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Abstract.

We investigate vector autoregressive processes and find the condition under which the processes are I(2). A representation theorem for such processes is proved and the interpretation of the AR model as an error correction model is discussed.

## 1. *Introduction*

The basic papers by Granger (1983) and Engle and Granger (1987) have started an intense research in the topic of cointegration and its connections with error correction models as originally formulated by Sargan (1964). Most of the work has been connected with processes integrated of order 1, where certain linear combinations are stationary, the so-called cointegrating relations. There are, however, indications that certain economic series are integrated of order 2, and the theory of higher order cointegration has been treated in Yoo (1986), Johansen (1988), Davidson (1988), Engle and Yoo (1989) and Granger and Lee (1988).

It is well-known that a process $X_t \in R^p$ is called integrated of order d if $\Delta^d X_t$ is a stationary invertible process, i.e an I(0) process, and that $X_t$ is cointegrated if for some $\lambda \in R^p$ the process $\lambda' X_t$ is integrated of a lower order than $X_t$. For I(2) processes one can thus be looking for linear combinations that are stationary, but there is clearly also the possibility that some linear combinations are reduced only to I(1) processes. In this case it may occur that these I(1) processes cointegrate with the differences of the process, which is also an I(1) process. An example is given by Granger and Lee (1988), and examples are also given in Johansen (1988).

Just to fix ideas consider two price variables $p_1$ and $p_2$. There is evidence that such series could be I(2) and one could imagine that $p_1 - p_2$ would be more stable, say I(1). The process $\Delta p_1$ is also I(1), and if $p_1 - p_2 - c\Delta p_1$ is stationary, we have an example of what Yoo (1986) calls polynomial cointegration, that is the coefficient of the variables are polynomials in the lag operator.

The present paper poses and solves the following problem: Given an autoregressive multivariate process $X_t$, under what conditions on the coefficients is the process I(2) and how does one calculate the cointegration vectors, and when does one have polynomial cointegration, and what kind of error correction model can be formulated which allow for adjustment to the various equilibrium relations. Part of these problems are apparent already for a real valued process given by

(1.1) $$X_t = \rho_1 X_{t-1} + \rho_2 X_{t-2} + \epsilon_t, \quad t = 1, \ldots, T,$$

where $\epsilon_t$ are i.i.d. Gaussian variables with mean zero and variance $\sigma^2$. It is well-known that the process is stationary, if

$$\rho_2 < 1 - \rho_1, \ \rho_2 < 1 + \rho_1, \text{ and } \rho_2 > -1.$$

If

$$\rho_1 + \rho_2 = 1,$$

the process is non-stationary, but only if also $\rho_2 > -1$ is the process $X_t$ an I(1) process. If instead $\rho_2 = -1$, then $\rho_1 = 2$ and the process is an I(2) process. Thus in order to test whether the process $X_t$ is an I(2) process or an I(1) process one must have precise conditions on the coefficients of the AR process, in order to derive the likelihood ratio test for the null of I(2)-ness. What is presented in this paper is the neccessary mathematical background for the understanding of the properties of the process $X_t$ under the various hypotheses.

Consider therefore the vector autoregressive model with Gaussian errors in p dimensions

$$A(L)X_t = \epsilon_t, \ t = 1,\ldots,T,$$

where $A(z) = \sum_{i=0}^{k} A_i z^i$. Define $\Pi = A(1)$, $\Psi = -\,dA(z)/dz|_{z\,=\,1}$, and $\Phi = ½d^2A(z)/dz^2|_{z\,=\,1}$. By expanding the polynomial A(z) around z = 1, the model can be written as

$$\Pi X_t + \Psi\Delta X_t + \Phi\Delta^2 X_t + A_3(L)\Delta^3 X_t = \epsilon_t, \tag{1.2}$$

where $A_3(L)$ is defined by the equation

$$A(z) = \Pi + \Psi(1-z) + \Phi(1-z)^2 + A_3(z)(1-z)^3. \tag{1.3}$$

ASSUMPTION 1. *The roots of* $|A(z)| = 0$ *are either outside the unit disc or equal to 1.*

It is well-known that under this assumption a necessary and sufficient conditon for $X_t$ to be stationary is that there are no unit roots:

THEOREM 1. *Under Assumption 1 a necessary and sufficient condition for* $X_t$ *to be stationary is that*

(1.4) $\Pi$ *has full rank.*

*In this case* $X_t$ *has the representation*

(1.5) $$X_t = A^{-1}(L)\epsilon_t = C_0(L)\epsilon_t,$$

*where the matrix valued function* $C_0(z)$ *has exponentially decreasing coefficients.*

Next we want to see what condition is needed on the parameters of the process in order that the process be an I(1) process. Clearly $\Pi$ has to be singular, but that is not enough, and the results can be formulated as Grangers representation theorem:

THEOREM 2. *Under Assumption 1 a necessary and sufficient condition for* $X_t$ *to be I(1) is that there exist matrices* $\alpha$ *and* $\beta$ $(p\times r)$, $r < p$, *of full rank such that*

(1.6) $$\Pi = \alpha\beta'$$

(1.7) $\alpha_\perp'\Psi\beta_\perp$ *has full rank.*

*Here* $\alpha_\perp$ *and* $\beta_\perp$ *are* $p\times(p-r)$ *matrices of full rank such that* $\alpha'\alpha_\perp = \beta'\beta_\perp = 0$. *In this case* $X_t$ *has the representation*

(1.8) $$X_t = \beta_\perp(\alpha_\perp'\Psi\beta_\perp)^{-1}\alpha_\perp'\sum_{i=1}^{t}\epsilon_i + C_1(L)\epsilon_t.$$

*Note that* $\beta'X_t$ *is stationary, and that* $\Delta X_t$ *is stationary. The matrix function* $C_1(z)$ *has exponentially decreasing coefficients.*

The proof of this is given in Johansen (1990) and can be briefly described as follows: Define $\beta^- = \beta(\beta'\beta)^{-1}$ such that $\beta'\beta^- = I$, and $\beta^-\beta' = P_\beta$, the projection onto the space spanned by $\beta$. Next define

$$Y_t = \beta' X_t \tag{1.9}$$

and

$$U_t = \beta_\perp' \Delta X_t, \tag{1.10}$$

so that

$$X_t = \beta^- Y_t + \beta_\perp^- \Delta^{-1} U_t = \beta^- Y_t + \beta_\perp^- \sum_{i=1}^{t} U_i. \tag{1.11}$$

The transformation from $(\beta,\beta_\perp)'X_t$ to $(Y_t,U_t)$ is given by the matrix polynomial

$$\begin{bmatrix} Y_t \\ U_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \beta' X_t \\ \beta_\perp' X_t \end{bmatrix} \tag{1.12}$$

with determinant

$$\Delta^{p-r}.$$

Thus no extra roots inside or outside the unit disc are introduced by this transformation. It turns out that the condition (1.7) guarantees that the AR model for $(Y_t,U_t)$ is invertible, and that Theorem 1 can be applied to the process $(Y_t,U_t)$.

The condition (1.7) is needed, but one can of course alternatively assume that the process is I(1). It seems reasonable, however, to formulate the condition in terms of the coefficients of the polyniomial A(z), since these are readily estimated, and since the condition (1.7) suggests a test for I(1)-ness, namely that the matrix $\alpha_\perp'\Psi\beta_\perp$ has full rank. We then need the properties of the processes under the null of reduced rank, and hence we shall investigate in the next section what happens when (1.7) fails. For the process $X_t$ given by (1.1) it is well known that the condition for non-stationarity is that $\rho_1 + \rho_2 = 1$, and by formulating this explicitly in terms of the coefficients, one is lead to the usual Dickey-Fuller test. To understand this test one then needs the properties of $X_t$ under the null of non-stationarity. Similarly if one wanted to test that the process is I(1) one could formulate the null hypothesis that $\rho_1 + \rho_2 = 1$ and $\rho_2 = -1$, and under this null the process would be I(2).

## 2. *The representation of I(2) processes*

In order to formulate the results we need some notation. We define

(2.1) $$M_{ab} = (a'a)^{-1}a'Mb(b'b)^{-1} = \bar{a}'M\bar{b},$$

where M,a, and b are matrices of matching dimensions. This means that M has the representation

$$M = (P_\alpha + P_{\alpha_\perp})M(P_\beta + P_{\beta_\perp}) = \alpha M_{\alpha\beta}\beta' + \alpha_\perp M_{\alpha_\perp\beta}\beta' + \alpha M_{\alpha\beta_\perp}\beta_\perp' + \alpha_\perp M_{\alpha_\perp\beta_\perp}\beta_\perp'.$$

If we are interested in I(2) processes, then the matrix given in (1.7) has to have reduced rank, and we assume, see (2.9), that $\Psi_{\alpha_\perp\beta_\perp} = \varphi\eta'$ for some (p-r)×s matrices $\varphi$ and $\eta$ of full rank. This gives rise to the following natural coordinate system: Let $\alpha_1 = \alpha_\perp\varphi$ and $\beta_1 = \beta_\perp\eta$ and supplement with $\beta_2 = (\beta,\beta_1)_\perp$ and $\alpha_2 = (\alpha,\alpha_1)_\perp$, such that $(\alpha,\alpha_1,\alpha_2)$ as well as $(\beta,\beta_1,\beta_2)$ are orthogonal and span $R^p$.

Note that

$$\begin{aligned}\alpha_2'\Psi\beta_\perp(\beta_\perp'\beta_\perp)^{-1} &= \alpha_2'(\alpha(\alpha'\alpha)^{-1}\alpha' + \alpha_\perp(\alpha_\perp'\alpha_\perp)^{-1}\alpha_\perp')\Psi\beta_\perp(\beta_\perp'\beta_\perp)^{-1}\\ &= \alpha_2'\alpha_\perp\Psi_{\alpha_\perp\beta_\perp} = \alpha_2'\alpha_\perp\varphi\eta' = \alpha_2'\alpha_1\eta' = 0,\end{aligned}$$

and similarly one gets

(2.2) $$(\alpha_\perp'\alpha_\perp)^{-1}\alpha_\perp'\Psi\beta_2 = 0.$$

It is illustrative to rewrite the model (1.2) in the coordinates given by $(\alpha,\alpha_1,\alpha_2)$ and $(\beta,\beta_1,\beta_2)$, that is by multiplying the matrices by $(\bar{\alpha},\bar{\alpha}_1,\bar{\alpha}_2)'$ and $(\bar{\beta},\bar{\beta}_1,\bar{\beta}_2)$. We then find the first three matrices to be

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \Psi_{\alpha\beta} & \Psi_{\alpha\beta_1} & \Psi_{\alpha\beta_2} \\ \Psi_{\alpha_1\beta} & \Psi_{\alpha_1\beta_1} & 0 \\ \Psi_{\alpha_2\beta} & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \Phi_{\alpha\beta} & \Phi_{\alpha\beta_1} & \Phi_{\alpha\beta_2} \\ \Phi_{\alpha_1\beta} & \Phi_{\alpha_1\beta_1} & \Phi_{\alpha_1\beta_2} \\ \Phi_{\alpha_2\beta} & \Phi_{\alpha_2\beta_1} & \Phi_{\alpha_2\beta_2} \end{bmatrix}.$$

It turns out that the matrix

(2.3) $$\Theta = \Phi - \Psi\beta(\beta'\beta)^{-1}(\alpha'\alpha)^{-1}\alpha'\Psi$$

plays an important role in the formulation of the results, and it is convenient to have a special notation for it.

Finally introduce the variables

(2.4) $$U_t = \beta_1'\Delta X_t,$$

(2.5) $$V_t = \beta_2'\Delta^2 X_t,$$

(2.6) $$Y_t = \beta' X_t + \Psi_{\alpha\beta_2}\Delta^{-1}V_t = \beta' X_t + \Psi_{\alpha\beta_2}\beta_2'\Delta X_t,$$

such that

(2.7) $$X_t = \bar{\beta} Y_t + \bar{\beta}_1\Delta^{-1}U_t + \bar{\beta}_2\Delta^{-2}V_t - \bar{\beta}\Psi_{\alpha\beta_2}\Delta^{-1}V_t.$$

The idea in the following is to show that $(Y_t,U_t,V_t)$ is a stationary process under suitable restrictions on the parameters, and the representation (2.7) then determines the order of integration of the process $X_t$ in the various directions $(\beta,\beta_1,\beta_2)$. Thus if $\mu \in R^p$, then $\mu' X_t$ is I(2) unless $\mu \in sp(\beta,\beta_1)$, i.e. orthogonal to $\beta_2$. If $\mu \in sp(\beta,\beta_1)$ then $\mu' X_t$ is I(1), unless $\mu$ is orthogonal to the vectors in $\bar{\beta}\Psi_{\alpha\beta_2}$, in which case the process $\mu' X_t$ is stationary, see Corollary 4.

THEOREM 3. *Under Assumption 1, a necessary and sufficient condition that* $X_t$ *be* $I(2)$ *is that there exist matrices* $\alpha,\beta$ $(p\times r)$, $r < p$, *and* $\varphi,\eta$ $(p-r)\times s$, $s < p-r$, *of full rank such that*

$$\Pi = \alpha\beta', \tag{2.8}$$

$$\Psi_{\alpha_\perp\beta_\perp} = \varphi\eta', \tag{2.9}$$

*and such that*

$$\Phi_{\alpha_2\beta_2} - \Psi_{\alpha_2\beta}\Psi_{\alpha\beta_2} = \theta_{\alpha_2\beta_2} \tag{2.10}$$

*has full rank.*

*In this case the variables* $(Y_t, U_t, V_t)$ *given by* (2.4), (2.5) *and* (2.6) *are all stationary and one finds the representations*

$$V_t = \theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2'\epsilon_t + C_v(L)\Delta\epsilon_t. \tag{2.11}$$

$$U_t = (\Psi^{-1}_{\alpha_1\beta_1}\bar{\alpha}_1' - \theta_{\alpha_1\beta_2}\theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2')\epsilon_t + C_u(L)\Delta\epsilon_t. \tag{2.12}$$

$$\begin{aligned} X_t = {} & (\bar{\beta}_1\Psi^{-1}_{\alpha_1\beta_1}\bar{\alpha}_1' - \bar{\beta}_1\theta_{\alpha_1\beta_2}\theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2' - \bar{\beta}\Psi_{\alpha\beta_2}\theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2')\sum_{i=1}^{t}\epsilon_i \\ & + \bar{\beta}_2(\theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2'\sum_{s=1}^{t}\sum_{i=1}^{s}\epsilon_i + C_v(L)\sum_{i=1}^{t}\epsilon_i) + C_2(L)\epsilon_t. \end{aligned} \tag{2.13}$$

PROOF: The transformation from $(\beta,\beta_1,\beta_2)'X_t$ to $(Y_t,U_t,V_t)$ has matrix

$$\begin{array}{cl} \begin{bmatrix} I & 0 & \Delta\Psi_{\alpha\beta_2} \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta^2 \end{bmatrix} & \begin{array}{l} r \\ s \\ p-r-s \end{array} \end{array}$$

with determinant $\Delta^s\Delta^{2(p-r-s)}$ which is only zero for $L = 1$. Thus no extra roots are induced either inside or outside the unit disc by this transformation. Hence the AR representation for the new variables has no roots outside the unit disc, by Assumption 1, and we only have to check that there are now no roots for $z = 1$. We insert the expression for $X_t$ (2.7) into the model (1.2). This gives a relation involving $\Delta^{-2}, \Delta^{-1}, \Delta^{0}, \Delta, \ldots$. We show that the choice of $(Y_t, U_t, V_t)$ makes the coefficients of $\Delta^{-2}$ and $\Delta^{-1}$ vanish.

The term $\Delta^{-2}V_t$ enters only in the expression for the levels of $X_t$, and the coefficient is

$$\alpha\beta'\bar{\beta}_2 V_t = \alpha\beta'\beta_2(\beta_2'\beta_2)^{-1}V_t = 0.$$

The coefficient of $\Delta^{-1}$ can be found to be

$$\alpha\beta'(\bar{\beta}_1 U_t - \bar{\beta}\Psi_{\alpha\beta_2}V_t) + \Psi\bar{\beta}_2 V_t.$$

The coefficient to $U_t$ is $\alpha\beta'\bar{\beta}_1 = 0$, and the coefficient to $V_t$ is

$$-\alpha\Psi_{\alpha\beta_2} + \Psi\bar{\beta}_2.$$

If we multiply by $\alpha'$ we get

$$-\alpha'\Psi\beta_2(\beta_2\beta_2)^{-1} + \alpha'\Psi\bar{\beta}_2 = 0.$$

And when we multiply by $\alpha_\perp'$ we find

$$\alpha_\perp'\Psi\bar{\beta}_2 = 0,$$

by the construction of $\beta_2$, see (2.2).

Finally we investigate the coefficient matrix to the levels $\Delta^0$. We find

$$\alpha Y_t + \Psi(\bar{\beta}_1 U_t - \bar{\beta}\Psi_{\alpha\beta_2}V_t) + \Phi\bar{\beta}_2 V_t$$

Now multiply by the matrix $(\bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2)'$ and obtain the coefficient matrix to $(Y_t, U_t, V_t)$

$$\begin{bmatrix} I & \Psi_{\alpha\beta_1} & (\Phi_{\alpha\beta_2} - \Psi_{\alpha\beta}\Psi_{\alpha\beta_2}) \\ 0 & \Psi_{\alpha_1\beta_1} & (\Phi_{\alpha_1\beta_2} - \Psi_{\alpha_1\beta}\Psi_{\alpha\beta_2}) \\ 0 & 0 & (\Phi_{\alpha_2\beta_2} - \Psi_{\alpha_2\beta}\Psi_{\alpha\beta_2}) \end{bmatrix} = \begin{bmatrix} I & \Psi_{\alpha\beta_1} & \theta_{\alpha\beta_2} \\ 0 & \Psi_{\alpha_1\beta_1} & \theta_{\alpha_1\beta_2} \\ 0 & 0 & \theta_{\alpha_2\beta_2} \end{bmatrix}$$

Under the assumptions of the Theorem this matrix is invertible and we find that the leading terms are

$$V_t = \theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2\epsilon_t,$$

and

$$U_t = \Psi^{-1}_{\alpha_1\beta_1}\bar{\alpha}_1'\epsilon_t - \theta_{\alpha_1\beta_2}\theta^{-1}_{\alpha_2\beta_2}\bar{\alpha}_2'\epsilon_t$$

from which the representations (2.11), (2.12) and (2.13) follow from (2.7).

Note that the representation (2.7) gives directly the leading term involving $\Delta^{-2}\epsilon_t$, but the term involving $\Delta^{-1}\epsilon_t$ comes from three sources, partly from $\Delta^{-1}U_t$, partly from $\Delta^{-1}V_t$, but also from $\Delta^{-2}V_t$ which would require the expression for the coefficient to $\Delta\epsilon_t$ in the expression of $V_t$. This can clearly be derived by going into detail with the inversion of the stationary process for $(Y_t, U_t, V_t)$, but we shall not give the result here, since whenever a linear combination of the process $X_t$ is considered which lies in the space spanned by $\beta_2$, then $\Delta^{-2}\epsilon_t$ is dominating, and whenever the linear combination is chosen orthogonal to $\beta_2$, then the leading term will be the coefficient to $\Sigma_i^t\epsilon_i$ in the first term of (2.13).

COROLLARY 4. *If the conditions of Theorem 3 hold and if further*

(2.14) $$\Psi_{\alpha\beta_2} = \xi\tau'$$

*where $\xi$ (r×m) is of full rank, m < r, then $\xi_\perp'\beta'X_t$ is stationary.*

PROOF: This follows from the stationarity of $Y_t$ by multiplying by $\xi_\perp'$.

As $\Psi_{\alpha\beta_2}$ is r×(p−r−s) such a $\xi$ can always be found if r > p−r−s. Thus the linear combinations $\xi_\perp'\beta X_t$ are the combinations of the I(2) processes that are stationary.

In the special case where $\Psi_{\alpha\beta_2} = 0$ all the combinations $\beta'X_t$ are stationary. In this case the condition for the process to be I(2) as given by (2.10) reduces to the condition that $\Phi_{\alpha_2\beta_2}$ be of full rank. This case is termed the balanced case in Johansen (1988), and is here seen to be a rather special, and perhaps not too interesting case. In the balanced case (2.7) shows that $X_t$ can be decomposed into the directions $(\beta,\beta_1,\beta_2)$ giving I(0), I(1) and I(2) processess respectively.

In general $\beta'X_t$ will be I(1) and only by involving $\beta_1'\Delta X_t$ can we get a stationary process. Thus $X_t$ is here cointegrated with its differences. It is this phenomenon which is called multicointegration by Granger and Lee (1988) and polynomial cointegration by Yoo (1986).

If we have the further condition that $\Psi_{\alpha_\perp\beta_\perp} = 0$ we get another model that has been studied before: namely the model of multicointegration discussed by Yoo (1986), see also Engle and Yoo (1989). He found by application of the Smith-McMillan form for matrix polynomials, that the following error correction model appeared in a natural way

(2.15) $$\Delta^2X_t = (\alpha + \tilde{\alpha}\Delta)(\beta + \tilde{\beta}\Delta)'X_{t-1} + A_2(L)\Delta^2X_{t-1} + \epsilon_t.$$

$$= \alpha\beta' X_{t-1} + (\alpha\tilde{\beta}' + \tilde{\alpha}\beta')\Delta X_{t-1} + A_2^{*}(L)\Delta^2 X_{t-1},$$

which is seen to satisfy the condition that $\Psi_{\alpha_\perp\beta_\perp} = 0$, and any model with this property can be written in the form (2.15).

Some of the results of Theorem 3 and Corollary 4 are related to Theorem 4.3 by Davidson (1988), but they are here given in a more explicit form, as conditions directly on the coefficients of the AR model.

Note that from Theorem 3 one can easily find the asymptotic properties of the process $X_t$, thus for instance one finds that

$$T^{-3/2} X_{[Tt]} \overset{w}{\to} \beta_{\bar{2}} \theta^{-1}_{\alpha_2\beta_2} \alpha_{\bar{2}}' \int_0^t W(u)du$$

$$= \beta_2\{\alpha_2'[\Phi - \Psi\beta(\beta'\beta)^{-1}(\alpha'\alpha)^{-1}\alpha\Phi]\beta_2\}^{-1}\alpha_2' \int_0^t W(u)du$$

as $T \to \infty$.

## 3. *The error correction model*

Consider first the case of I(1) variables. The model (1.2) can, under the condition (1.6), be written in the form

$$\Delta X_t = \alpha\beta' X_{t-1} + A_1(L)\Delta X_{t-1} - \epsilon_t \tag{3.1}$$

which gives the interpretation of the model as an error correction model, where the current values of the changes react to the disequilibrium error $e_t = \beta' X_t$ lagged one period, with adjustment coefficients $\alpha$. Note that all terms in (3.1) are stationary.

Next consider the case of I(2) variables. In this situation the error correction model is a lot more complicated, since one can imagine that adjustment can occur to any of the many different stationary relations we have found. Under condition (2.9) and (2.14) model (1.2) can be written as

$$(3.2)\quad \Delta^2 X_t = (\alpha\xi_\perp)(\beta\xi_\perp^-)'X_{t-2} + (\alpha\xi)((\beta\xi^-)'X_{t-2} + \tau'\beta_2'\Delta X_{t-1}) + (2\alpha + \Psi\beta(\beta'\beta)^{-1}, \alpha_1 + \alpha\Psi_{\alpha\beta_1})(\beta,\beta_1)'\Delta X_{t-1} + A_2(L)\Delta^2 X_t - \epsilon_t.$$

This formulation gives the possibility to interprete the stationary processes

$$e_{1t} = (\beta\xi_\perp^-)'X_t,$$
$$e_{2t} = (\beta\xi^-)'X_{t-1} + \tau'\beta_2'\Delta X_t,$$

and

$$e_{3t} = (\beta,\beta_1)'\Delta X_t,$$

as disequilibrium errors affecting the second difference of the process through the adjustment coefficients

$$\alpha\xi_\perp,\ \alpha\xi\ ,\ (2\alpha + \Psi\beta(\beta'\beta)^{-1}, \alpha_1 + \alpha\Psi_{\alpha\beta_1})$$

respectively. The vector $(\beta,\beta_1)$ reduce the order of the process from 2 to 1 and in this sense $(\beta,\beta_1)'X_t$ represents a stable relation among the variables. The polynomial vector $(\beta\xi^-, \tau'\beta_2\Delta)$ represents a polynomial cointegrating vector which reduces the process to I(0), with adjustment coefficients $\alpha\xi$, and finally $\beta\xi_\perp$ are the linear combinations that reduce the process to stationarity, with adjustmnent coefficients $\alpha\xi_\perp$.

Note that the choice of lags in the representation (3.2) is not so important, since if for example we prefer $\beta'X_{t-1} + \Psi_{\alpha\beta_2}\beta_2'\Delta X_{t-1}$ instead of $\beta'X_{t-2} + \Psi_{\alpha\beta_2}\beta_2'\Delta X_{t-1}$, the difference involves $\Delta\beta'X_{t-1}$ which can be absorbed in the coefficient to $\beta'\Delta X_{t-1}$.

*Example 1.* Consider the model for the two dimensional process $X_t$ given by

$$-\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X_t + \begin{bmatrix} 0 & ½ \\ ½ & 2+a \end{bmatrix} \Delta X_t + \begin{bmatrix} 0 & -½ \\ -½ & -1 \end{bmatrix} \Delta^2 X_t = \epsilon_t, \ t = 0,1,\ldots T.$$

The determinant of the characteristic polynomial is found to be

$$|A(z)| = (a + 1-z - (1-z)z^2/4)(1-z),$$

which is seen to have no roots inside the unit disc if either $a = 0$ or if $a \geq 3$, say. The matrix $\Pi$ has reduced rank, and we can define $\alpha = \beta = (1,2)'$, so that we can choose $\alpha_\perp = \beta_\perp = (-2,1)'$. It is easily seen that $\alpha_\perp' \Psi \beta_\perp = a$, such that if $a > 0$ the process $X_t$ is I(1) and the cointegrating relation is found from the first row of the $\Pi$ matrix

$$\beta' X_1 = X_{1t} + 2X_{2t}.$$

If $a = 0$ condition (2.9) is satisfied with $\varphi = \eta = 0$, and in this case $\beta_2 = \alpha_2 = \beta_\perp = \alpha_\perp$. One can check that the condition (2.10) is satisfied and hence that the process is I(2) in this case. Thus any linear combination which is not orthogonal to $-2X_{1t} + X_{2t}$ is I(2), and the combination $\beta' X_t = X_{1t} + 2X_{2t}$ is I(1). There is cointegration between the levels and the differences, since

$$\beta' X_t + \Psi_{\alpha\beta_2} \beta_2' \Delta X_t = 2X_{1t} + X_{2t} + (-2\Delta X_{1t} + \Delta X_{2t})/10,$$

is stationary, but no linear combination of levels is stationary.

*Example 2* This model was proposed by Hendry and von Ungern-Sternberg (1981) and discussed by Johansen (1988). Let $X_t = (c_t, l_t, z_t)$ denote the logarithm of consumption, personal sector liquid assets and disposable income respectively. The model considered takes the form

$$\Delta c_t = \beta\Delta z_t + \gamma_{11}(z_{t-1}-c_{t-1}) + \gamma_{12}(z_{t-1}-l_{t-1}) - \epsilon_{1t}$$
$$\Delta l_t = \gamma_{21}(z_{t-1}-c_{t-1}) - \epsilon_{2t}.$$

In order to get a full system and to illustrate the methods of this paper we add the following equation

$$\Delta^2 z_t = \epsilon_{3t}.$$

We find

$$\Pi = \begin{bmatrix} \gamma_{11} & \gamma_{12} & -\gamma_{11}-\gamma_{12} \\ \gamma_{21} & 0 & -\gamma_{21} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 1-\gamma_{11} & -\gamma_{12} & \gamma_{11}+\gamma_{12}-\beta \\ -\gamma_{21} & 1 & \gamma_{21} \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One easily finds that

$$\alpha' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \beta' = \begin{bmatrix} \gamma_{11} & \gamma_{12} & -\gamma_{11}-\gamma_{12} \\ \gamma_{21} & 0 & -\gamma_{21} \end{bmatrix},$$

such that $\alpha_\perp' = (0,0,1)$ and $\beta_\perp' = (1,1,1)$. In this case $\alpha_\perp'\Psi\beta_\perp = 0$, such that $\alpha_1 = \beta_1 = (0,0,0)$ and $\alpha_2 = \alpha_\perp$, and $\beta_2 = \beta_\perp$, and the condition (2.10) is satisfied, since $\alpha_2'\Psi = 0$ and $\alpha_2'\Phi\beta_2 = 1$. Hence the process $X_t$ is I(2). Thus the linear combinations $\beta'X_t$ has two components: $\gamma_{11}(z_t-c_t) + \gamma_{12}(z_t-l_t)$ and $\gamma_{21}(z_t-c_t)$, which are non-stationary I(1) processes, but they cointegrate with $\Delta X_t$, since $\Psi_{\alpha\beta_2} = 3^{-1}(1-\beta,1)'$ is non-zero. There is, however, a linear combination of $\beta'X_t$ that is stationary, since the vector $(-1,1-\beta)$ annihilates $\Psi_{\alpha\beta_2}$. This means that the combination

$$(\beta\bar{\xi}_\perp)'X_t = [(\gamma_{21}(1-\beta)-\gamma_{11}))(z_t-c_t) - \gamma_{12}(z_t-l_t)]/[1+(1-\beta)^2]$$

is a stationary relation between the I(2) variables $(c_t, l_t, z_t)$, which in the error correction form (3.2) has adjustment coefficients

$$\alpha\xi_\perp = (-1, 1-\beta, 0)'.$$

## 4. *References*

Davidson, J. (1988). The cointegration properties of VAR models. UCSD Discussion paper.

Engle, R.F. and Granger, C.W.J. (1987). Cointegration and Error Correction: Representation, Estimation and Testing, *Econometrica*, 55, 251-271.

Engle, R.F. and Yoo, B.S. (1989). Cointegrated economic time series: A survey with new results. UCSD Discussion paper.

Granger, C.W.J. and Lee, T-H. (1988). Multicointegration. UCSD Discussion paper # 24.

Granger, C.W.J. and Lee, T-H. (1988). Investigation of Production, Sales and Inventory Relationships using Multicointegration and Nonsymmetric Error Correction Models. UCSD Discussion paper # 38.

Granger, C.W.J. (1983). Cointegrated Variables and Error Correction Models UCSD Discussion paper # 13.

Hendry, D. F. and von Ungern-Sternberg, T. (1981). Liquidity and inflation effects of consumer expenditure. In A.S. Deaton (ed.) *Essays in the theory and measurement of consumers' behaviour*. Cambridge University Press.

Johansen, S. (1988). The mathematical structure of error correction models. *Contempory mathematics*, 80, 359-386.

Johansen, S. (1990). Estimation and Hypothesis testing of cointegration vectors in gaussian vactor autoregressive models. University of Copenhagen Preprint

Sargan, J. D. (1964). Wages and prices in the United Kingdom. A study in econometric methodology. In P.E.Hart, G. Mills and J.K. Whitaker, (eds.) *Econometric analysis for national economic planning*. Butterworths, London.

Yoo, B. S. (1986). Multi-cointegrated time series and a generalized error-correction models. UCSD Discussion paper

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