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AN I(2) COINTEGRATION ANALYSIS OF THE PURCHASING POWER PARITY BETWEEN AUSTRALIA AND USA

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Abstract. Cointegration analysis of autoregressive models allowing for I(1) and I(2) variables is briefly reviewed and the methods are illustrated by an analysis of the purchasing power parity between Australia and USA.

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0. Introduction

This paper is an illustration of a technique for analyzing time series data that allows for processes that are integrated of order 2. The theory is presently being developed, see Johansen (1991b) and has been illustrated by Johansen (1991c) and Juselius (1991). The paper contains 3 sections. In section 1 we discuss cointegration in particular in connection with the autoregressive model. Section 2 is a brief explanation of the statistical technique centered around reduced rank regression, as developed by Anderson (1951), and later applied to time series by Ahn and Reinsel (1988) and Johansen (1988). The last section contains an application of the methods to an analysis of the purchasing power parity between Australia and the United States. We have chosen a data set that consist of log prices and interest rates in Australia and United States as well as the exchange rate. We are interested in the purchasing power parity relation that says that $p_{AU} - p_{US} - \text{exch}$ is a stationary relation. It is found that the PPP relation is not stationary by itself, but if we include interest rates we can achieve a stationary relation. The analysis presented focuses on the determination of the cointegrating ranks and the long–run relations, and no attempt is made to formulate a final econometric model.

1. The autoregressive models

Autoregressive models are useful tools for describing the statistical variation of systems of economic time series. This section contains a discussion of some of the properties of autoregresssive processes and models, with special emphasis on the order of integration and the concept of cointegration. It turns out that in an analysis of a system of economic time series many of the parameters of the autoregressive model can be given a meaningful interpretation which helps explain or at least describe the complicated interdependence in the economy.

We make a distinction between the properties of a particular process with specified parameter values and the properties of a model as specified by a set of parameters. Thus we discuss cointegration for a multivariate process and define two classes of models, the I(1) models and the I(2) models.

We formulate the I(1) models as parameter restrictions of the general autoregressive model and the I(2) models as further parameter restrictions.

A p–dimensional autoregressive process is generated by the equations

(1.1) $$X_t = \Pi_1 X_{t-1} + \dots + \Pi_k X_{t-k} + \mu + \Psi D_t + \epsilon_t, \qquad t = 1,\dots,T$$

where the ϵ_t are independent Gaussian p–dimensional with mean zero and variance Ω, and D_t are seasonal dummies.

The VAR model is defined by letting the parameters

(1.2) $$(\Pi_1,\dots,\Pi_k, \mu, \Psi, \Omega)$$

be unrestricted. There are two different reparameterizations that are convenient representations of the same model. The first one is defined by the equations

(1.3) $$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \mu + \Psi D_t + \epsilon_t$$

where $\Pi = \Sigma_1^k \Pi_i - I$, and $\Gamma_i = -\Sigma_{i+1}^k \Pi_j$, $i = 1,\dots,k-1$, such that the parameters

$$(\Pi, \Gamma_1,\dots,\Gamma_{k-1}, \mu, \Psi, \Omega)$$

are unrestricted.

The next is defined by

(1.4) $$\Delta^2 X_t = \Gamma \Delta X_{t-1} + \Pi X_{t-2} + \sum_{i=1}^{k-2} \Phi_i \Delta^2 X_{t-i} + \mu + \Psi D_t + \epsilon_t$$

where $\Gamma = \Sigma_1^{k-1} \Gamma_i - I + \Pi$, $\Phi_i = -\Sigma_{i+1}^{k-1} \Gamma_j$, $i = 1,\dots,k-2$, such that the parameters

$$(\Pi, \Gamma, \Phi_1,\dots,\Phi_{k-2}, \mu, \Psi, \Omega)$$

are unrestricted.

The properties of a process generated by any of the above equations depend on the parameter values, and to discuss this it is convenient to introduce the characteristic polynomial

(1.5) $$A(z) = I - \Sigma_1^k \Pi_i z^i.$$

It is then well known that if the roots of the equation $\det(A(z)) = 0$ have modulus greater than one, then the process described by any of the above equations is stationary. Since we are interested in non–stationary processes we will allow for unit roots: $z = 1$. This gives rise to a class of non–stationary processes with the property that they become stationary by differencing. A non–stationary process with stationary differences is called an I(1) process. Similarly we call a process I(2) if it is non–stationary and its difference is I(1), that is $\Delta^2 X_t$ is stationary.

1.1 Cointegration of I(1) processes and the definition of I(1) models

As an example of an I(1) process consider

$$X_t = C \sum_{i=1}^{t} \epsilon_i + C(L)\epsilon_t, \tag{1.6}$$

where C is a p×p matrix and $C(z) = \Sigma_0^\infty C_i z^i$. The process X_t is composed of a random walk representing the permanent shocks to the system or the common trends and a stationary process. Thus X_t is clearly non–stationary and since ΔX_t is stationary the process is I(1).

Now assume that C is singular, and that the matrix ν is such that $\nu' C = 0$. By multiplying (1.6) by ν' we find that $\nu' X_t$ is stationary, since the term involving the random walk vanishes. Thus the common trends $\Sigma_1^t \epsilon_i$ are eliminated by taking suitable linear combinations of the process. This phenomenon is called cointegration by Granger (1981) and was investigated systematically by Engle and Granger (1987), and later by many other authors. The intuition behind this is that the common trends or driving forces drive the economic variables in a non–stationary way, whereas the combinations $\nu' X_t$ are stationary. The relations $\nu' X = 0$ represent the "stable" economic laws, and $\nu' X_t$ measures the disequilibrium error. Note that in particular that if $\nu = (1,0,...,0)$ is a cointegrating vector then $\nu' X_t$, the first coordinate of X_t is stationary. Thus stationarity of the individual components is a special case of cointegration. If in particular $C = 0$, then any vector ν is a cointegrating vector, or in other words the process is stationary.

The above example (1.6) provides a simple example of the notion of cointegration, but it is more convenient from a statistical point of view to apply the autoregressive representation (1.3) of the process and express integration and cointegration in terms of the parameters of the autoregressive model. The reason for this is of course that the autoregressive parameters are easily estimated by regression techniques. Thus we formulate the I(1) models as a condition on the parameters of the model (1.3).

DEFINITION 1. *The I(1) models H_r, $r = 0,1,\ldots,p$ are defined by equation (1.3) together with the reduced rank condition*

(1.7) $$\Pi = \alpha\beta'$$

where α and β are $p \times r$ matrices.

Note that the definition of the I(1) models implies that we have a set of nested models

$$H_0 \subset \ldots \subset H_r \subset \ldots \subset H_p,$$

The interpretation of H_0 is that $\Pi = 0$ and hence that (1.3) is a VAR model for the differences ΔX_t, where as H_p is the full VAR model for the process in levels. In general H_r allows for at most r cointegrating vectors.

The motivation for this definition can be found in Granger's Theorem, see Engle and Granger (1987). It states that under a suitable extra condition, see Johansen (1990a), the process described by equation (1.3) with the restriction (1.7) for α and β of full rank is I(1) and given by the a representation of the form (1.6), with a matrix C of the form $C = \beta_\perp \alpha_\perp'$, for a suitable choice of $p \times (p-r)$ matrices $\alpha_\perp$ and $\beta_\perp$ of full rank, such that $\alpha'\alpha_\perp = \beta'\beta_\perp = 0$. The parameters β are the long–run parameters or the cointegrating vectors, since $\beta' X_t$ is stationary. The space spanned by $\beta_\perp$ is called the attractor set, since the process tends through the equations to be driven back towards the attractor set. The parameter α, i.e. the coefficient to the disequilibrium error $\beta' X_{t-1}$, is interpreted as the force of adjustment of the process to the attractor set.

The variable $\alpha_\perp' X_t$ can be interpreted as the common trends driving the economy, since by multiplying (1.3) by $\alpha_\perp'$ one finds that the variables evolve without adjusting to the disequilibrium error, since the term $\alpha_\perp' \alpha\beta' X_{t-1} = 0$.

Thus the common trends $\alpha_\perp' X_t$ represent the driving forces in the economy that move the variables in a non–stationary way around the attractor set spanned by $\beta_\perp$. The agents react to the disequilibrium error $\beta' X_t$ through the adjustment coefficients α in order to re–establish the equilibrium.

The constant term μ in equation (1.3) has a double role to play. From Granger's representation theorem we find that the process X_t has a linear trend given by $C\mu t = \beta_\perp \alpha_\perp' \mu t$. The coefficient μ also enters into the levels of the stationary process $\beta' X_t$. The process has no trend if $\alpha_\perp' \mu = 0$. We formulate the hypothesis H_r^* of no trend as the restriction $\alpha_\perp' \mu = 0$.

Thus the parameters of the cointegration model H_r in the autoregressive formulation admits a meaningful interpretation which facilitates the formulation of economic questions of relevance in terms of the statistical parameters of the model.

1.2 *Cointegration of I(2) processes and definition of I(2) models.*

Consider the example of an I(2) process of the same form as (1.6)

$$X_t = C_2 \sum_{j=1}^{t} \sum_{i=1}^{j} \epsilon_i + C_1 \sum_{i=1}^{t} \epsilon_i + C(L)\epsilon_t,\ t = 1,\dots,T. \tag{1.8}$$

For I(2) processes the notion of cointegration is not so simple. To see this consider first the case where C_2 has reduced rank, and where $\nu' C_2 = 0$. Clearly $\nu' X_t$ is no longer I(2) but only I(1), since the first term of (1.8) vanishes. If also $\nu' C_1 = 0$, or $C_1 = 0$, then the process $\nu' X_t$ is stationary and ν reduces the order from 2 to 0, but there is one more situation that can occur and which is of interest for the applications.

Consider vectors ν_1 and ν_2, such that $\nu_1' C_2 = 0$, and $\nu_1' C_1 + \nu_2' C_2 = 0$, then a simple calculation shows that $\nu_1' X_t + \nu_2' \Delta X_t$ is stationary.

This phenomenon, which is called polynomial cointegration, has been studied by Granger and Lee (1989), Engle and Yoo (1989) and Gregoire and Laroque (1991), see also Johansen (1988). Thus for I(2) processes the notion of "equilibrium" or "stable" relation may involve not only the levels but also the differences. There are as well common I(2) trends as common I(1) trends and the formulation of these concepts in terms of the autoregressive parameters is given below.

In order to define the I(2) models we need some notation. For any $p \times r$ matrix α of full rank r we define $\alpha_\perp$ as a $p \times (p-r)$ matrix of full rank such that $\alpha' \alpha_\perp = 0$. We also define $\bar{\alpha} = \alpha(\alpha'\alpha)^{-1}$, such that $\alpha'\bar{\alpha} = I$ and $P_\alpha = \alpha\bar{\alpha}'$ is the projection onto the space spanned by the columns of α.

DEFINITION 2. *The I(2) models $H_{r,s}$, $s = 0,1.,,,.p-r$, $r = 0,\dots,p$ are defined by equation (1.4) together with the conditions*

(1.9) $\quad \Pi = \alpha\beta'$,

(1.10) $\quad \alpha_\perp' \Gamma \beta_\perp = \varphi\eta'$.

Here α and β are $p \times r$ matrices of rank r and φ and η are of dimension $(p-r) \times s$.

Again we have the inclusions

$$H_{r,0} \subset \dots \subset H_{r,s} \subset \dots H_{r,p-r}.$$

Note that $H_{r,p-r}$ leaves $\alpha_\perp' \Gamma \beta_\perp$ unrestricted, but still assumes that there are exactly r cointegrating relations, thus $H_{r,p-r} \subset H_r$. A process satisfying conditions (1.9) and (1.10) with φ and η of full rank has a representation of the form (1.8), see Johansen (1990b) provided an extra condition is satisfied. Thus under these conditions X_t is I(2).

With the autoregressive parameterization (1.9) and (1.10) one can express the various cointegrating properties based on the above mentioned result in the following way: The cointegrating vectors that reduce the order of the process from 2 to 1 are given by the r+s vectors $(\beta, \beta_\perp \eta)$. The p–r–s vectors $\beta_\perp^2 = \bar{\beta}_\perp \eta_\perp$ show which variables are I(2). The coefficients β have the further property that $\beta' X_t$ cointegrates with the differences ΔX_t with coefficients given by $\bar{\alpha}'\Gamma$, such that

$$\beta' X_t + \bar{\alpha}'\Gamma\Delta X_t \tag{1.11}$$

is stationary. Since $(\beta,\beta_\perp\eta)'\Delta X_t$ is already stationary, we also get that

$$\beta' X_t + \bar{\alpha}'\Gamma\bar{\beta}^2_\perp\beta^{2\prime}_\perp\Delta X_t \tag{1.12}$$

is stationary. In equation (1.12) there are r relations. They involve p–r–s I(1) variables $\beta^{2\prime}_\perp\Delta X_t$. If p–r–s < r then these can be eliminated by choosing ξ such that $\xi'\alpha'\Gamma\beta^2_\perp = 0$. In this case we define $\beta_{stat} = \beta\xi$, and (1.12) then implies that $\beta'_{stat}X_t$ is stationary.

The coefficients $(\alpha,\alpha_\perp\varphi)$ have the interpretation as adjustment coefficients to the various disequilibrium errors defined by (1.12) and $\eta'\bar{\beta}'_\perp\Delta X_t$, see (2.8). The common I(2) trends can be defined as $\alpha^2_\perp = \bar{\alpha}_\perp\varphi_\perp$, since the linear combinations $\alpha^{2\prime}_\perp X_t$ evolve without adjusting to any disequilibrium term. The expression for the matrices C_1 and C_2 is rather involved, but some of the structure is apparent from the formula $C_2 = \beta^2_\perp\alpha^{2\prime}_\perp$. The precise formulation of the above results can be found in Johansen (1990b).

2. Statistical analysis of the I(1) and the I(2) model

The Gaussian errors in the models imply that likelihood analysis is feasible. For I(1) models this leads to reduced rank regression of differences on levels corrected for lagged differences and deterministic terms. Likelihood analysis of I(2) models is not so simple, and will not be given here. Instead we show by analyzing the equations defining the model, that by first making the above reduced rank regression in order to estimate r, α and β, and then analyzing the common trends by reduced rank regression of $\hat{\alpha}'_\perp\Delta^2X_t$ on $\hat{\beta}'_\perp\Delta X_{t-1}$ suitably corrected, we can make inferences in I(2) models. The details are given in Johansen (1991b), and illustrated in Johansen (1991c) and Juselius (1991).

2.1 Statistical inference in the I(1) model

Equation (1.3) with the restriction (1.7) can be written as

$$\Delta X_t = \alpha\beta' X_{t-1} + \Sigma_1^{k-1} \Gamma_i \Delta X_{t-i} + \mu + \Psi D_t + \epsilon_t, \tag{2.1}$$

where all parameters are varying freely. This is clearly a linear regression model except for the reduced rank matrix $\alpha\beta'$. The analysis of H_r consists of a preliminary regression of X_t and ΔX_t on lagged differences, constant and seasonal dummies D_t. This gives residuals R_{0t} and R_{1t} and the next step is a reduced rank regression of R_{0t} on R_{1t}. To describe this in more detail define

$$S_{ij} = T^{-1}\Sigma_1^T R_{it} R_{jt}', \quad i,j = 0,1 \tag{2.2}$$

and solve the eigenvalue problem

$$|\lambda S_{11} - S_{10} S_{00} S_{01}| = 0, \tag{2.3}$$

for eigenvalues $1 > \lambda_1 > \ldots > \lambda_p > 0$ and eigenvectors $V = (v_1,\ldots,v_p)$ normalized by $V'S_{11}V = I$. The estimate of β is given by $\hat{\beta} = (v_1,\ldots,v_r)$, the estimate of α is $\hat{\alpha} = S_{01}\hat{\beta}$ and finally the estimate of Ω is $\hat{\Omega} = S_{00} - \hat{\alpha}\hat{\alpha}'$. The maximized likelihood function is apart from a constant given by

$$L_{\max}^{-2/T} = |\hat{\Omega}| = |S_{00}| \prod_{i=1}^{r} (1-\lambda_i). \tag{2.4}$$

This procedure solves the problem for all values of r and the test of H_r in H_p is given by

$$Q_r = -T \sum_{i=r+1}^{p} \ln(1-\lambda_i). \tag{2.5}$$

The asymptotic distribution of Q_r under the assumption of precisely r cointegrating relations depends on the number of non–stationary components, p–r, and on the presence of the linear trend, but does not involve any of the other parameters of the model, see Johansen (1990a). It is non–standard and tabulated by simulation in Johansen and Juselius (1990). If we describe data by a model allowing for a linear trend, that is $\alpha_\perp \mu \neq 0$, then Table A.1 in Johansen and Juselius (1990) gives the quantiles c_{p-r}, say. The value of r is then estimated by the procedure

$$\{\hat{r} = r\} = \{ Q_0 > c_p,\ldots,Q_{r-1} > c_{p-r+1}, Q_r < c_{p-r}\}. \tag{2.6}$$

The analysis of H_r^* which restricts the constant term by $\alpha_\perp'\mu = 0$ is similar and performed by a reduced rank regression of ΔX_t on $(X_{t-1}',1)'$ corrected for lagged differences and seasonal dummies.

2.2 The statistical analysis of the I(2) model

The likelihood analysis of the I(2) model is much more complicated due to the two reduced rank conditions (1.9) and (1.10), see Johansen (1990c). We here apply a different analysis which consists of two reduced rank regressions of the type described for the I(1) model, see Johansen (1991b). The first analysis is the analysis of the I(1) model, that is without the restriction on the matrix Γ as given by (1.10). This determines r, α and β. The next step is an analysis of the I(2) model for fixed values of r, α and β.

To see why this works assume for a moment that r, α and β were known, and consider the equation (1.4). Note that the levels only enter through the term $\alpha\beta' X_{t-2}$, such that if we multiply the equation by $\alpha_\perp'$, the term involving the levels vanishes, and we obtain the equation in differences

$$\alpha_\perp' \Delta^2 X_t = \alpha_\perp' \Gamma \Delta X_{t-1} + \Sigma_1^{k-2} \alpha_\perp' \Phi_i \Delta^2 X_{t-i} + \alpha_\perp' \mu + \alpha_\perp' \Psi D_t + \alpha_\perp' \epsilon_t. \tag{2.7}$$

Now apply the identity

$$P_\beta + P_{\beta_\perp} = \beta\bar{\beta}' + \beta_\perp \bar{\beta}_\perp' = I,$$

to introduce the variables $\bar{\beta}' \Delta X_{t-1}$ and $\bar{\beta}_\perp' \Delta X_{t-1}$ through the expression

$$\begin{aligned} \alpha_\perp' \Gamma \Delta X_{t-1} &= \alpha_\perp' \Gamma (\beta\bar{\beta}' + \beta_\perp \bar{\beta}_\perp') \Delta X_{t-1} \\ &= \alpha_\perp' \Gamma \beta (\bar{\beta}' \Delta X_{t-1}) + \varphi\eta' (\bar{\beta}_\perp' \Delta X_{t-1}). \end{aligned} \tag{2.8}$$

Here we have used condition (1.10) to replace the coefficient matrix $\alpha_\perp' \Gamma \beta_\perp$ by $\varphi\eta'$, thus allowing these to enter directly into the equations. By combining (2.7) and (2.8), it is seen that the analysis for fixed r, α and β consists of a reduced rank regression of the variables $\alpha_\perp' \Delta^2 X_t$ on $\bar{\beta}_\perp' \Delta X_{t-1}$, corrected for lagged second differences, $\bar{\beta}' \Delta X_{t-1}$ together with constant and seasonal dummies.

In view of the fact that the constant term in (2.7) gives rise to a linear trend in the differences, and a quadratic trend in the levels of the process it seems reasonable to restrict the constant by assuming that $\varphi_{\perp}' \alpha_{\perp}' \mu = 0$. The analysis of this model is accomplished by a reduced rank regression of $\alpha_{\perp}' \Delta^2 X_t$ on $((\beta_{\perp}' \Delta X_{t-1})',1)'$ corrected for $\beta' \Delta X_{t-1}$, lagged second differences and seasonal dummies.

The procedure suggested is to analyze the data using the I(1) analysis with μ unrestricted to determine estimates of r, α and β and then analyze equation (2.7) by reduced rank regression for the estimated values of r, α, and β with the constant term restricted using (2.8). The properties of this procedure are given in Johansen (1991b), where the most important result is that the asymptotic distribution of the test statistic for $H_{r,s}$ in $H_{r,p-r}$ is distributed as that of H_r^* in H_p only with p–r–s degrees of freedom. Thus the same tables can be used for the I(1) analysis as for the I(2) analysis. It is also proved that likelihood inference concerning the parameters can be made using the χ^2 distribution, since the parameters are either asymptotically Gaussian or mixed Gaussian.

3. Purchasing power parity between Australia and the United States as an illustration of the I(2) analysis.

The data are quarterly series from 1972:1 to 1991:1 taken from the data base DX (Time Series Data Express v2.1). They consist of the consumer price index for Australia, p_{AU}, and the United States, p_{US}, the exchange rate, exch, measured as the log of the prices of Australian dollars in US dollars, and the 5 year treasury bond rate in both countries, i_{AU} and i_{US}.

The price series are seasonally adjusted. Since there was no series for Australia that covered the whole period, p_{AU} is spliced from two series giving the weighted average for 6, respectively 8, state capital cities. Where the series overlap the difference was +/– 0.1. the data are plotted in levels and differences in the Appendix. It is clear that the series are non–stationary, and that a linear trend is needed to describe the price series. It is not so obvious if the differences are non–stationary, which would require an I(2) analysis. What we can safely assume, however, is that the processes are not I(3), which is the basic assumption for the I(2) analysis to be valid. Questions about the order of integration of the individual variables are then formulated inside the model as restrictions on the parameters.

3.1 The fitting of an autoregressive model with 2 lags

A VAR(2) model was fitted to the data, and some summary statistics are given in Table 1.

TABLE 1

The autocorrelations and diagnostic statistics for the residuals after fitting an AR(2) model.

	B–P(18)	Arch(2)	Skew.	Ex.Kurt
p_{AU}	14.8	1.54	.70	1.097
p_{US}	23.4	1.66	.20	.406
exch	16.5	.21	2.95	1.090
i_{AU}	13.9	3.30	2.54	.757
i_{US}	8.6	20.28	.27	1.707

The test statistics are the Box–Pierce statistic $T\Sigma_1^{18} r_i^2$ which should be compared with the quantiles of a $\chi^2(16)$ distribution, the ARCH statistic which is approximately distributed as $\chi^2(2)$, and finally the skewness and excess kurtosis normalized to be asymptotically distributed as $\chi^2(1)$.

It is seen that there is no autocorrelation left in the residuals, but that the US interest rate has a large ARCH statistic. The asymptotic theory underlying the limit results certainly allows for distributions other than the Gaussian. The main requirement is that their cumulative sums converge to Brownian motions. I do not know how the

ARCH effects influence the results, and the first thing that will be investigated below is whether the data allows a parameter restriction that implies that we can analyze the series conditional on the US interest rate.

3.2 The determination of the cointegration ranks r and s

The I(1) analysis as described in section 2.1 gives the eigenvalues, eigenvectors and adjustment coefficients in Table 2 calculated from (2.3), normalized by the coefficient to p_{AU}.

TABLE 2

The eigenvalues, eigenvectors, and their adjustment coefficients from the I(1) analysis

			Eigenvalues		
	.484	.262	.215	.074	.058
			Long–run coefficients β		
p_{AU}	1.00	1.00	1.00	1.00	1.00
p_{US}	–.95	–1.98	–1.12	–1.65	–1.01
$exch$	.38	.76	–.81	.14	–.35
i_{AU}	–11.75	2.77	4.16	.42	–.98
i_{US}	9.34	3.88	2.03	3.28	–1.39
			Adjustment coefficients α		
Δp_{AU}	–.030	–.013	–.007	.012	–.018
Δp_{US}	.004	–.034	–.001	.028	.008
$\Delta exch$	–.035	–.124	.159	.043	–.064
Δi_{AU}	.028	–.043	.005	–.007	–.009
Δi_{US}	–.008	–.052	–.000	–.027	.018

The test statistics Q_r for testing the hypothesis H_r in H_p are calculated from (2.5) and for each value of r and the corresponding estimates of α and β given in the first r columns of Table 2, we analyze (2.7) by reduced rank regression and restricted constant term, and calculate the statistics $Q^*_{r,s}$ equivalent to (2.5) for testing the hypothesis $H^*_{r,s}$ in $H^*_{r,p-r}$. The results are given in Table 3.

TABLE 3

Test statistics for the determination of the cointegration ranks r and s in the I(2) model

r			$Q^*_{r,s}$			Q_r	p–r	c_{p-r}
0	261.50 s=0	157.44 s=1	95.96 s=2	45.89 s=3	10.91 s=4	101.38	5	68.91
1		178.48 s=0	84.48 s=1	32.61 s=2	9.18 s=3	51.78	4	47.18
2			86.26 s=0	28.10 s=1	4.21 s=2	28.43	3	29.51
3				28.82 s=0	10.04 s=1	10.24	2	15.20
4					8.84 s=0	4.45	1	3.96
p–r–s	5	4	3	2	1			
c^*_{p-r-s}	75.33	53.35	35.07	20.17	9.09			

Table 3 contains information on the cointegrating ranks r and s and should be read as follows: First we determine the rank r. The test statistics Q_r for testing H_r in H_p, the unrestricted VAR model are listed in column 7 next to the degrees of freedom, p–r, and the last column has the 95% quantiles taken from Table A.1 in Johansen and Juselius (1990) which is the relevant one to use if the model allows for an unrestricted constant term, and the linear trend is present. It is seen that the formal procedure for determining $\hat{r}$ at the 5% level gives $\hat{r} = 2$, since $Q_0 > c_5$, $Q_1 > c_4$, but $Q_2 < c_3$. The long–run coefficients given in

Table 2, show that the third eigenvector has roughly the coefficients we are looking for, for the two prices and the exchange rate. The third eigenvalue is a borderline case, and since the asymptotic tables are at best giving the order of magnitude of the actual quantiles, we should be careful not to make too strong decisions based upon the formal test alone. Let us, however, for the moment assume that the best value of $\hat{r}$ is 2, but keep in mind that $\hat{r} = 3$ could be just as good.

Next we investigate the hypotheses $H^*_{0,2}$, $H^*_{1,2}$, and $H^*_{2,2}$. The test statistics $Q^*_{2,0}$ $Q^*_{2,1}$ and $Q^*_{2,2}$, are calculated from formula (2.5) based on reduced rank analysis of equation (2.7) with restricted constant term. The values are given in the row corresponding to r = 2 in Table 3. The quantiles are taken from Table A.3 in Johansen and Juselius (1990). It is seen that s = 0 is strongly rejected since the test statistic $Q^*_{2,0} =$ 86.26 is much greater than the quantile 35.07. Similarly $Q^*_{2,1} = 28.10$ is larger than the quantile 20.17, but $Q^*_{2,2} = 4.21$ is less than the 95 % quantile given as 9.09. If $H^*_{2,2}$ is rejected then the matrix in condition (1.10) is found to have full rank, and the process is integrated of order 1. Thus the statistic $Q^*_{2,2}$ displays the information about I(2)–ness in the data, and $H^*_{2,2}$ is clearly accepted.

We continue the analysis under the assumption that r = 2, s = 2 and p–r–2 = 1, which leaves 1 common I(2) trend in the data. Note that of $\hat{r} = 3$ were chosen then the test statistics $Q^*_{3,0} = 28.82$ and $Q^*_{3,1} = 10.04$ would reject $H^*_{3,0}$ and $H^*_{3,1}$ showing that there are no I(2) trends in the data. We feel that the analysis based upon all of Table 3, rather than just Q_r, helps pick up the correct value not only of s but also of r, by pointing out where the singularity of the matrix in (1.10) is most pronounced.

3.3 Determination of parameter estimates

It is one of the results of Johansen (1991b) that even though the process has I(2) components it still holds that the tests carried out on α and β in the I(1) analysis will be asymptotically distributed as χ^2 due to the fact that the estimator $\hat{\beta}$ derived from the I(1) analysis is asymptotically mixed Gaussian and that of $\hat{\alpha}$ is asymptotically Gaussian. Thus we can test hypotheses on α and β, using the procedures described in Johansen and Juselius (1991).

The hypothesis we want to check is that the last row of α is zero. The reason for this is that in this case the conditional model given i_{US} will yield the estimate of α and β. It is hoped that the conditional model would be slightly better fitted by an autoregressive model since the problematic variable, i_{US}, is kept fixed.

The likelihood ratio test statistic for the hypothesis that $\alpha_{51} = \alpha_{52} = 0$ is 5.66 which evaluated in a χ^2 distribution with 2 degrees of freedom corresponds to a p–value of 6%. We continue the analysis under the assumption that $\alpha_{5.} = 0$, in which case the new estimates of β and α are given in Table 4

TABLE 4

The estimates of α and β under the assumption that $r = 2$, and that $\alpha_{5.} = 0$ normalized by the coefficient to p_{AU}

	$\hat{\beta}$		$\hat{\alpha}$	
p_{AU}	1.000	1.000	–.027	–.006
p_{US}	–.806	–1.087	.006	–.001
$exch$	.323	–.885	–.030	.154
i_{AU}	–13.685	4.244	.027	–.005
i_{US}	9.975	1.961	.000	.000

Note that the second cointegration vector has approximately the coefficients (1,–1,–1,*,*) indicating that the PPP relation needs the interest rates to become stationary. The corresponding adjustment coefficients are very small except for the exchange rate equation indicating that the prices hardly adjust to a deviation from the PPP as measured by the second column of $\hat{\beta}$.

We now proceed to estimate the various vectors which describe the cointegration properties of the process under the assumption that r = 2, and s = 2, such that there is p–r–s = 1 common I(2) trend, and s = 2 common I(1) trends. We apply the estimates of α and β given in Table 4 from the I(1) analysis.

TABLE 5

$\beta^2_\perp = \bar{\beta}_\perp \eta_\perp$	$\alpha^2_\perp = \bar{\alpha}_\perp \varphi_\perp$	β_{stat}
1.742	–.057	1.000
.872	–.064	–1.092
.089	–.004	–.905
–.326	–.047	4.089
.340	–.061	1.829

We have chosen to give only some of the estimates from the I(2) analysis in Table 5 since they are the ones that are most easily interpreted.

The first column is the vector which appears in the Granger representation theorem (1.8) since C_2 is proportional to $\beta^2_\perp \alpha^{2\prime}_\perp$. Thus $\beta^2_\perp$ shows which variables are actually I(2). It points towards the price series, but we do not have information on the variances of the individual coefficients.

Similarly $\alpha^2_\perp$ is interpreted as that linear combination that describes the common I(2) trend. It is seen to put equal weight on prices series and interest rates and the I(2)–ness can thus not be ascribed to any one of the variables.

The final column β_{stat} is found as that linear combination of the r = 2 relations (1.12) which eliminates the contribution from the term $\beta^{2\prime}_\perp \Delta X_t$.

This then is the closest we can come to a stationary relation between the variables in the system under the assumption of an I(2) model. It is seen that the prices and the exchange rates appear roughly with the coefficients (1,–1,–1) as would be expected from the law of one price, but the PPP relation by itself is not stationary, but a relation involving the interest rates can be stationary. This is in accordance with the investigation in Johansen and Juselius (1990) of the Purchasing Power Parity between Denmark and Germany.

6. *Conclusion*

We have analyzed the five series using an I(2) model. It should be pointed out that the analysis by means of a VAR model assumes constant parameters throughout the period and that the shocks can be described by the random ϵ_t's. The non–stationarity of the series is described as the cumulative effect of the shocks, that is, as I(1) or I(2) processes. An alternative description would be as a stationary process with a shift in level around 1980. A careful investigation of these assumptions is not made here.

We find, under the I(2) assumptions, that a linear combination (1,–1,–1,.*,*) is stationary and the Appendix contains a plot of the process $\beta'_{stat}X_t$ together with the PPP relation and the Australian prices. It is seen that the PPP relation is more stable than p_{AU} but that the improvement from PPP to β_{stat}, which involves the interest rates is only slight.

One would perhaps expect that the interest rate differential would be stationary, but any test that we have performed indicates that this is not the case. It is seen from the plots that although they move together for the first half of the period, the US interest rate comes down again, whereas the Australian stay at the high level. Thus there is no comovement in the interest rates. It is seen that the PPP relation needs a lift in the end of the period to become more stable, and this is what the interest rates do. Since the Australian interest rate is best for this it has a higher coefficient than the US interest rate.

Any statistical analysis rests on assumptions, not all of which have been checked in the present application. The point I want to make with this investigation is that the I(2) analysis helps the understanding of the structure of the data and thus improves the chances of building effective economic models.

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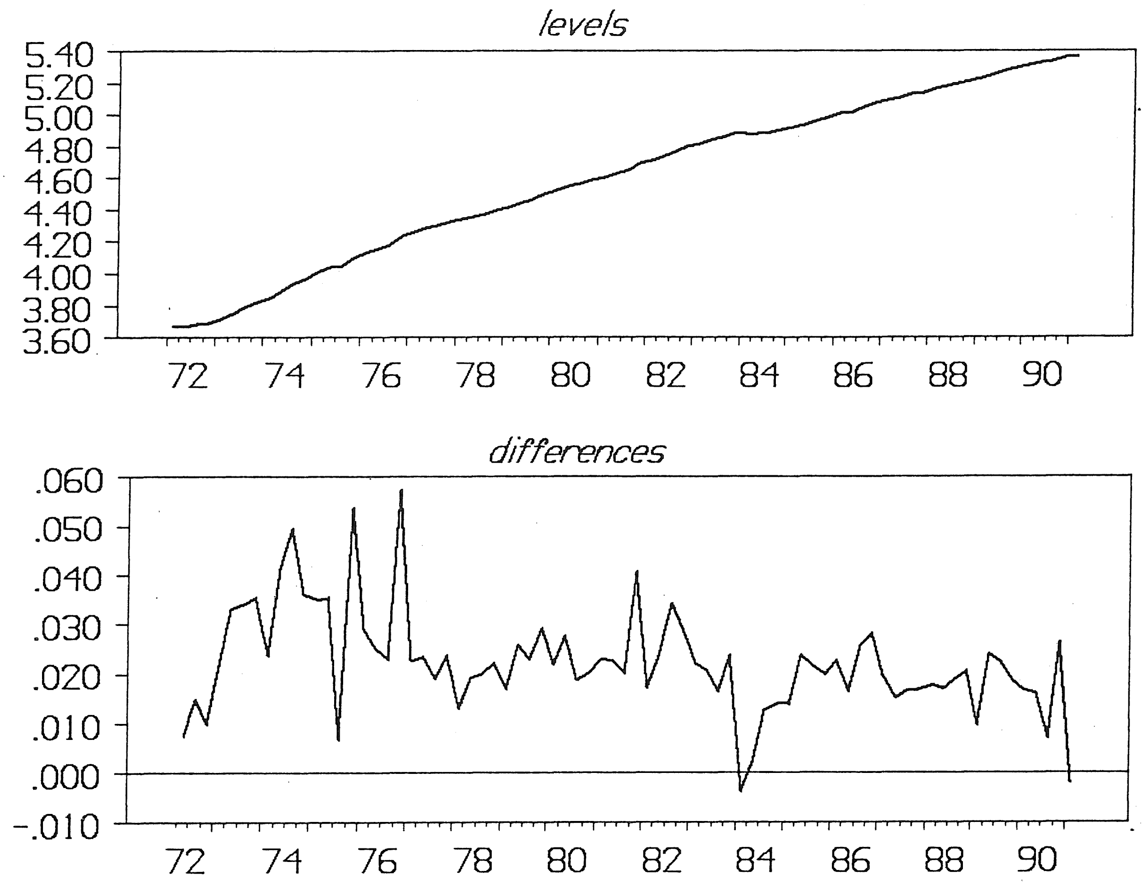
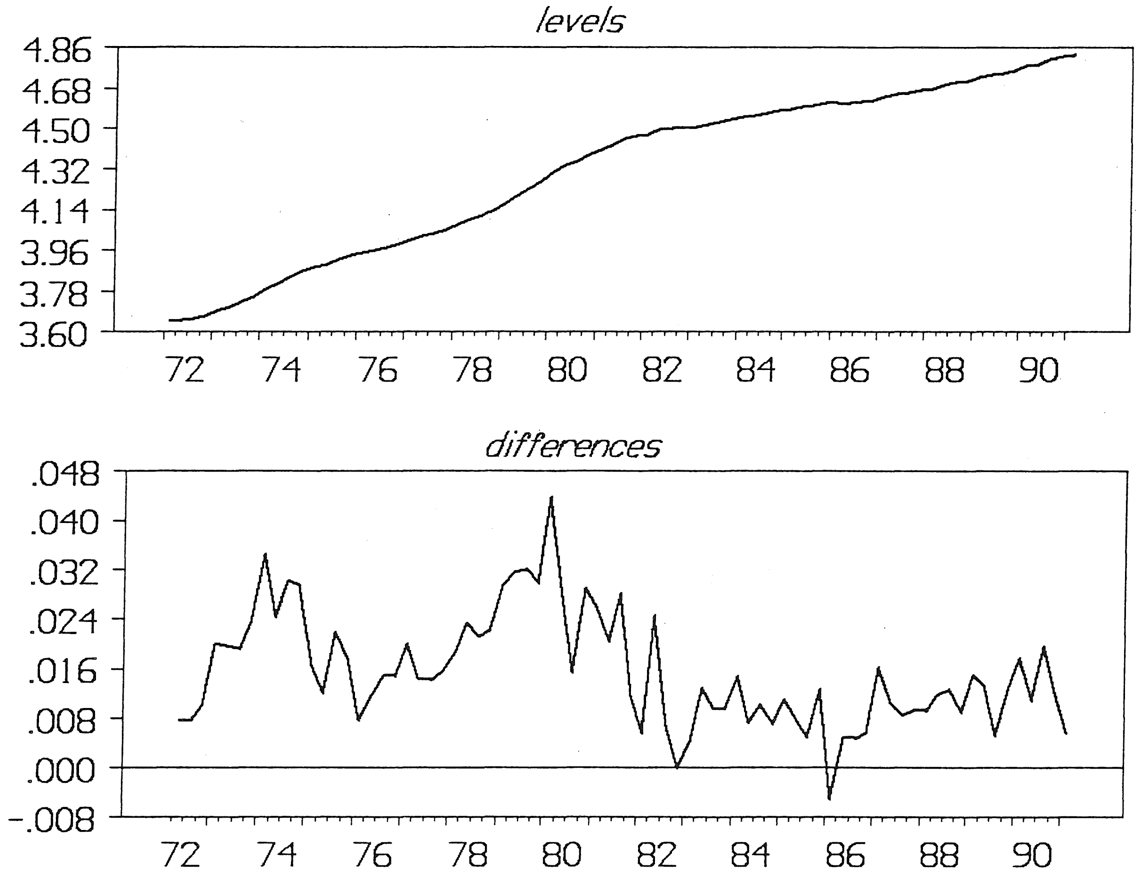
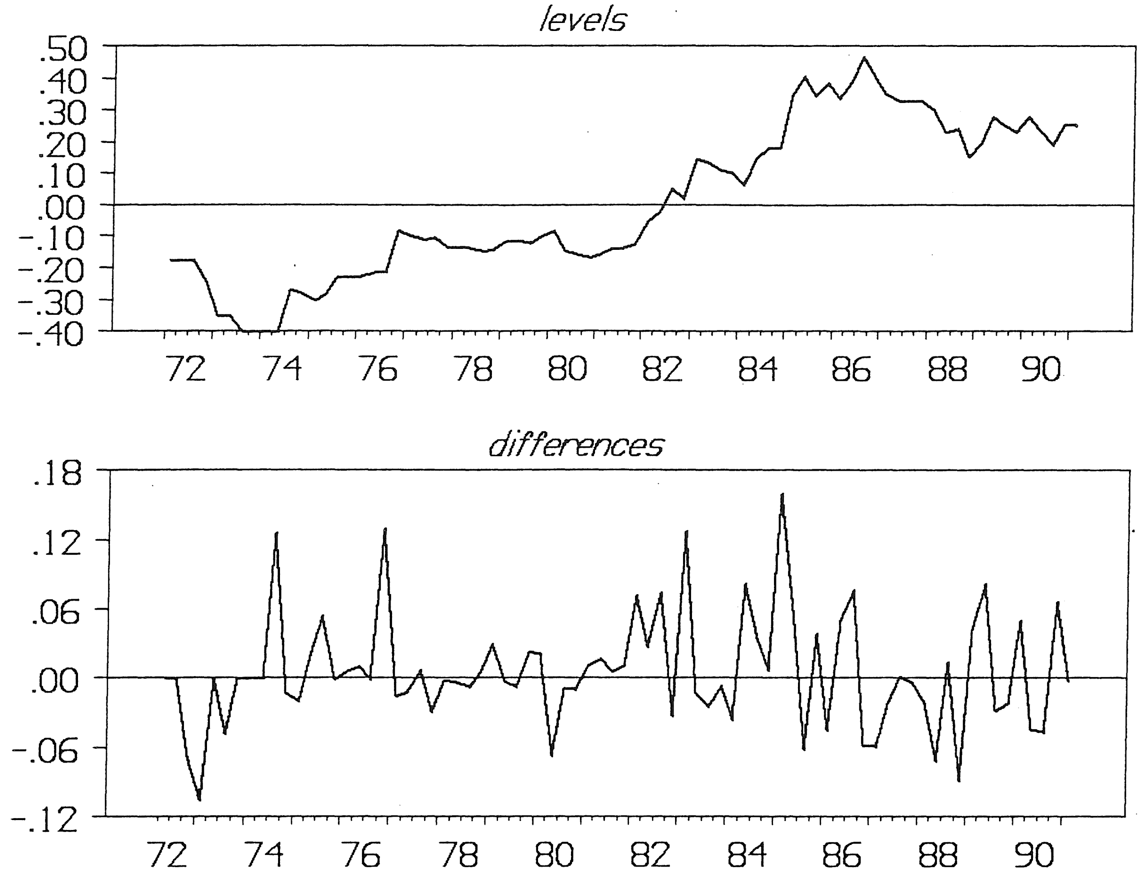
Australian log(CPI)



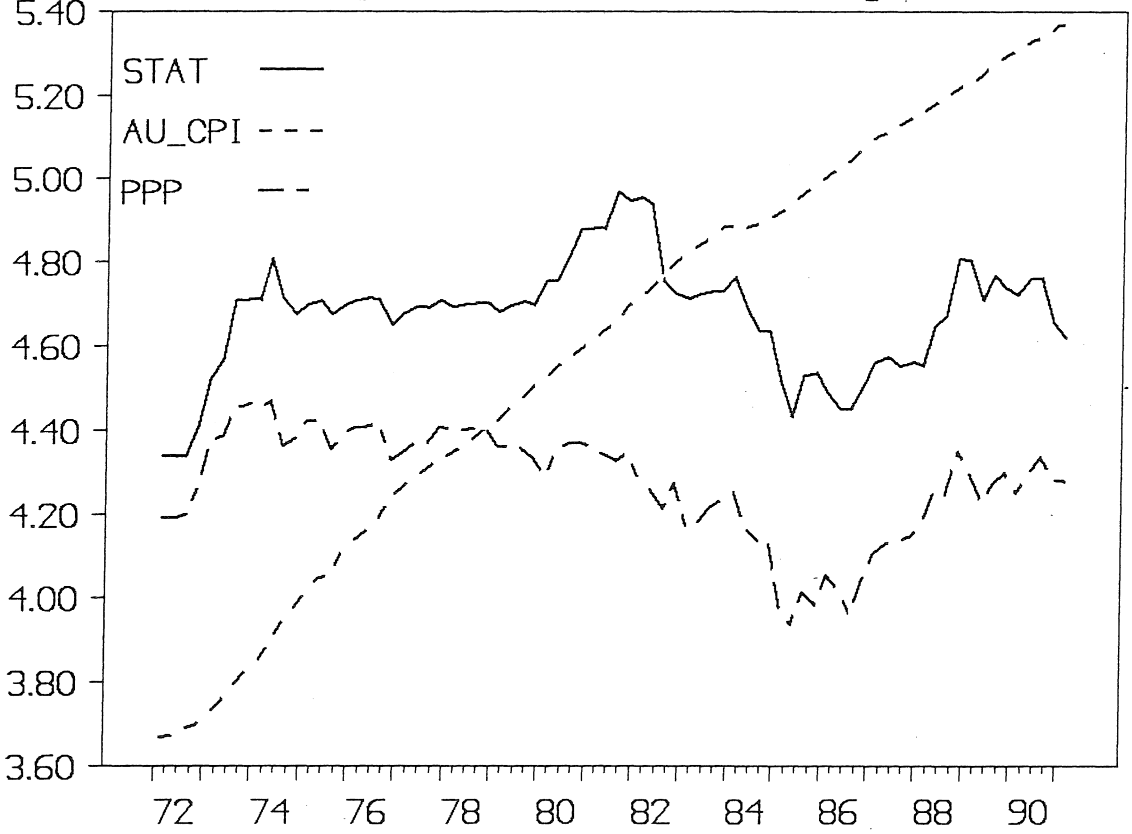
United States log(CPI)



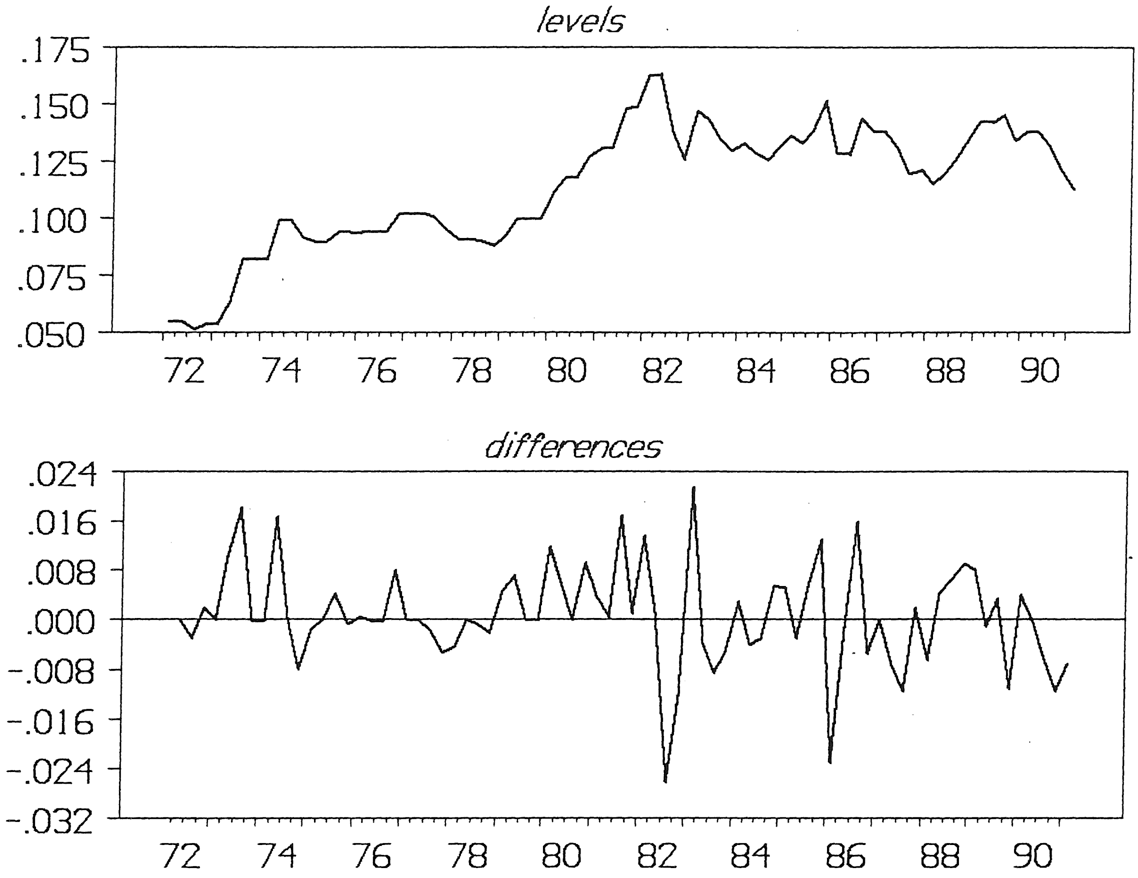
Exchange rate



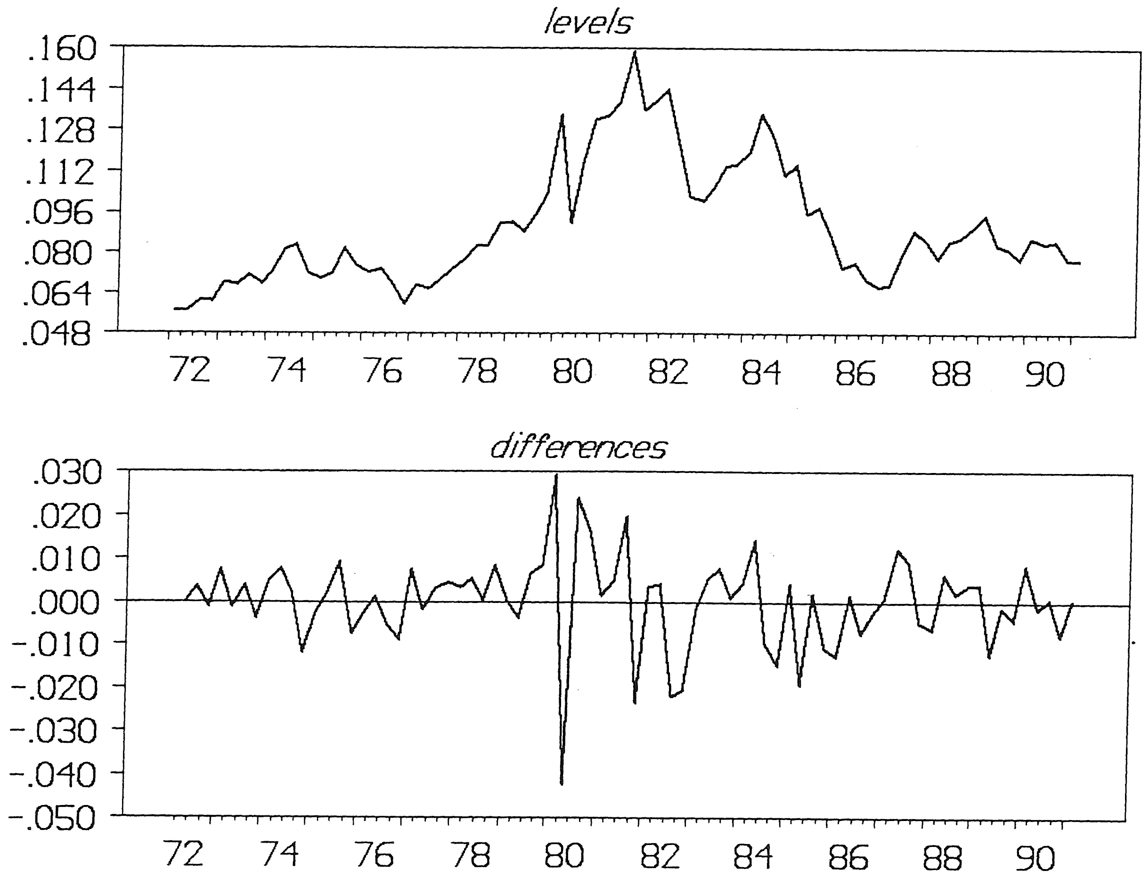
Stationary relation, PPP, AU-log(CPI)



Australian interest rate



United States interest rate



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