+t)A - \int_0^s BX_u du - \int_0^t B(\alpha(u) + \beta(u)X_s)du,$$

which reduces to U_s provided

$$(2.9) \qquad \alpha(t) = tA + \int_0^t B\alpha(u)du$$

$$(2.10) \qquad \beta(t) - I = \int_0^t B\beta(u)du$$

Here (2.10) is trivial since $\beta(u) = e^{uB}$, and (2.9) is equivalent to (2.4).

Suppose now that X is adapted and continuous, and that U defined by (2.8) is a, necessarily continuous, local martingale with $[U,U]_t = tC$. Find an orthogonal matrix S such that $\tilde{C} = SCS^T$ is diagonal, and introduce the process $\tilde{U} = SU$. Then $\tilde{U}$ is a continuous local martingale with $\tilde{U}_0 = 0$, $[\tilde{U},\tilde{U}]_t = tC$. If, as we may assume,

$$\tilde{C} = \text{diag}\,(\lambda_1,\ldots,\lambda_{n'},0,\ldots,0)$$

with $\lambda_1,\ldots,\lambda_{n'}$ all > 0, it follows from Lévy's theorem that

$$W = \left[\frac{1}{\sqrt{\lambda_1}}\tilde{U}_1,\ldots,\frac{1}{\sqrt{\lambda_{n'}}}\tilde{U}_{n'}\right]$$

is a BM(n') - process. Also, for $n' < \ell \leq n$, $\tilde{U}_\ell$ is a continuous local martingale of vansishing quadratic variation, and hence

$$\tilde{U}_{n'+1} \equiv \ldots \equiv \tilde{U}_n \equiv 0.$$

Consequently

$$dU_t = S^T d\tilde{U}_t = DdW_t$$

with D the $n \times n'$ - matrix

$$S^T \begin{bmatrix} K \\ 0 \end{bmatrix}$$

where $K = \mathrm{diag}\,(\sqrt{\lambda_1},\ldots,\sqrt{\lambda_{n'}})$ and 0 is the $(n-n')\times n'$ null matrix. Using (2.8), we see that

$$dX_t = (A + BX_t)dt + DdW_t.$$

Since $DD^T = C$, it follows from Theorem 2.1 that X is an (A,B,C) - diffusion. □

From the proof we read off the following

2.11 Corollary Suppose X is an (A,B,C) - diffusion defined on $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, and let $n' = \text{rank}(C)$. Then there exists an adapted $BM(n')$ - process W such that, using this W, X solves the stochastic differential equation (2.2). □

Suppose again that X is an (A,B,C) - diffusion, Markov with respect to the given filtration $(\mathcal{F}_t)$, and define

$$V_t = e^{-tB}X_t.$$

2.12 Proposition V is a Gaussian process with independent increments, i.e. for any $s,t \geq 0$, $V_{s+t} - V_s$ is Gaussian and independent of $\mathcal{F}_s$.

Proof Using (2.3), the conditional characteristic function of an increment is found to be

$$(2.13)\quad \begin{aligned} &E(\exp\{i(v,V_{s+t} - V_s)\}|\mathcal{F}_s) \\ &= \exp\{i(v,e^{-(s+t)B}\alpha(t)) - \tfrac{1}{2}v^T e^{-(s+t)B}\Sigma(t)e^{-(s+t)B^T}v\}, \end{aligned}$$

which is non-random as required. □

For $s \leq t$, define $\xi(s,t) = E(V_t - V_s)$ and $\Gamma(s,t) = \text{Var}\,(V_t - V_s)$. Because V has independent increments, Γ is additive, while ξ is additive automatically:

$$\xi(s,u) = \xi(s,t) + \xi(t,u), \quad \Gamma(s,u) = \Gamma(s,t) + \Gamma(t,u)$$

for $s \leq t \leq u$. We can read off ξ and Γ from (2.13), and, using (1.3), (1.5), write them in additive form,

$$\xi(s,t) = e^{-tB}\alpha(t) - e^{sB}\alpha(s) = \int_s^t e^{-uB}A\, du,$$
$$\Gamma(s.t) = e^{-tB}\Sigma(T)e^{-tB^T} - e^{-sB}\Sigma(s)e^{-sB^T} = \int_s^t e^{-uB}Ce^{-uB^T} du,$$

see also (1.11), (1.12).

In general, V is not a GHD because the increments are non-stationary. It is easily verified that V has stationary increments iff

$$BA = 0, \quad BC + CB^T = 0,$$

in which case V is an (A,0,C) - diffusion, cf Example 1.24.

It may also be noted, that the general V is a martingale iff $A = 0$.

3. Nonsingularity and singularity of (A,B,C) - diffusions

Let X be an (A,B,C) - diffusion, so that the covariances $\Sigma(t)$ for the transition probabilities $p_t(x,\cdot)$ are given by either of the expressions (1.12) or (1.16).

3.1 Definition X is nonsingular if $\Sigma(t)$ is nonsingular for all $t > 0$, and singular if $\Sigma(t)$ is singular for all $t > 0$. □

The definition is exhaustive by the first statement in the following

3.2 Theorem (i) An (A,B,C) - diffusion is either nonsingular or singular. It is nonsingular if and only if

$$\text{rank}(D, BD, B^2D, \cdots, B^{n-1}D) = n, \tag{3.3}$$

where D is any matrix such that $DD^T = C$.

(ii) The linear subspace of $\mathbb{R}^n$ determined by the affine support for the Gaussian probability $p_t(x,\cdot)$, is the same for all $t > 0$ and all $x \in \mathbb{R}^n$.

Notation If $M_1, \cdots, M_k$ are matrices with the same number of rows, rank $(M_1, \cdots, M_k)$ denotes the dimension of the subspace spanned by the column vectors in all the M_i.

Remarks The condition (3.3) is necessary and sufficient for the infinitesimal generator of the transition semigroup $(p_t)_{t \geq 0}$ to be hypoelliptic, see Section 6 of Ichihara and Kunita [8] or Chaleyat - Maurel and Elie [3]. It is also the condition for the pair (B,D) of matrices to be controllable, see Zakai and Snyders [20], Section 4.2 of Davis [21] or Section 5.6 A in Karatzas and Shreve [11]. Also Okabe [18], Theorem 4.1, arrives at (3.3) for the particular GHD's that he studies.

Theorem 3.1 (i) is contained in Proposition 6.2 of [8], and Proposition 4.2.5 of [21]. □

Proof From (1.5) it is seen that $t \to \Sigma(t)$ is increasing in the sense that for $s,t \geq 0$, $\Sigma(t+s) - \Sigma(t)$ is symmetric and positive semidefi-,nite. In particular, for any $a \in \mathbb{R}^n$, the non-negative function

$$t \to a^T\Sigma(t)a \tag{3.4}$$

is non-decreasing.

The subspace determined from the affine support for $p_t(x,\cdot)$ does not depend on x, and equales $L(t) = M^{\perp}(t)$, where

$$M(t) = \{a \in \mathbb{R}^n : a^T\Sigma(t)a = 0\}.$$

Since the function in (3.4) is increasing, the subspaces $L(t)$ increase with t. Introduce $L = \bigcap_{t>0} L(t)$ and conclude that $L(t) = L$ for t sufficiently small and hence that $a \in L^{\perp}$ iff $a^T\Sigma(t)a = 0$ for t sufficiently small. But (1.16) provides a Taylor expansion for $t \to a^T\Sigma(t)a$, converging for all t, so if $a \in L^{\perp}$, since the function vanishes on a non-degenerate interval, it vanishes everywhere and thus we see that

$L^{\perp} \subseteq M(t)$ for all $t > 0$, i.e. $L \supseteq L(t)$ and the assertion $L = L(t)$ for all $t > 0$ follows. We have shown (ii) and also that an (A,B,C) - diffusion is either non-singular or singular.

To establish the condition (3.3) for an (A,B,C) - diffusion X to be nonsingular, we assume, as we may, that X is obtained as a solution to (2.2) where $DD^T = C$. With $\tilde{X} = X - E(X)$, from (2.2) we have

$$(3.5) \qquad \tilde{X}_t - e^{tB}\tilde{X}_0 = J_t + DW_t,$$

where

$$J_t = \int_0^t e^{(t-s)B} BDW_s \, ds.$$

Suppose that X is singular. Then for any $a \in L^{\perp}\backslash 0$, $a^T\Sigma(t)a = 0$, i.e. $\mathrm{Var}(a^T\tilde{X}_t) = 0$ or $a^T\tilde{X}_t = 0$ (since $E(a^T\tilde{X}_t) = 0$) for all t. Now, assuming for convenience that $\tilde{X}_0 \equiv 0$,

$$da^T\tilde{X}_t = a^T(BJ_t + BDW_t)dt + a^TDdW_t$$

and since $[a^T\tilde{X}, a^T\tilde{X}] \equiv 0$, we see that $a^TD = 0$ and then, since $a^T\tilde{X} \equiv 0$, that

$$a^TBJ + a^TBDW \equiv 0.$$

The quadratic variation of this process is therefore 0, i.e. $a^TBD = 0$, and then $a^TBJ \equiv 0$ follws. Since

$$da^T BJ_t = a^T B(BJ_t + BDW_t)dt$$

also

$$a^T B^2 J + a^T B^2 DW \equiv 0$$

whence $a^T B^2 D = 0$ etc. Continuing, it is seen that $a^T B^k D = 0$ for all $k = 0,1, \cdots,$ and thus the subspace spanned by all columns in the $B^k D$ has dimension $< n$, so (3.3) does not hold. If conversely (3.3) does not hold since the subspace spanned by all columns in all matrices $B^k D$ is the same as that spanned by the first n, $D,\cdots,B^{n-1}D$, we can find $a \neq 0$ such that $a^T B^k D = 0$ for all k. But then (1.12) or (1.16) shows that $a^T \Sigma(t)a = 0$ for all t, i.e. the diffusion is singular. □

In the sequel we shall exclusively discuss nonsingular (A,B,C) - diffusions. The reader is reminded that this subclass of $\mathcal{C}_n$ agrees with all GHD's in n dimensions for which all the transitions are nonsingular Gaussian.

Note that if X is nonsingular, then for all k and all $0 < t_1 \cdots < t_k$, the kn - dimensional random variable $(X_{t_1},\cdots,X_{t_k})$ follows a Gaussian distribution with a covariance of full rank kn.

Consider a nonsingular (A,B,C) - diffusion with C singular. Even though the diffusion is nonsingular in the sense of Definition 3.1, it is singular in the sense that parts of the process are completely determined from others. More specifically we have

3.6 <u>Proposition</u> Let X be a nonsingular (A,B,C) - diffusion with C singular. Then there exists a nonsingular $n \times n$ - matrix F such that with

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = FX$$

where Y_1 contains the $n' = \text{rank}(C)$ first and Y_2 the $n - n'$ last components of FX, the process Y_2 is adapted to the filtration $(\mathcal{F}_t^0(Y_1) \vee Y_{2,0})_{t \geq 0}$. Furthermore, Y_1 solves an equation of the form

$$dY_{1,t} = Z_t dt + dW_t,$$

where W is a $BM(n')$ - process and Z_t is $\mathcal{F}_t^0(Y_1) \vee Y_{2,0}$ - measurable.

<u>Notation</u> $\mathcal{F}_t^0(Y_1) \vee Y_{2,0}$ denotes the σ-algebra generated by the random variables $Y_{1,s}$ for $0 \leq s \leq t$ and $Y_{2,0}$. □

<u>Proof</u> There exists F nonsingular so that FX is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion with

(3.7)
$$\tilde{C} = \begin{bmatrix} I_{n'} & 0 \\ 0 & 0 \end{bmatrix}$$

see Propositions 5.3 and 5.4 below. Without loss of generality we may therefore assume that C itself is given by (3.7). Further, by Corollary 2.11 there exists a $BM(n')$-process W such that

$$dX_t = (A + BX_t)dt + DdW_t$$

where

$$D = \begin{bmatrix} I_{n'} \\ 0 \end{bmatrix}.$$

Writing

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where e.g. X_1 consists of the first n' components of X and B_{11} is the upper left $n' \times n'$ - submatrix of B, it is seen that

(3.8) $$dX_{1,t} = (A_1 + B_{11}X_{1,t} + B_{12}X_{2,t})dt + dW_t,$$

$$dX_{2,t} = (A_2 + B_{21}X_{1t} + B_{22}X_{2,t})dt.$$

The latter is an ordinary differential equation for X_2 with a unique solutuion if $X_{2,0}$ and X_1 are specified. The desired conclusions now follow easily. □

4. (A,B,C) - diffusisions with smooth components

From Theorem 3.2 it is clear that there exist nonsingular (A,B,C) - diffusions with a singular C. In this section we shall discuss a characteristeric path property of such diffusions. As usual, D denotes any $n \times n'$- matrix with $DD^T = C$.

4.1. Definition (a) An (A,B,C) - diffusion X has a smooth component if there exists $\ell: \mathbb{R}^n \to \mathbb{R}$ linear, $\ell \neq 0$, such that the paths of the process $\ell(X)$ are continuously differentiable.

(b) A smooth component $\ell(X)$ is of degree $d \geq 1$ provided the paths of $\ell(X)$ are d times continuously differentiable and the normalized d'th derivative $\ell(X)^{(d)} - \ell(X)_0^{(d)}$ has non - vanishing quadratic variation. □

Notation Instead of writing $\ell(x)$, where $\ell: \mathbb{R}^n \to \mathbb{R}$ is linear, we shall write $a^T x$ with a a n-column vector.

4.2 Theorem Let X be a non-singular (A,B,C) - diffusion. Then

(a) X has a smooth component if and only if C is singular;

(b) for $a \neq 0$, $a^T X$ is smooth of order d if and only if

$$(4.3) \qquad a^T D = a^T BD = \ldots = a^T B^{d-1} D = 0, \quad a^T B^d D \neq 0;$$

and in that case the derivatives are given by

$$(4.4) \qquad (a^T X)^{(p)} = a^T B^{p-1} A + a^T B^p X \quad (1 \leq p \leq d)$$

(c) X has a smooth component of order at least d if and only if

$$\text{rank}(D, BD, \ldots, B^{d-1}D) < n.$$

Proof Throughout the proof we assume that X solves the SDE (2.2) with W a BM(n')-process. Also, for convenience we assume that $X_0 \equiv 0$.

Obviously (a) is a special case of (c) (for d=1) and (b) implies (c). Even though we need only prove (b), we find it instructive to present a simple direct proof af (a).

Because of (2.2), for any $a \in \mathbb{R}^m$,

$$d(a^T X) = a^T(A+BX)dt + a^T DdW. \tag{4.5}$$

But if C is singular, there exists $a \neq 0$ such that $a^T D = 0$, and we see that $a^T X$ is continuously differentiable. Conversely, if $a^T X$ is smooth and $a \neq 0$, then $[a^T X, a^T X] \equiv 0$ so (4.5) shows that $a^T C\, a = 0$, i.e. C is singular.

To prove (b), assume first that $a \neq 0$ satisfies (4.3). From the argument just given since $a^T D = 0$ we know that $a^T X$ is continuously differentiable, and now proceed to show by induction, that if $a^T X$ is $p - 1$ times continuously differentiable, where $2 \leq p \leq d$, it is also p times continuously differentiable with p'th dervative $(a^T X)^{(p)}$ given by (4.4) (which for $p = 1$ is just (4.5) when $a^T D = 0$).

But with (4.4) valid for $p - 1$, it follows from (2.2) that

$$d(a^T X)^{(p-1)} = a^T B^{p-1} dX$$

$$= (a^T B^{p-1} A + a^T B^p X)dt$$

since $a^TB^{p-1}D = 0$. Thus $(a^TX)^{(p-1)}$ is differentiable with derivative given by (4.4).

Using (4.4) for $p = d$, one finds

$$d(a^TX)^{(d)} = (a^TB^dA + a^TB^{d+1}X)dt + a^tB^dDdW,$$

in particular

$$[(a^TX)^{(d)},(a^TX)^{(d)}]_t = t\, a^T B^dC\, (B^T)^da$$

does not vanish and thus a^TX is smooth of degree d.

If conversely a^TX is smooth of degree d, we show that (4.4) holds and that $a^TB^{p-1}D = 0$ for $1 \leq p \leq d$. Indeed, since a^TX is smooth, the right hand side of (4.5) is of bounded variation on finite intervals, hence $a^TD = 0$ and (4.4) holds for $p = 1$. And if $a^TB^{p-2}D = 0$ and (4.4) holds for $p - 1$, where $2 \leq p \leq d$, then

$$(a^TX)^{(p)}dt = d(a^TX)^{(p-1)} = (a^TB^{p-1}A + a^TB^pX)dt + a^TB^{p-1}DdW,$$

in particular the right hand side is of bounded variation wherefore $a^TB^{p-1}D = 0$ and (4.4) holds for p.

Finally, from (4.4) for $p = d$

$$[(a^TX)^{(d)}, (a^TX)^{(d)}]_t = t\, a^TB^dC\, (B^T)^da,$$

and since by assumption this quadratic variation must not vanish, $a^TB^dD \neq 0$. □

<u>Remarks</u> Combining (c) with Theorem 3.2, we see that all smooth components of a nonsingular (A,B,C) - diffusion are of degree $\leq n - 1$. Furthermore, since $\text{rank}(D, BD, \ldots, B^{d-1}D)$ increases strictly with d until the value n is reached, we see that if $\text{rank}(C) = r < n$, all smooth components are of degree $\leq n - r$.

Singular (A,B,C) - diffusions may be characterized as those with C^∞ - smooth components: if X is singular, we know from Section 3 that there exists $a \neq 0$ with $a^T\Sigma(t)a = 0$ for all t, i.e. $\text{Var}(a^TX_t|X_0) = 0$ and consequently, a.s, simultaneously for all t

$$a^TX_t = E(a^TX_t|X_0) = a^T(\alpha(t) + \beta(t)X_0),$$

with the rightmost expression differentiable infinitely often.

Dym [5] studies a particular class of n-dimensional GHD's with a smooth component of order $n - 1$. In our notation, these processes are (0,B,C) - diffusions with B of the form

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ b_n & b_{n-1} & b_{n-2} & & b_2 & b_1 \end{bmatrix}$$

and $C = DD^T$ with $D^T = (0 \ldots 0\ 1)$. Essentially the same class of processes reappear in Okabe [18], Section 4. (See also Example 4.6 and Proposition 4.10 below). □

4.6. Example The simplest example of a nonsingular X with a singular C is obtained for $n = 2$ with

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We may take $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in which case $BD = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so (3.3) holds and the diffusion is indeed nonsingular.

Solving (2.2) with W a BM(1)-process gives

$$dX_{1,t} = dW_t, \quad dX_{2,t} = X_{1,t}dt$$

so if we add the initial condition $X_0 \equiv 0$,

$$X_{1,t} = W_t, \quad X_{2,t} = \int_0^t W_s ds.$$

The example is easily generalized to n dimensions, where smooth components of order $1,\ldots,n-1$ are obtained as successive integrals of a one-dimensional Brownian motion: take $A = 0$ and

$$B = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

use $D^T = (1 \; 0 \; \ldots \; 0)$ and with W as above, obtain

$$dX_{1,t} = dW_t, \quad dX_{p,t} = X_{p-1,t}dt \qquad (2 \le p \le n).$$

□

4.7 Example In the previous example with $n = 2$, since $e^{-tB} = I - tB$, one finds that the associated process with independent increments is given by (see Proposition 2.12)

$$V_t - V_0 = \begin{bmatrix} W_t \\ - t\, W_t + \int_0^t W_s ds \end{bmatrix}.$$

Note that V is a martingale, cf. the final comment of Proposition 2.12.

Suppose also that $V_0 \equiv 0$. Then $tV_{1,t} + V_{2,t}$ is differentiable i.e. V is an example of a continuous multidimensional martingale $M = (M_1,\dots,M_n)$ for which there exists continuous functions $f_1,\dots,f_n$ giving the process

$$\sum_{i=1}^{n} f_i M_i \tag{4.8}$$

smooth paths, while still the process in non-trivial (non-constant). (Of course, if $f_1,\dots,f_n$ are constants, (4.8) defines a continuous martingale, and therefore, if it has smooth paths, it is constant). □

4.9 Example For $n = 2$, $A = 0$,

$$B = \begin{bmatrix} b & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $b \neq 0$, we obtain using $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with W a BM(1),

$$dX_{1,t} = b\, X_{1,t} dt + dW_t,$$

$$dX_{2,t} = X_{1,t} dt$$

which, for $b < 0$, makes X_1 a recurrent Ornstein-Uhlenbeck process, and X_2 the classical Ornstein-Uhlenbeck velocity process. □

We shall conclude this section with some comments on the connection between the results above and the theory of higher order Markov processes (see the introduction for references). The reader is first reminded that (Doob [4], p. 272), a real valued process $Y = (Y_t)_{t \geq 0}$ is p'th order Makov in the restricted sense if the sample paths are $p-1$ times differentiable and for all $s \geq 0$, $t > 0$ the conditional distribution of Y_{s+t} given $(Y_u)_{0 \leq u \leq s}$ depends only on Y_s, $Y_s^{(1)}, \ldots, Y_s^{(p-1)}$.

In particular 1'st order Markov is just Markov. Note that if Y is p'th order Markov, it is also Markov of any order higher than p, as long as the required path derivatives exist.

4.10 Proposition Le X be a n-diminsional nonsingular (A,B,C) - diffusion, and let $a^T X$ be a linear component.

(a) If a^TX is smooth of order $d \geq 0$, then a^TX is p'th order Markov in the restricted sense, where $1 \leq p \leq d + 1$, provided

$$(4.11) \qquad a^TB^p \in \text{span}\{a^TB^\ell : 0 \leq \ell \leq p - 1\}.$$

(b) If a^TX is a linear component which is smooth of maximal order $n - 1$, then a^TX is n'th order Markov in the restricted sense.

Remarks (a) applies in particular if a^TX is smooth of order 0, i.e. if a^TX has a non vanishing quadratic variation. In that case we see that a^TX is Markov iff $a^TB = c\, a^TB$ for some constant c. The reader may verify that this is the same as the condition (5.16) below when $Fx = a^Tx$.

It was noted above that for an (A,B,C) - diffusion in n dimensions, all components are smooth of order $d \leq n - 1$. (b) tells us what happens for those rather special diffusions that posess a smooth component of order $n - 1$, such as the diffusions studied by Dym [5] and Okabe [18] (who started with a n'th order process and then arrived at his diffusions).

Proof (a) Let a^TX be smooth of order d and fix p, $1 \leq p \leq d + 1$. In particular, the derivatives $(a^TX)^{(\ell)}$ exist for $1 \leq \ell \leq p - 1$. Clearly, to show that a^TX is p'th order Markov, it suffices to show that for s and $t > 0$ given

$$\mathscr{L}(a^TX_{s+t}|\mathscr{F}_s) = \mathscr{L}(a^TX_{s+t}|(a^TX)_s^{(\ell)},\ 0 \leq \ell \leq p - 1),$$

where $\mathcal{L}$ stands for distribution. But both conditional distributions are Gaussian with, as a consequence, non random conditional variances. It therefore suffices to show that there exist constants c_ℓ and γ such that

$$E(a^T X_{s+t} | \mathcal{F}_s) = \sum_{\ell=0}^{p-1} c_\ell (a^T X)_s^{(\ell)} + \gamma$$

or, using (4.4), that

$$a^T(\alpha(t) + \beta(t)X_s) = \sum_{\ell=0}^{p-1} c_\ell \; a^T B^\ell \; X_s + \sum_{\ell=1}^{p-1} c_\ell \; a^T B^{\ell-1} A + \gamma.$$

But this is possible if $c_\ell = c_\ell(t)$ can be found such that

$$a^T \beta(t) = \sum_{\ell=0}^{p-1} c_\ell \; a^T B^\ell,$$

i.e. precisely when

$$a^T \beta(t) \in \text{span } \{a^T B^\ell : 0 \leq \ell \leq p - 1\},$$

and that this is true for all $t > 0$ is immediate from (4.11) which implies that $a^T B^m$ is in the linear span of the vectors $a^T B^\ell$ $(0 \leq \ell \leq p - 1)$ for all m.

(b) If $a^T X$ is smooth of order $n-1$,

$$(4.12) \qquad a^T B^\ell D = 0 \quad (0 \leq \ell \leq n - 2), \quad a^T B^{n-1} D \neq 0$$

by Theorem 4.2(b). In particular

$$\operatorname{rank}(D, BD, \ldots, B^{n-2}D) < n.$$

On the other hand, (3.3) holds and therefore $\rho_\ell = \operatorname{rank}(D, BD, \ldots, B^{\ell-1}D)$ increases strictly with ℓ as long as $1 \leq \ell \leq n$. This is possible only if $\rho_\ell = \ell$, in particular $\operatorname{rank}(D) = 1$ and we may choose D to be a $n \times 1$ column vector and then deduce that the n vectors $D, BD, \ldots, B^{n-1}D$ are linearly independent and span $\mathbb{R}^n$. Using this we shall show that

$$\text{(4.13)} \qquad \operatorname{span}\{a^T B^\ell : 0 \leq \ell \leq n-1\} = \mathbb{R}^n,$$

which implies (4.11) for $p = n$, completing the proof.

That (4.13) holds, follows if we show that if $x \in \mathbb{R}^n$ satisfies $a^T B^\ell x = 0$, $0 \leq \ell \leq n-1$, then $x = 0$. Write x in the form

$$\sum_{\ell=0}^{n-1} e_\ell B^\ell D.$$

Using (4.3) with $d = n$ we obtain

$$0 = a^T x = e_{n-1} a^T B^{n-1} D$$

forcing $e_{n-1} = 0$ and

$$x = \sum_{\ell=0}^{n-2} e_\ell B^\ell D.$$

But then, again using (4.3),

$$0 = a^T B\, x = e_{n-2} a^T B^{n-1} D$$

so $e_{n-2} = 0$. Continuing we find $e_\ell = 0$, $0 \leq \ell \leq n-1$, and thus $x = 0$ as desired. □

5. Affine equivalence of (A,B,C) - diffusions

In this section we discuss conditions that permit us to transform, in a simple manner, the paths of an (A,B,C) - diffusion into those of another. We shall focus on transformations that are coordinatewise affine with, possibly, timedependent coefficients.

Let Φ denote the space of continuous functions $\varphi\colon [0,\infty) \to \mathbb{R}^n$, and let Ψ denote the space of continuous functions ψ from $[0,\infty)$ to the space of nonsingular $n \times n$-matrices. Further, let X be an (A,B,C) - and $\tilde{X}$ an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion, both n-dimensional and nonsingular.

5.1. Definition X and $\tilde{X}$ are affinely equivalent, denoted $X \sim \tilde{X}$, if there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that the process $\varphi + \psi X$ is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion. □

Notation The process $\varphi + \psi X$ at time t takes the value $\varphi(t) + \psi(t)X_t$. □

For $\varphi + \psi X$ to be an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion, it must be continuous. With X nonsingular, the continuity of φ and ψ is then forced and is therefore assumed in the definition of Φ and Ψ. As will be seen below, only φ and ψ that are C^∞ need to be considered.

Obviously, for any $\varphi \in \Phi$, $\psi \in \Psi$, $\tilde{X} = \varphi + \psi X$ is a Gaussian diffusion with, possibly, non-stationary transition probabilities. These are determined from the conditional expectations and covariances for $\tilde{X}$ and using the nonsingularity of X it is then seen that the requirement that $\tilde{X}$ be an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion is equivalent to φ,ψ satisfying the equations

$$(5.2)\qquad \begin{aligned} \varphi(s+t)+\psi(s+t)\alpha(t) &= \tilde{\alpha}(t)+\tilde{\beta}(t)\varphi(s) \\ \psi(s+t)\beta(t) &= \tilde{\beta}(t)\psi(s) \\ \psi(s+t)\Sigma(t)\psi^T(s+t) &= \tilde{\Sigma}(t) \end{aligned}$$

for any $s,t \geq 0$. (Of course $\tilde{\alpha},\tilde{\beta},\tilde{\Sigma}$ are the functions given by (1.10) - (1.12), using $\tilde{A},\tilde{B},\tilde{C}$ as input). Fixing s, it is seen from the second equation that ψ is C^∞ to the right of s and the first then shows the same to be true for φ. Thus φ and ψ are C^∞ on $[0,\infty)$. The three functional equations as an easy consequence yield e.g. differential equations in φ and ψ, some of which will be explored further below.

Of course, if φ and ψ are constant, $\varphi + \psi X$ belongs to $\mathscr{C}_n$ and is nonsingular, see Proposition 5.3 below. To establish Proposition 5.4 it is necessary to allow for timedependent φ. Allowing also for timedependent ψ makes it possible to construct certain (A,B,C) - diffusions in a simple manner, see Example 5.13.

The terminology used in Definition 5.1 suggests that $\sim$ is an equivalence relation. Formally this may be argued as follows: we saw above that if X is (A,B,C), then $\varphi + \psi X$, where $\varphi \in \Phi$, $\psi \in \Psi$, is $(\tilde{A},\tilde{B},\tilde{C})$ iff (5.2) holds. Consequently, if X^* is another (A,B,C) - diffusion and (5.2) holds, $\varphi + \psi X^*$ is $(\tilde{A},\tilde{B},\tilde{C})$. Now, to show for instance that $\sim$ is symmetric, assume that $X \sim \tilde{X}$ with X (A,B,C), and find φ,ψ such that $\hat{X} = \varphi + \psi X$ is $(\tilde{A},\tilde{B},\tilde{C},)$. Since $-\psi^{-1}\varphi + \psi^{-1}\hat{X} = X$ is (A,B,C), by what was just said, so is $-\psi^{-1}\varphi + \psi^{-1}\tilde{X}$ and hence $\tilde{X} \sim X$.

A final remark in connection with (5.2): differentiating the last equation with respect to t, for $t = 0$ gives

$$\psi(s)\, C\, \psi^T(s) = \tilde{C},$$

in particular it is seen that $X \sim \tilde{X}$ forces C and $\tilde{C}$ to have the same rank.

5.3. <u>Proposition</u> Let X be an (A,B,C) - diffusion and let $\varphi(t) = E$ and $\psi(t) = F$ be constant with F nonsingular. Then $\tilde{X} = E + FX$ is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion, where

$$\tilde{A} = FA - FBF^{-1}E$$
$$\tilde{B} = FBF^{-1}$$
$$\tilde{C} = FCF^T. \qquad \square$$

The proof is trivial.

5.4. <u>Proposition</u> Any nonsingular (A,B,C) - diffusion is affinely equivalently to an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion, where $\tilde{A} = 0$ and $\tilde{C}$ is a diagonal matrix of the form

$$\mathrm{diag}(1,\ldots,1,0,\ldots,0). \qquad (5.5)$$

<u>Proof</u> Let X be an (A,B,C) - diffusion and choose F orthogonal such that FCF^T is diagonal. By Proposition 5.3, FX is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion with $\tilde{C} = FCF^T$. Using a further linear transformation that permutes and rescales the coordinates appropriately, shows that X is equivalent to an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion with $\tilde{C}$ as in (5.5).

Next, let again X be (A,B,C) and obtained as a solution to (2.2). Consider $\tilde{X} = \varphi + X$, where (cf. (2.6)),

$$\varphi(t) = -\int_0^t e^{sB}A\,ds$$

From (2.2) it is seen that $\tilde{X}$ solves

$$d\tilde{X} = B\tilde{X}dt + DdW,$$

i.e. $\tilde{X}$ is a (0,B,C) - diffusion. Since C is not affected by the transformation $X \to \tilde{X}$, it follows that any diffusion with C of the form (5.5) is equivalent to a diffusion with the same C and A = 0. The proposition is proved. □

We shall say that an (A,B,C) - diffusion is on standard form if A = 0 and C is given by (5.5). For the purpose of exploring the equivalence classes for ~, Proposition 5.4 shows that it is enough to consider equivalence of two diffusions on standard form. We shall now do this for two diffusions X, $\tilde{X}$ with $C = \tilde{C} = I$, $A = \tilde{A} = 0$. Obviously, by Proposition 5.3, $X \sim \tilde{X}$ if there exists F orthogonal such that $\tilde{B} = FBF^{-1}$. The following result shows in particular, that it is possible to obtain $X \sim \tilde{X}$ in some additional cases.

5.6 <u>Proposition</u> Let B, $\tilde{B}$ be given $n \times n$ matrices and let X be a (0,B,I) - diffusion. Then $\varphi + \psi X$, where $\varphi \in \Phi$, $\psi \in \Psi$ are differentiable, is a $(0,\tilde{B},I)$ - diffusion if and only if

(5.7) $$\varphi' = \tilde{B}\varphi,$$

(5.8) $$\psi' = \tilde{B}\psi - \psi B,$$

(5.9) $$\psi\psi^T = I.$$

Remark φ', ψ' denote the derivatives of φ and ψ (componentwise). Note that (5.9) requires each $\psi(t)$ to be orthogonal.

Proof By Theorem 2.7, $\varphi + \psi X$ is a $(0,\tilde{B},I)$ - diffusion iff

$$U_t = \varphi(t) + \psi(t)X_t - \varphi(0) - \psi(0)X_0 - \int_0^t \tilde{B}(\varphi(s) + \psi(s)X_s)ds$$

is a BM(n) - process. Also, X satisfies

$$dX = BXdt + dW$$

with W a BM(n), so by Ito's formula

$$dU = \{\varphi' - \tilde{B}\varphi + (\psi' - \tilde{B}\psi + \psi B)X\}dt + \psi dW,$$

which is a local martingale iff (5.7) and (5.8) hold (recall that X is nonsingular). Since $[U,U] = \psi\psi^T$, it is seen that U is BM(n) iff (5.7) - (5.9) hold. □

Remarks Of course (5.7) has the complete solution

$$\varphi(t) = e^{t\tilde{B}}\varphi(0).$$

The proposition may be rephrased as follows: for $A = \tilde{A} = 0$, $C = \tilde{C} = I$, φ and ψ solve the functional equations (5.2) iff they solve (5.7) - (5.9). It is easy to derive the latter directly from (5.2), but more tedious to provide an analytic argument going the other way. □

The next result discusses the problem of finding the pairs $(B,\tilde{B})$ for which (5.8) and (5.9) are solvable and then determining the solution. The problem may be reduced slightly by imposing the boundary condition $\psi(0) = I$: suppose $\tilde{\psi}$ with $\tilde{\psi}(0) = I$ solves (5.8) and (5.9) for the pair $(B,F^T\tilde{B}F)$ with F orthogonal. Then $\psi = F\tilde{\psi}$ solves (5.8) and (5.9) for the pair $(B,\tilde{B})$, and $\psi(0) = F$.

With $B,\tilde{B}$ as above, introduce

$$Q = \tilde{B} - B, \quad R = \frac{1}{2}(B - B^T), \quad \overline{B} = \frac{1}{2}(B + B^T),$$

in particular R is skewsymmetric, $\overline{B}$ is symmetric and

$$B = \frac{1}{2}(\overline{B} + R).$$

Also define the matrix-valued function

$$\delta(t) = e^{t(Q+R)}e^{-tR}.$$

5.10. <u>Proposition</u> (a) Given B and $\tilde{B}$, there is at most one solution to the equations (5.8) and (5.9) that satisfy $\psi(0) = I$. The solution exists if and only if (ai) Q is skewsymmetric

(aii) $\bar{B}$ and $\delta(t)$ commute for all t,

and in that case is given by

$$\psi(t) = \delta(t).$$

(b) In particular $\psi = \delta$ solves (5.8) and (5.9) provided Q is skewsymmetric and either

(bi) $\bar{B}$ commutes with Q and R

or

(bii) Q commutes with $\bar{B}$ and R.

Proof Clearly (bi) implies (aii). And if (bii) holds,

$$\delta(t) = e^{tQ}$$

since Q and R commute, and again (aii) follows. Thus (b) follows from (a).

For the proof of (a), we shall repeatedly use the fact that if M is skewsymmetric, $M + M^T = 0$, then e^M is orthogonal.

Suppose ψ solves (5.8) and (5.9) with $\psi(0) = I$. The differential equation (5.8) has the unique solution

$$\psi(t) = e^{t\tilde{B}}e^{-tB},$$

but of course this $\psi(t)$ is not orthogonal in general. But with (5.9) true, also $\psi'\psi^T + \psi\psi'^T = 0$, or

$$\psi' = -\psi\psi'^T\psi$$

and inserting this in (5.8), solving for ψ'^T using (5.9) and transposing gives

$$\psi' = -\tilde{B}^T\psi + \psi B^T. \tag{5.11}$$

For $t = 0$ this and (5.8) gives $Q = -Q^T$, i.e. (ai) holds. But then taking the average of (5.8) and (5.11) gives

$$\psi' = (Q + R)\psi - \psi R, \tag{5.12}$$

which, when $\psi(0) = I$, has δ as its unique solution. Subtracting (5.11) from (5.8) and using that Q is skewsymmetric, shows finally that (aii) holds.

If conversely (ai) and (aii) are true, $\psi(t) = \delta(t)$ is orthogonal since Q and R are skewsymmetric, so (5.9) holds. And since (5.12) is true, adding $\bar{B}\psi - \psi\bar{B}$, which is 0 by (aii), to the right hand side of (5.12), we see that (5.8) follows. □

<u>Remarks</u> An equivalent formulation of (aii) is that for all p, $\bar{B}$ commutes with $\delta^{(p)}(0)$, where $\delta^{(p)}$ is the p'th derivative of δ. In particular, for $p = 1$ this shows that $\bar{B}$ and Q commute.

If Q is skewsymmetric and (bii) holds, as already noted $\delta(t) = e^{tQ}$. Thus $(\delta(t))_{t\geq 0}$ defines a semigroup of orthogonal transformations: $\delta(s + t) = \delta(s)\delta(t)$.

In the two-dimensional case, the space of skewsymmetric matrices is one-dimensional, so that Q and R always commute. Hence $\delta(t) = e^{tQ}$ and with Q skewsymmetric, (aii) holds iff Q commutes with $\overline{B}$. Using that $\overline{B}$ is symmetric it is readily checked that with $Q \neq 0$ (i.e. $B \neq \overline{B}$ which is the only interesting case), this is possible iff $\overline{B} = kI$ for some constant k. □

5.13. Example Suppose that $B = bI$ for some $b \in \mathbb{R}$. Proposition 5.10 (a) shows that (5.8) and (5.9) are solvable, with $\psi(0) = I$, iff $\tilde{B} = B + Q$ with Q skewsymmetric and in that case the solution is $\psi(t) = e^{tQ}$.

Thus, if X has independent components which are $(0,b,c)$ - diffusions (one-dimensional Ornstein-Uhlenbeck processes with $a = 0$, see Example 1.21), $\tilde{X}_t = e^{tQ}X_t$ defines a $(0, bI + Q, cI)$ - diffusion whenever Q is skewsymmetric. If $b = 0$ we call $\tilde{X}$ a rotating Brownian motion and, for $b \neq 0$, a rotating Ornstein-Uhlenbeck process.

In particular, if $b = 0$, $c = 1$ and $X_0 \equiv 0$, X is a BM(n) - process, and $\tilde{X}$ becomes a $(0,Q,I)$ - diffusion. Note that since $||\tilde{X}|| = ||X||$ (with $||\cdot||$ the Euclidean norm), it follows that the radial part of a rotating Brownian motion in n dimensions (with $c = 1$) is a n-dimensional Bessel process. However, one cannot expect to find a skew product representation of $\tilde{X}$ since after the time change required to make the directional part of $\tilde{X}$ Markovian, the rate of rotation will depend on the radial part.

For $n = 2$ a simple expression is available for the transformations $\delta(t) = e^{tQ}$: necessarily

$$Q = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$

for some θ, and one finds

$$e^{tQ} = \begin{bmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{bmatrix}. \qquad \square$$

It is easy to find other examples where Proposition 5.10 applies, but they appear mostly to involve transformations of little known processes into others that are equally unknown!

The reduction to standard form performed in Proposition 5.4 is not entirely satisfactory, since it does not reduce the matrix B. As we shall now see, in certain cases, this is possible.

The following result is a modification of Gzyl's theorem [6] (which deals only with stationary GHD's). We believe that there is a mistake in his reasoning, which leads him to consider deterministic processes in two dimensions of Doob's type $M(e^{i}\theta)$ ([4], p. 240) rather than rotating Ornstein-Uhlenbecks.

5.14. <u>Proposition</u> Let X be a (0,B,I) - diffusion, and suppose that B is normal. Then there exists F orthogonal such that FX is a $(0,\tilde{B},I)$ - diffusion with $\tilde{B}$ of the form

$$\tilde{B} = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{bmatrix}$$

where each diagonal block matrix is either a scalar or a 2×2 matrix of the form

$$B_j = \begin{bmatrix} b_j & \theta_j \\ -\theta_j & b_j \end{bmatrix}.$$

Proof Use Proposition 5.3, and the fact used by Gzyl, that there exists F orthogonal such that $FBF^{-1} = \tilde{B}$ (Schmidt [19], Theorem 3.2 (b)). □

Remarks Recall that B is normal if $BB^T = B^TB$. In particular any B symmetric, skewsymmetric or orthogonal is normal.

It may be noted, that the transformation $\delta(t)$ carrying a $(0,B,I)$ - diffusion into a $(0,\tilde{B},I)$ - diffusion if conditions (ai), (aii) of Proposition 5.10 are satisfied, leaves the class of diffusions with normal B invariant: if B is normal, so is $\tilde{B} = B + Q$ by (ai), (aii) and the first remark following Proposition 5.10.

Proposition 5.14 has the following alternative formulation: if $C = I$, B is normal, after a suitable orthogonal transformation and conditionally on $X_0 = x$ say, X splits into independent components, each of which is either a one-dimensional Ornstein-Uhlenbeck process, a scaled BM(1) - process, a two-dimensional rotating Ornstein-Uhlenbeck or a two-dimensional rotating Brownian motion (cf. Example 5.13). □

We shall conclude this section with a result that deals with (constant) affine transformations, but not with affine equivalence.

Let $\ell\colon \mathbb{R}^n \to \mathbb{R}^m$ be affine and given by a m-column vector E and a $m \times n$-matrix F,

$$\ell x = E + Fx.$$

Also, let $\ker(F) = \{x \in \mathbb{R}^n\colon Fx = 0\}$ denote the kernel space for F.

5.15. Proposition If X is a non-singular (A,B,C) - diffusion, then $\tilde{X} = E + FX$ is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion for some $\tilde{A},\tilde{B},\tilde{C}$ if and only if

$$B : \ker(F) \to \ker(F). \tag{5.16}$$

If this condition is satisfied, necessarily

$$\tilde{B}F = FB \tag{5.17}$$

$$\tilde{A} = FA - \tilde{B}E \tag{5.18}$$

$$\tilde{C} = FCF^T. \tag{5.19}$$

In particular, if $m \leq n$ and F is of full rank m, $\tilde{A},\tilde{B},\tilde{C}$ are uniquely determined by these equations, and $\tilde{X}$ is nonsingular.

Proof Assume as usual that X solves (2.2). By Theorem 2.7, $\tilde{X}$ is an $(\tilde{A},\tilde{B},\tilde{C})$ - diffusion iff

$$V_t = \tilde{X}_t - \tilde{X}_0 - t\tilde{A} - \int_0^t \tilde{B}\tilde{X}_s ds$$

is a local martingale with $[V,V]_t = t\tilde{C}$. But

$$dV = \{FA - \tilde{A} - \tilde{B}E + (FB - \tilde{B}F)X\}dt + FDdW,$$

and since X is nonsingular, it follows that $\tilde{X}$ is $(\tilde{A},\tilde{B},\tilde{C})$ iff (5.17) - (5.19) hold.

That (5.17) holds for some $\tilde{B}$ is equivalent to saying that $x \to FBx$ depends on x only through Fx, a condition which is equivalent to (5.16).

The proof of the last assertion is trivial. □

6. Stationarity

Let X be a nonsingular (A,B,C) - diffusion on $\mathbb{R}^n$. We shall say that X is <u>stationary</u>, or has a <u>stationary version</u>, if the transition probabilities

$$p_t(x,\cdot) = N(\alpha(t) + \beta(t)x, \Sigma(t))$$

for X admit an invariant probability, i.e. there is a probability π on $\mathbb{R}^n$ such that

(6.1) $$\pi = \int \pi(dx) p_t(x,\cdot)$$

for all t.

Let

$$\mathrm{spec}(B) = \{\lambda \in \mathbb{C} : Bz = \lambda z \quad \text{for some } z \in \mathbb{C}^n \backslash 0\}$$

denote the spectrum for B when viewed as a linear mapping on $\mathbb{C}^n$.

6.2 <u>Theorem</u> (a) For a nonsingular (A,B,C) - diffusion X to be stationary it is necessary and sufficient that

(6.3) $$\mathrm{spec}(B) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}.$$

In particular it is necessary that B be nonsingular.

(b) If (6.3) holds, the invariant probability π is uniquely determined as the Gaussian law with mean μ and nonsingular covariance Γ given by

$$(6.4) \qquad A + B\mu = 0, \quad C + B\Gamma + \Gamma B^T = 0.$$

(c) If (6.3) holds,

$$\lim_{t\to\infty} (\alpha(t) + \beta(t)x) = \mu \qquad (x \in \mathbb{R}^n),$$

$$\lim_{t\to\infty} \Sigma(t) = \Gamma,$$

that is, for every $x \in \mathbb{R}^n$, as $t \to \infty$, $p_t(x,\cdot)$ converges weakly to the invariant probability $\pi = N(\mu,\Gamma)$.

(d) If (6.3) holds and X has initial distribution $\pi = N(\mu,\Gamma)$, then the covariance function for the stationary process X is $R(t) = E(X_0 - \mu)(X_t - \mu)^T$ where

$$R(t) = \Gamma\beta^T(t).$$

<u>Remarks</u> Condition (6.3) was noted by Doob [4] (e.g. p. 265), and it is well known that if it holds, there is a unique invariant Gaussian law, see Zakai and Snyders [20] or Section 6 of Ichihara and Kunita [8]. □

<u>Proof</u> We shall show that X has an invariant <u>Gaussian</u> probability π iff (6.3) holds, and that with $\pi = N(\mu,\Gamma)$, (6.4) and (c) and (d) are true. If then $\tilde{\pi}$ is an <u>arbitrary</u> invariant probability, and (6.3) holds,

since $\tilde{\pi} = \tilde{\pi}p_t \to \pi$ weakly as $t \to \infty$, it follows that $\tilde{\pi} = \pi$. The proof is then completed by showing that the existence of an invariant probability implies (6.3).

If π is Gaussian, so is πp_t as given by the expression on the right of (6.1). It follows that $\pi = N(\mu,\Gamma)$ is invariant iff for all t

$$\mu = \alpha(t) + \beta(t)\mu, \tag{6.5}$$

$$\Gamma = \Sigma(t) + \beta(t)\Gamma\beta^T(t). \tag{6.6}$$

Suppose $N(\mu,\Gamma)$ is invariant. By (6.6), $\Gamma \geq \Sigma(t)$ and since $\Sigma(t)$ is nonsingular, so is Γ.

Let now $\lambda \in \mathrm{spec}(B^T) = \mathrm{spec}(B)$ and find $z \in \mathbb{C}^n\setminus 0$ such that $B^Tz = \lambda z$. Then $B^T\bar{z} = \bar{\lambda}\,\bar{z}$ and

$$\beta^T(t)z = e^{t\lambda}z, \quad \beta^T(t)\bar{z} = e^{t\bar{\lambda}}\bar{z}$$

and from (6.6) and (1.12) it follows that

$$\bar{z}^T\Gamma z = \int_0^t \bar{z}^T e^{s\bar{\lambda}} C e^{s\lambda} z\,ds + \bar{z}^T e^{t\bar{\lambda}}\Gamma e^{t\lambda} z$$

$$= \bar{z}^T C z \int_0^t e^{2s\mathrm{Re}(\lambda)}ds + \bar{z}^T\Gamma z e^{2t\mathrm{Re}(\lambda)},$$

in particular the right hand side is bounded as a function of t, and consequently, since Γ is nonsingular, $\mathrm{Re}(\lambda) \leq 0$ always, and if

$\mathrm{Re}(\lambda) = 0$, necessarily $\bar{z}^T C z = 0$, i.e. $\bar{z}^T D = 0$ for D determined so that $DD^T = C$. But since X is nonsingular we can rule out this possibility: $\bar{z}^T D = 0$ forces $\bar{z}^T B^k D = \bar{\lambda}^k \bar{z}^T D = 0$ for all k in contradiction with (3.3). We have shown that if there is an invariant Gaussian law, then (6.3) holds.

Now suppose that (6.3) holds. In particular B is nonsingular and therefore (6.4) has a unique solution in a vector μ and a symmetric matrix Γ. Also because of (6.3), the limits

$$\tilde{\mu} = \lim_{t\to\infty} \alpha(t) = \int_0^\infty e^{sB} A ds,$$
$$\tilde{\Gamma} = \lim_{t\to\infty} \Sigma(t) = \int_0^\infty e^{sB} C e^{sB^T} ds,$$

(cf. (1.11) and (1.12)), exist and are finite with, evidently, $\tilde{\Gamma}$ positive semidefinite. But, using (1.3), (1.5) it is readily checked that $\tilde{\mu}$, $\tilde{\Gamma}$ satisfy (6.5) and (6.6), i.e. $N(\tilde{\mu},\tilde{\Gamma})$ is invariant. Differentiating (6.5), (6.6) at $t = 0$ we see that $\tilde{\mu}$, $\tilde{\Gamma}$ solve (6.4) and therefore deduce that $\mu = \tilde{\mu}$, $\Gamma = \tilde{\Gamma}$ and that $N(\mu,\Gamma)$ is invariant. Assertion (c) is now also immediate since by (6.3), $\beta(t)x \to 0$ for all x, while (d) follows from (1.20).

It remains to show that if $\tilde{\pi}$ is any invariant probability, then (6.3) holds. Suppose $\tilde{\pi}$ is invariant and that (6.3) does not hold. Find $\lambda \in \mathrm{spec}(B^T)$ with $\mathrm{Re}(\lambda) \geq 0$ and find $z \in \mathbb{C}^n \setminus 0$ such that $B^T z = \lambda z$. Let $\tilde{X}$ be a timehomogeneous Markov process with initial law $\tilde{\pi}$ and transitions p_t, and consider the complex-valued process $\tilde{Z}$, where

$$\tilde{Z}_t = z^T\tilde{X}_t.$$

Since $\tilde{\pi}$ is invariant, the distribution of $\tilde{Z}_t$ does not depend on t. On the other hand, the conditional variances V_{Re} and V_{Im} given $\tilde{X}_0 = x$ for the realvalued variables $Re(\tilde{Z}_t)$ and $Im(\tilde{Z}_t)$ satisfy, cf. (1.12),

$$V_{Re}(t) + V_{Im}(t) = \bar{z}^T Cz \int_0^t e^{2sRe(\lambda)} ds.$$

Reusing an earlier argument, since $B^T z = \lambda z$ and (A,B,C) - diffusions are nonsingular, we see that $D^T z \neq 0$ for D with $C = DD^T$. Because $Re(\lambda) \geq 0$ therefore $V_{Re}(t) + V_{Im}(t) \to \infty$ as $t \to \infty$ which is enough to guarantee that for all $x \in \mathbb{R}^n$ and all $r \geq 0$

$$\lim_{t\to\infty} P(|\tilde{Z}_t| \leq r \mid \tilde{X}_0 = x) = 0.$$

(Recall that given $\tilde{X}_0 = x$, $Re(\tilde{Z}_t)$ and $Im(\tilde{Z}_t)$ are both Gaussian with a variance not depending on x). But then also

$$\tilde{\pi}\{x \in \mathbb{R}^n\colon\ |z^T x| \leq r\} = \int \tilde{\pi}(dy) P(|\tilde{Z}_t| \leq r \mid \tilde{X}_0 = y)$$

tends to 0 as $t \to \infty$ for all $r \geq 0$, which is impossible with $\tilde{\pi}$ a probability. □

Let again X be a nonsingular (A,B,C) - diffusion and let F be a $m \times n$ - matrix. Assume that $m \leq n$ and that F has full rank m. We shall say that X has <u>F as a stationary component</u> if there exists a Gaussian law $\pi = N(\mu,\Gamma)$ such that if X is run with initial distribution π, the m-dimensional process FX is strictly stationary: the distribution of $(FX_{s+t})_{t\geq 0}$ does not depend on s.

The following result is read off from Proposition 5.15 and Theorem 6.2.

6.7 <u>Proposition</u> If X is nonsingular and F is of full rank $m \leq n$, then F is a stationary component provided

$$B : \ker(F) \to \ker(F)$$

and

$$\operatorname{spec}(\tilde{B}) \subseteq \{\lambda \in \mathbb{C}^m : \operatorname{Re}(\lambda) < 0\},$$

where $\tilde{B}$ is the uniquely determined $m \times n$ - matrix satisfying $\tilde{B}F = FB$.

□

7. Equivalence of measures

Let Ω^o denote the space of continuous paths $w: \mathbb{R}_+ \to \mathbb{R}^n$, write X_t^o for the projection $X_t^o(w) = w(t)$ and introduce the σ-algebras

$$\mathcal{F}_t^o = \sigma(X_s^o)_{s\leq t}, \quad \mathcal{F}^o = \sigma(X_s^o)_{s\geq 0}.$$

Fix an initial state $x^o \in \mathbb{R}^n$ and for A,B,C given, let $P^{x^o}_{A,B,C}$ denote the probability on $(\Omega^o, \mathcal{F}^o)$ that makes $X^o = (X_t^o)_{t\geq 0}$ an (A,B,C) - diffusion starting at x^o. For $t \geq 0$, write $P^{t,x^o}_{A,B,C}$ for the restriction to $\mathcal{F}_t^o$ of $P^{x^o}_{A,B,C}$.

In this section we shall discuss the problem of equivalence between two measures $P^t = P^{t,x^o}_{A,B,C}$ and $\tilde{P}^t = P^{t,x^o}_{\tilde{A},\tilde{B},\tilde{C}}$, and in the case where $\tilde{P}^t \sim P^t$, determine the Radon-Nikodym derivative $d\tilde{P}^t/dP^t$. Note that since $\tilde{P}^t$ and P^t are both Gaussian, $\tilde{P}^t \sim P^t$ iff $\tilde{P}^t \ll P^t$.

In agreement with the shorthand notation just introduced, we shall write P for $P^{x^o}_{A,B,C}$, $\tilde{P}$ for $P^{x^o}_{\tilde{A},\tilde{B},\tilde{C}}$

Let $t > 0$. Subject to P, the quadratic variation on $[0,t]$ of X^o is tC and hence

$$\lim_{n\to\infty} \sum_{k=1}^{[2^n t]} (X^o_{k2^{-n}} - X^o_{(k-1)2^{-n}})(X^o_{k2^{-n}} - X^o_{(k-1)2^{-n}})^T = tC$$

with convergence in P-probability. Passing to a P - a.s. convergent subsequence, if $\tilde{P}^t \ll P^t$ that subsequence must also converge $\tilde{P}$ - a.s. to tC. It follows that if $\tilde{P}^t \ll P^t$, then $\tilde{C} = C$.

If C is nonsingular, standard results show that this condition is also sufficient for $\tilde{P}^t \ll P^t$. If however $C = \tilde{C}$ is singular, under both P and $\tilde{P}$ some components of X^o are completely determined from others, cf. Proposition 3.6, and for $\tilde{P}^t \ll P^t$ to hold, it is then necessary that on $[0,t]$ these components should be determined from the others in the same manner, regardless of whether P or $\tilde{P}$ is used.

In the result to be stated now, we use the same block matrix notation as in Proposition 3.6 so that we write

$$X^o = \begin{bmatrix} X_1^o \\ X_1^o \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with X_1^o the first n' components of X^o, B_{11} the upper left $n' \times n'$ submatrix of B etc. Also, as usual, D is any $n \times n'$ - matrix with $DD^T = C$.

7.1 Theorem (a) Either $\tilde{P}^t \sim P^t$ for all $t > 0$ or $\tilde{P}^t \perp P^t$ for all $t > 0$.

(b) If C is nonsingular, $\tilde{P}^t \sim P^t$ if and only if $\tilde{C} = C$ and in that case

$$(7.2) \quad \frac{d\tilde{P}^t}{dP^t} = \exp\left[\int_0^t (\tilde{Z}_s^T - Z_s^T)C^{-1}dX_s^o - \frac{1}{2}\int_0^t (\tilde{Z}_s^T C^{-1}\tilde{Z}_s - Z_s^T C^{-1} Z_s)ds\right],$$

where

$$Z_s = A + BX_s^o, \quad \tilde{Z}_s = \tilde{A} + \tilde{B}X_s^o.$$

(c) If C is singular and (A,B,C) - diffusions are nonsingular, $\tilde{P}^t \sim P^t$ if and only if $\tilde{C} = C$ and

(7.3) $$a^T\tilde{A} = a^TA, \quad a^T\tilde{B} = a^TB$$

for all $a \in \mathbb{R}^n$ with $a^TD = 0$. In particular, if

(7.4) $$C = \begin{bmatrix} I_{n'} & 0 \\ 0 & 0 \end{bmatrix}$$

where $n' = \text{rank}(C) < n$, $\tilde{P}^t \sim P^t$ if and only if

$$\tilde{A}_2 = A_2, \quad \tilde{B}_{21} = B_{21}, \quad \tilde{B}_{22} = B_{22}$$

and in that case

(7.5) $$\frac{d\tilde{P}^t}{dP^t} = \exp\left[\int_0^t (\tilde{Z}_{1,s} - Z_{1,s})^T dX^o_{1,s} - \frac{1}{2}\int_0^t (\tilde{Z}^T_{1,s}\,\tilde{Z}_{1,s} - Z^T_{1,s}\,Z_{1,s})ds\right]$$

where

$$Z_{1,s} = A_1 + B_{11}X^o_{1,s} + B_{12}X^o_{2,s}$$

$$\tilde{Z}_{1,s} = \tilde{A}_1 + \tilde{B}_{11}X^o_{1,s} + \tilde{B}_{12}X^o_{2,s}.$$

Remark In Corollary 2.1.2 of [1], Le Breton and Musiela give the Radon-Nikodym derivative on $[0,t]$, in the case where $A = \tilde{A} = 0$, $C = \tilde{C}$

is allowed to be singular and $B = (I - CC^{+})\tilde{B}$ with C^{+} a generalized inverse of C. It is obvious that in this case the condition (7.3) is satisfied. □

Proof (a) For a given $t > 0$ either $\tilde{P}^t \sim P^t$ or $\tilde{P}^t \perp P^t$ since both $\tilde{P}^t$ and P^t are Gaussian. That equivalence either holds for all or none of $t > 0$ is a consequence of (b) or (c) since the conditions given there do not depend on t. (If the (A,B,C) - diffusions are singular one must first transform linearly into a nonsingular diffusion before using this argument).

(b) That $\tilde{P}^t << P^t$ implies $\tilde{C} = C$ was argued in the introductory remarks of this section. Suppose now that $\tilde{C} = C$ is nonsingular and find F nonsingular so that $FCF^T = I$. Consider

$$Y = F(X^{o} - x^{o}) \tag{7.6}$$

which subject to P $(\tilde{P})$ is a $(F(A + Bx^{o}), FBF^{-1}, I)$ - diffusion $((F(\tilde{A} + \tilde{B}x^{o}), F\tilde{B}F^{-1}, I)$ - diffusion) starting at the origin, cf. Proposition 5.3. In particular, under P there is a BM(n) - process W defined on Ω^{o} such that

$$dY_t = (F(A + Bx) + FBF^{-1} Y_t)dt + dW_t$$

with a similar expression valid under $\tilde{P}$. From the multivariate analogue of Theorem 7.7 in Liptser and Shiryayev [14] it then follows that if μ^t $(\tilde{\mu}^t)$ is the distribution of $(Y_s)_{0\leq s\leq t}$ under P $(\tilde{P})$, we have $\tilde{\mu}^t \sim \mu^t$ with

$$\frac{d\tilde{\mu}^t}{d\mu^t} = \exp\left[\int_0^t (\tilde{Z}^T_{F,s} - Z^T_{F,s})dX^o_s - \frac{1}{2}\int_0^t (\tilde{Z}^T_{F,s}\tilde{Z}_{F,s} - Z^T_{F,s}Z_{F,s})ds\right]$$

where

$$Z_{F,s} = F(A + Bx^o) + FBF^{-1}X^o_s,$$
$$\tilde{Z}_{F,s} = F(\tilde{A} + \tilde{B}x^o) + F\tilde{B}F^{-1}X^o_s.$$

But then $\tilde{P}^t \sim P^t$ and

$$\frac{d\tilde{P}^t}{dP^t} = \frac{d\tilde{\mu}^t}{d\mu^t} \circ f$$

where $f: \Omega^o \to \Omega^o$ is the transformation corresponding to (7.6), i.e. $X^o_t \circ f = F(X^o_t - x^o)$. Using this and $C^{-1} = F^TF$, (7.2) follows easily.
(c) If $\tilde{P}^t << P^t$ we know that $\tilde{C} = C$. With (A,B,C) - diffusions nonsingular it also follows directly that $(\tilde{A},\tilde{B},C)$ - diffusions are nonsingular. Let now $a \neq 0$ satisfy $a^TD = 0$. Then a^TX^o is smooth under both P and $\tilde{P}$ by Theorem 4.2 (b) and

(7.7)
$$(a^TX^o)^{(1)} = a^TA + a^TBX^o \qquad P - a.s.$$
$$(a^TX^o)^{(1)} = a^T\tilde{A} + a^T\tilde{B}X^o \qquad \tilde{P} - a.s.$$

and consequently $\tilde{P}^t << P^t$ forces that $\tilde{P}$ - a.s.

$$a^T(\tilde{B} - B)X^o_s = \text{constant} \qquad (0 \leq s \leq t).$$

Since X^o is nonsingular under $\tilde{P}$ it follows that $a^T\tilde{B} = a^TB$, and inserting this in (7.7) then gives $a^T\tilde{A} = a^TA$.

Suppose conversely that $\tilde{C} = C$ and $a^T\tilde{A} = a^TA$, $a^T\tilde{B} = a^TB$ for all a with $a^TD = 0$. Considering $Y = F(X^o - x^o)$ for a suitable nonsingular F, we may without loss of generality assume that $x^o = 0$ and that C has the form (7.4) where $n' = \text{rank}(C)$. We may then use

$$D = \begin{bmatrix} I_{n'} \\ 0 \end{bmatrix},$$

which is $n \times n'$, and find that the assumptions on $A, \tilde{A}, B, \tilde{B}$ translate into

$$\tilde{A}_2 = A_2, \quad \tilde{B}_{21} = B_{21}, \quad \tilde{B}_{22} = B_{22}.$$

By Proposition 3.6, under both P and $\tilde{P}$, X_2^o is adapted to the filtration generated by X_1^o. Combining this with (3.8) and Theorem 7.7 in [14], we see that if P_1, $\tilde{P}_1$ denote respectively the P - and $\tilde{P}$ - distribution of X_1^o (so P_1, $\tilde{P}_1$ are probabilities on the space Ω_1^o of n' - dimensional continuous paths), then $\tilde{P}_1^t \ll P_1^t$

$$\frac{d\tilde{P}_1^t}{dP_1^t} = \exp\left[\int_0^t (\tilde{Z}_{1,s}^T - Z_{1,s}^T)dX_{1,s}^o - \frac{1}{2}\int_0^t (\tilde{Z}_{1,s}^T\tilde{Z}_{1,s} - Z_{1,s}^T Z_{1,s})ds\right]$$

where (with a convenient notational abuse) $X_{1,s}^o$ now denotes the projection $w_1 \to w_1(s)$ on Ω_1^o) and e.g.

$$Z_{1,s} = A_1 + B_{11}X^{o}_{1,s} + B_{12}X^{o}_{2,s}.$$

Again invoking the fact that $X^{o}_{2,t}$ is a function of $(X^{o}_{1,s})_{0\leq s\leq t}$ it follows that $\tilde{P}^t << P^t$ and that (7.5) holds. □

7.8 Example Let $n = 2$, $A = 0$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, cf. Example 4.6. Then $\tilde{P}^t \sim P^t$ for all $t > 0$ iff $\tilde{C} = C$ and

$$\tilde{A} = \begin{bmatrix} \tilde{a} \\ 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 \\ 1 & 0 \end{bmatrix}$$

in which case

$$\frac{d\tilde{P}^t}{dP^t} = \exp\left(\int_0^t (\tilde{a} + \tilde{b}_1X^{o}_{1,s} + \tilde{b}_2X^{o}_{2,s})dX^{o}_{1,s} - \frac{1}{2}\int_0^t (\tilde{a} + \tilde{b}_1X^{o}_{1,s} + \tilde{b}_2X^{o}_{2,s})^2 ds\right)$$

with

$$X^{o}_{2,t} = x^{o}_2 + \int_0^t X^{o}_{1,s}ds$$

P - a.s. and $\tilde{P}$ - a.s. □

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